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06b. Convolutions

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Convolutions

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1. *Convolution on $L^1(\mathbb{R}^n)$*

The formulaic definition of *convolution* of $f, g \in L^1(\mathbb{R}^n)$ is as a pointwise (a.e.) function

$$(f * g)(x) \;=\; \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

For each fixed x, it is not *a priori* clear that the integral converges. If $f, g \in L^2$, then Cauchy-Schwarz-Bunyakowsky could be invoked to show that the integral converges absolutely, but on $\mathbb{R}^n$ there are L^1 functions that are not L^2. So we need

[1.1] Claim: For $f, g \in L^1(\mathbb{R}^n)$, $f * g \in L^2(\mathbb{R}^n)$, and

$$|f * g|_{L^1} \;\le\; |f|_{L^1} \cdot |g|_{L^1}$$

Proof: [... iou ...] ///

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2. *Structural meaning of convolution*

The *right translation* action of $\mathbb{R}^n$ on functions on $\mathbb{R}^n$ is

$$(R_g f)(x) \;=\; f(x+g) \qquad \text{(for } x, g \in \mathbb{R}^n\text{)}$$

The right invariance of the measure/integral immediately gives the invariance of the L^2 norm, for example:

$$|R_g f|^2_{L^2} \;=\; \int_{\mathbb{R}}^n |f(g+x)|^2\, dx \;=\; \int_{\mathbb{R}}^n |f(x)|^2\, dx \;=\; |f|^2_{L^2}$$

[2.1] Claim: The map $\mathbb{R}^n \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $g \times f \to R_g f$ is continuous.

Proof: Fix $f \in L^2(\mathbb{R}^n)$, and take $\varepsilon > 0$. By density of $C^o_c(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, take $\varphi \in C^o_c(\mathbb{R}^n)$ such that $|f - \varphi|_{L^2} < \varepsilon$. Since φ is compactly supported, φ is *uniformly* continuous: for all $\varepsilon' > 0$, there is a neighborhood N of $e \in \mathbb{R}^n$ such that $|\varphi(xh) - \varphi(x)| < \varepsilon'$ for all $h \in N$, for all $x \in \mathbb{R}^n$. For $g \in N$,

$$|R_g f - f|_{L^2} \;\le\; |R_g f - R_g \varphi|_{L^2} + |R_g \varphi - \varphi|_{L^2} + |\varphi - f|_{L^2}$$

$$\leq \; |f - \varphi|_{L^2} + \varepsilon' \cdot \text{meas}\,(\text{spt}\,\varphi) + |\varphi - f|_{L^2} \;=\; \varepsilon + \varepsilon' \cdot \text{meas}\,(\text{spt}\,\varphi) + \varepsilon$$

Given ε and φ, shrink N so that $\varepsilon' \leq \text{meas}\,(\text{spt}\,\varphi)$, so $|R_g f - f|_{L^2} < 3\varepsilon$ for $g \in N$. ///

Integral-operator action of $C_c^o(\mathbb{R}^n)$ on functions on $\mathbb{R}^n$: Let $\varphi \in C_c^o(\mathbb{R}^n)$ act on functions on $\mathbb{R}^n$ by

$$(\varphi \cdot f)(x) \;=\; \int_{\mathbb{R}}^n \varphi(g) \cdot f(x+g)\, dg$$

Convolution We do not need to *define* convolution of $C_c^o(\mathbb{R}^n)$ functions, but, rather, *discover* what kind of product on such functions is compatible with repeated application of the integral operators. That is, for $\varphi, \psi \in C_c^o(\mathbb{R}^n)$, we want

$$(\varphi * \psi) \cdot f \;=\; \varphi \cdot \Big(\psi \cdot f\Big)$$

It hardly matters what topological vector space f lies in, whether or not it is a space of functions on $\mathbb{R}^n$, since the same identity should hold regardless.

Compute directly, granting[1] that continuous operators commute with these integrals and, of course, scalars commute with linear operators:

$$\varphi \cdot \Big(\psi \cdot f\Big) \;=\; \int_{\mathbb{R}}^n \varphi(g) R_g \int_{\mathbb{R}}^n \psi(h)\, R_h\, f\; dh\, dg \;=\; \int_{\mathbb{R}}^n \int_{\mathbb{R}}^n \varphi(g)\psi(h) R_g\, R_h\, f\; dh\, dg$$

$$= \int_{\mathbb{R}}^n \int_{\mathbb{R}}^n \varphi(g)\psi(h) R_{g+h}\, f\; dh\, dg$$

At this point, there are two possible courses of action, either replace g by $g-h$, or h by $h-g$. Both choices are completely reasonable, but in the non-commutative case the *appearances* would be different. Let's replace g by $g-h$, assuming that dg refers to a *right* invariant measure on $\mathbb{R}^n$. First interchanging the order of integration, do the change of variables, and then change back:

$$\varphi \cdot \Big(\psi \cdot f\Big) \;=\; \int_{\mathbb{R}}^n \Big(\int_{\mathbb{R}}^n \varphi(g-h)\psi(h) R_g\, f\; dg\Big)\, dh \;=\; \int_{\mathbb{R}}^n \Big(\int_{\mathbb{R}}^n \varphi(g-h)\psi(h)\; dh\Big) R_g f\; dg$$

$$= \Big(\int_{\mathbb{R}}^n \varphi(g-h)\psi(h)\; dh\Big) \cdot f$$

That is, we have more-or-less *proven*

[2.2] Proposition: Convolution

$$(\varphi * \psi)(g) \;=\; \int_{\mathbb{R}}^n \varphi(g-h)\psi(h)\; dh$$

for $\varphi, \psi \in L^1$ gives the associativity

$$(\varphi * \psi) \cdot f \;=\; \varphi \cdot \Big(\psi \cdot f\Big) \qquad\qquad (\text{for all } f \in L^2(\mathbb{R}^n))$$

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[2.3] Remark: In fact, the above discussion applies, with reasonable modifications, to arbitrary topological groups G in place of $\mathbb{R}^n$, and very general topological vector spaces in place of $L^2(\mathbb{R}^n)$.

[1] Since the present argument does not show that continuous linear operators move inside the integrals, it remains a heuristic. But, eventually, we can rigorize it, in terms of *Gelfand-Pettis vector-valued integrals*.

3. $\widehat{f*g} = \widehat{f}*\widehat{g}$

The idea is that Fourier transform converts convolution to pointwise multiplication.

[3.1] Claim: For $f,g \in L^1(\mathbb{R}^n)$,

$$\widehat{f*g} \;=\; \widehat{f}*\widehat{g}$$

Indeed, from above, $f*g \in L^1$, so the Fourier transform integral converges absolutely. Also, $\widehat{f}$ and $\widehat{g}$ are in fact *continuous*, by Riemann-Lebesgue, so there is certainly no ambiguity in talking about multiplication of point-wise values.

Proof: [... iou ...] ///

4. δ as unit in convolution algebra

With suitable hypotheses on f,

[4.1] Claim: $\delta * f = f * \delta = f$ and $\delta' * f = f * \delta' = f'$.

[... iou ...]

5. Cautionary example

Disturbingly, *associativity* does not hold for arbitrary triples of distributions:

[5.1] Claim: Let H be the Heaviside step function, 0 left of 0, and 1 right of 0. Since

$$(1*\delta')*H \;=\; 1'*H \;=\; 0*H \;=\; 0 \;\neq\; 1 \;=\; 1*\delta \;=\; 1*(\delta'*H)$$

associativity fails here. ///

The spectral theory for *normal compact* operators on Hilbert spaces, and basic properties of Gelfand-Pettis integrals of vector-valued functions, have immediate application: uniqueness of invariant (Haar) measure on compact abelian groups A, and then proof that

$$L^2(A) \;=\; \text{completion of} \bigoplus_{\chi:A\to\mathbb{C}^\times} \mathbb{C}\cdot\chi$$

where χ runs over continuous *characters* of A, that is, continuous group homomorphisms $A \to \mathbb{C}^\times$. These characters arise as *simultaneous eigenfunctions* for the integral operators

$$T_\varphi \;:\; f \;\longrightarrow\; \int_A \varphi(y)\,f(x+y)\,dy \qquad\qquad (\text{for } \varphi \in C_c^o(A) \text{ and } f \in L^2(A))$$

normalized to $\chi(0)=1$, writing A additively. This gives another approach to the L^2 theory of Fourier series on circles or products of circles, as well as harmonic analysis on the p-adic integers $\mathbb{Z}_p$, and more exotic items such as *solenoids* $\mathbb{A}/\mathbb{Q}$, where $\mathbb{A}$ is the adele group.

6. Approximate identities

One notion of *approximate identity* $\{\varphi_i\}$ on $\mathbb{R}^n$ is a sequence of *non-negative* $\varphi_i \in C_c^o(\mathbb{R}^n)$ whose supports shrink to $\{0\}$, in the sense that, given a neighborhood N of 0, there is i_o such that for all $i \geq i_o$ the support of φ_i is inside N. Further,

$$\int_{\mathbb{R}^n} \varphi_i(g)\, dg \;=\; 1$$

A less strict version replaces the shrinking of supports with the condition that, for every $\varepsilon > 0$ and $\delta > 0$, there is sufficiently large i_o such that for every $i \geq i_o$

$$\int_{|x|\geq\delta} \varphi_i(g)\, dg \;<\; \varepsilon$$

[6.1] Claim: For $f \in L^2(\mathbb{R}^n)$ and for approximate identity φ_i,

$$\varphi_i \cdot f \;\longrightarrow\; f$$

Proof: [... iou ...] ///
