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07. *Banach Spaces*

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Many natural spaces of functions, such as $C^o(K)$ for K compact, and $C^k[a,b]$, have natural structures of Banach spaces.

Abstractly, Banach spaces are less convenient than Hilbert spaces, but still sufficiently simple so many important properties hold. Several standard results true in greater generality have simpler proofs for Banach spaces.

Riesz' lemma is an elementary result often an adequate substitute in Banach spaces for the lack of sharper Hilbert-space properties. We include natural counter-examples to the *minimum principle* valid in Hilbert spaces, but not generally valid in Banach spaces.

The Banach-Steinhaus/uniform-boundedness theorem, open mapping theorem, and closed graph theorem are not elementary, since they invoke the Baire category theorem. The Hahn-Banach theorem is non-trivial, but does *not* use completeness.

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1. *Basic Definitions*

A real or complex [1] vectorspace V with a real-valued function, the *norm*,

$$| \, | : V \longrightarrow \mathbb{R}$$

with properties

$$\begin{array}{ll} |x+y| \le |x| + |y| & \text{(triangle inequality)} \\ |\alpha x| = |\alpha| \cdot |x| & (\alpha \text{ complex, } x \in V) \\ |x| = 0 \;\Rightarrow\; x = 0 & \text{(positivity)} \end{array}$$

is a *normed complex vectorspace*, or simply *normed space*. Because of the triangle inequality, the function

$$d(x,y) \;=\; |x-y|$$

is a *metric*. The *symmetry* comes from

$$d(y,x) \;=\; |y-x| \;=\; |(-1)\cdot(x-y)| \;=\; |-1|\cdot|x-y| \;=\; |x-y| \;=\; d(x,y)$$

[1] In fact, for many purposes, the scalars need not be $\mathbb{R}$ or $\mathbb{C}$, need not be locally compact, and need not even be commutative. The basic results hold for Banach spaces over non-discrete, complete, normed division rings. This allows scalars like the p-adic field $\mathbb{Q}_p$, or Hamiltonian quaternions $\mathbb{H}$, and so on.

When V is *complete* with respect to this metric, V is a *Banach space*.

Hilbert spaces are Banach spaces, but many natural Banach spaces are *not* Hilbert spaces, and may fail to enjoy useful properties of Hilbert spaces. *Riesz' lemma* below is sometimes a sufficient substitute.

Most norms on Banach spaces do *not* arise from inner products. Norms arising from inner products recover the inner product via the *polarization* identities

$$4\langle x, y\rangle \;=\; |x+y|^2 \;-\; |x-y|^2 \qquad \text{(real vector space)}$$

$$4\langle x, y\rangle \;=\; |x+y|^2 \;-\; |x-y|^2 \;+\; i|x+iy|^2 \;-\; i|x-iy|^2 \qquad \text{(complex vector space)}$$

Given a norm on a vector space, *if* the polarization expression gives an inner product, *then* the norm is produced by that inner product. However, checking whether the polarization expression is bilinar or hermitian, may be awkward or non-intuitive.

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2. *Riesz' Lemma*

The following essentially elementary inequality is sometimes an adequate substitute for corollaries of the Hilbert-space minimum principle and its corollaries. Once one sees the proof, it is not surprising, but,

[2.1] Lemma: *(Riesz)* For a non-dense subspace X of a Banach space Y, given $r < 1$, there is $y \in Y$ with $|y| = 1$ and $\inf_{x\in X} |x - y| \geq r$.

Proof: Take y_1 not in the closure of X, and put $R = \inf_{x\in X} |x - y_1|$. Thus, $R > 0$. For $\varepsilon > 0$, let $x_1 \in X$ be such that $|x_1 - y_1| < R + \varepsilon$. Put $y = (y_1 - x_1)/|x_1 - y_1|$, so $|y| = 1$. And

$$\inf_{x\in X} |x-y| \;=\; \inf_{x\in X} \Big| x + \frac{x_1}{|x_1-y_1|} - \frac{y_1}{|x_1-y_1|} \Big| \;=\; \inf_{x\in X} \Big| \frac{x}{|x_1-y_1|} + \frac{x_1}{|x_1-y_1|} - \frac{y_1}{|x_1-y_1|} \Big|$$

$$=\; \frac{\inf_{x\in X}|x-y_1|}{|x_1-y_1|} \;=\; \frac{R}{R+\varepsilon}$$

By choosing $\varepsilon > 0$ small, $R/(R+\varepsilon)$ can be made arbitrarily close to 1. ///

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3. *Counter-examples to unique norm-minimizing element*

The (true) *minimum principle* for Hilbert spaces is that a closed, convex subset has a unique element of minimum norm. This has many important elementary corollaries special to Hilbert spaces, such as existence of orthogonal complements to subspaces, and often fails for Banach spaces.

An important historical example of failure of functionals to attain their infs on closed, convex subsets of Banach spaces is the falsity of the *Dirichlet principle* as originally naively proposed. [2]

[2] The *Dirichlet principle*, invoked by Riemann but observed by Weierstraß to be false as stated, would assert that a solution of $\Delta u = f$ on an open set Ω in $\mathbb{R}^n$, with boundary condition $u|_{\partial\Omega} = g$ on $\partial\Omega$, is a minimizer of the *energy integral*

$$E(u) \;=\; \int_\Omega \tfrac{1}{2}|\nabla u|^2 + |u|^2$$

on the Banach subspace of $C^2(\Omega)$ functions u satisfying $u|_{\partial\Omega} = g$. However, the infimum need not be attained in that Banach space. Hilbert justified Dirichlet's principle in certain circumstances. Beppo Levi (1906) observed that using energy integrals to form the norm (squared) of a pre-Hilbert space in $C^2(\Omega)$, and *completing* to a Hilbert space, *does* guarantee existence of a solution *in that Hilbert space*.

The (true) minimizing principle in a Hilbert space V is that, in a closed, convex, non-empty subset $E \subset V$, there is a unique element of least norm. As an example corollary, for non-dense subspace W of a Hilbert space V, there is $v \in V$ with $|v| = 1$ and $\inf_{w \in W} |v - w| = 1$, by taking v to be a unit-length vector in the *orthogonal complement* to W. This minimization property typically *fails* in Banach spaces, as follows.

[3.1] Example: *Many* minimizing elements can exist: in the Banach space $L^1[a,b]$, in the closed, convex subset $E = \{f : \int_a^b f = 1\}$, there are infinitely-many norm-minimizing elements.

[3.2] Example: In the Banach space $Y = C^o[0,2]$, with closed convex subset

$$E \;=\; \{f \in C^o[0,2] \;:\; \int_0^1 f(x)\,dx - \int_1^2 f(x)\,dx = 1\}$$

there is *no* norm-minimizing element. To this end, let

$$s(x) \;=\; \begin{cases} 1 & (\text{for } 0 \le x \le 1) \\ -1 & (\text{for } 1 \le x \le 2) \end{cases}$$

and

$$\lambda(f) \;=\; \int_0^2 f(x) \cdot s(x)\,dx$$

Certainly $C^o[0,2] \subset L^2[0,2]$, so by Cauchy-Schwarz-Bunyakowsky,

$$|\lambda(f)| \;=\; |\langle f, s\rangle| \;\le\; |f|_{L^2[0,2]} \cdot |s|_{L^2[0,2]} \;=\; |f|_{L^2[0,2]} \cdot \sqrt{2}$$

with equality only for f a scalar multiple of s. Also, certainly

$$|f|_{L^2[0,2]} \;\le\; \Big(\int_0^2 |f|_Y^2\Big)^{\frac{1}{2}} \;=\; |f|_Y \cdot \sqrt{2} \qquad (\text{for } f \in C^o[0,2])$$

Since s is not continuous, non-zero $f \in Y$ is never a constant multiple of s, so Cauchy-Schwarz-Bunyakowsky gives a *strict* inequality

$$|\lambda(f)| \;<\; |f|_{L^2[0,2]} \cdot \sqrt{2} \;\le\; |f|_Y \cdot 2 \qquad (\text{for all } 0 \ne f \in Y)$$

Thus,

$$\frac{1}{2} \;<\; |f|_Y \qquad (\text{for } f \in E)$$

Yet it is easy to arrange continuous functions f with $\lambda(f) = 1$ and sup-norm $|f|_Y$ *approaching* $1/2$ from above, by approximating $\frac{1}{2}s(x)$ by continuous functions. For example, form a continuous, piecewise-linear function

$$g(x) \;=\; \begin{cases} \frac{1}{2} & (\text{for } 0 \le x \le 1-\varepsilon) \\ \frac{1}{2} - \dfrac{x-(1-\varepsilon)}{2\varepsilon} & (\text{for } 1-\varepsilon \le x \le 1+\varepsilon) \\ -\frac{1}{2} & (\text{for } 1+\varepsilon \le x \le 2) \end{cases}$$

The sup-norm of g is obviously $\frac{1}{2}$, and $\lambda(g) = 1 - \frac{1}{2}\varepsilon$. Thus, functions $f = g/(1 - \frac{1}{2}\varepsilon)$ have $\lambda(f) = 1$ and sup norms approaching $\frac{1}{2}$ from above. This proves the claimed failure. ///

4. *Normed spaces of linear maps*

There is a *natural norm* on the set of continuous linear maps $T : X \to Y$ from one normed space X to another normed space Y. Even when X, Y are Hilbert spaces, the set of continuous linear maps $X \to Y$ is generally only a *Banach* space.

Let $\mathrm{Hom}^o(X,Y)$ denote [3] the collection of continuous linear maps from the normed vectorspace X to the normed vectorspace Y. Use the same notation $|\,|$ for the norms on both X and Y, since context will make clear which is meant.

A linear (not necessarily continuous) map $T : X \to Y$ from one normed space to another has *uniform operator norm*

$$|T| \;=\; |T|_{\mathrm{uniform}} \;=\; \sup_{|x|\le 1} |Tx|$$

where we allow the value $+\infty$. Such T is called *bounded* if $|T| < +\infty$. There are several obvious variants of the expression for the uniform norm:

$$|T| \;=\; \sup_{|x|\le 1} |Tx| \;=\; \sup_{|x|<1} |Tx| = \sup_{|x|\ne 0} \frac{|Tx|}{|x|}$$

[4.1] Proposition: For a linear map $T : X \to Y$ from one normed space to another, the following conditions are equivalent:
- T is *continuous*.
- T is *continuous at* 0.
- T is *bounded*.

Proof: First, show that continuity *at a point* x_o implies continuity *everywhere*. For another point x_1, given $\varepsilon > 0$, take $\delta > 0$ so that $|x - x_o| < \delta$ implies $|Tx - Tx_o| < \varepsilon$. Then for $|x' - x_1| < \delta$

$$|(x' + x_o - x_1) - x_o| \;<\; \delta$$

By linearity of T,

$$|Tx' - Tx_1| \;=\; |T(x' + x_o - x_1) - Tx_o| \;<\; \varepsilon$$

which is the desired continuity at x_1.

Now suppose that T is continuous at 0. For $\varepsilon > 0$ there is $\delta > 0$ so that $|x| < \delta$ implies $|Tx| < \varepsilon$. For $x \ne 0$,

$$\Big|\frac{\delta}{2|x|} x\Big| \;<\; \delta$$

so

$$\Big|T\frac{\delta}{2|x|}\cdot x\Big| \;<\; \varepsilon$$

Multiplying out and using the linearity, boundedness is obtained:

$$|Tx| \;<\; \frac{2\varepsilon}{\delta}\cdot |x|$$

[3] Another traditional notation for the collection of continuous linear maps from X to Y is $B(X,Y)$, where B stands for *bounded*.

Finally, prove that boundedness implies continuity at 0. Suppose there is C such that $|Tx| < C|x|$ for all x. Then, given $\varepsilon > 0$, for $|x| < \varepsilon/C$

$$|Tx| < C|x| < C \cdot \frac{\varepsilon}{C} \;=\; \varepsilon$$

which is continuity at 0. ///

The space $\mathrm{Hom}^o(X,Y)$ of continuous linear maps from one normed space X to another normed space Y has a natural structure of vectorspace by

$$(\alpha T)(x) \;=\; \alpha \cdot (Tx) \qquad \text{and} \qquad (S+T)x \;=\; Sx + Tx$$

for $\alpha \in \mathbb{C}$, $S, T \in \mathrm{Hom}^o(X,Y)$, and $x \in X$.

[4.2] **Proposition:** With the uniform operator norm, the space $\mathrm{Hom}^o(X,Y)$ of continuous linear operators from a normed space X to a *Banach* space Y is *complete*, whether or not X itself is complete. Thus, $\mathrm{Hom}^o(X,Y)$ is a Banach space.

Proof: Let $\{T_i\}$ be a Cauchy sequence of continuous linear maps $T : X \to Y$. Try defining the limit operator T in the natural fashion, by

$$Tx \;=\; \lim_i Tx_i$$

First, check that this limit exists. Given $\varepsilon > 0$, take i_o large enough so that $|T_i - T_j| < \varepsilon$ for $i, j > i_o$. By the definition of the uniform operator norm,

$$|T_i x - T_j x| \;<\; |x|\varepsilon$$

Thus, the sequence of values $T_i x$ is Cauchy in Y, so has a limit in Y. Call the limit Tx.

We need to prove that the map $x \to Tx$ is *continuous* and *linear*. The arguments are inevitable. Given $c \in \mathbb{C}$ and $x \in X$, for given $\varepsilon > 0$ choose index i so that for $j > i$ both $|Tx - T_j x| < \varepsilon$ and $|Tcx - T_j cx| < \varepsilon$. Then

$$|Tcx - cTx| \le |Tcx - T_j cx| + |cT_j x - cTx| = |Tcx - T_j cx| + |c| \cdot |T_j x - Tx| < (1 + |c|)\varepsilon$$

This is true for every ε, so $Tcx = cTx$. Similarly, given $x, x' \in X$, for $\varepsilon > 0$ choose an index i so that for $j > i$ $|Tx - T_j x| < \varepsilon$ and $|Ty - T_j y| < \varepsilon$ and $|T(x+y) - T_j(x+y)| < \varepsilon$. Then

$$|T(x+y) - Tx - Ty| \le |T(x+y) - T_j(x+y)| + |T_j x - Tx| + |T_j y - Ty| < 3\varepsilon$$

This holds for every ε, so $T(x+y) = Tx + Ty$.

For continuity, show that T is *bounded*. Choose an index i_o so that for $i, j \ge i_o$

$$|T_i - T_j| \le 1$$

This is possible since the sequence of operators is Cauchy. For such i, j

$$|T_i - T_j x| \;\le\; |x|$$

for all x. Thus, for $i \ge i_o$

$$|T_i x| \;\le\; |(T_i - T_{i_o})x| + |T_{i_o} x| \;\le\; |x|(1 + |T_{i_o}|)$$

Taking a limsup,

$$\limsup_i |T_i x| \;\le\; |x|(1 + |T_{i_o}|)$$

This implies that T is bounded, and so is continuous.

Finally, we should see that $Tx = \lim_i T_i x$ is the operator-norm limit of the T_i. Given $\varepsilon > 0$, let i_o be sufficiently large so that $|T_i x - T_j x| < \varepsilon$ for all $i, j \geq i_o$ and for all $|x| \leq 1$. Then $|Tx - Tx_i| \leq \varepsilon$ and

$$\sup_{|x|\leq 1} |Tx - T_i x| \;\leq\; \sup_{|x|\leq 1} \varepsilon \;=\; \varepsilon$$

giving the desired outcome. ///

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5. *Dual spaces of normed spaces*

This section considers an important special case of continuous linear maps between normed spaces, namely continuous linear maps from Banach spaces to *scalars*. All assertions are special cases of those for continuous linear maps to general Banach spaces, but deserve special attention.

For X a normed vectorspace with norm $|\,|$, a *continuous* linear map $\lambda : X \to \mathbb{C}$ is a *(continuous linear) functional* on X. Let

$$X^* \;=\; \mathrm{Hom}^o(X, \mathbb{C})$$

denote the collection of all such (continuous) functionals.

As more generally, for *any* linear map $\lambda : X \to \mathbb{C}$ of a normed vectorspace to $\mathbb{C}$, the *norm* $|\lambda|$ is

$$|\lambda| \;=\; \sup_{|x|\leq 1} |\lambda x|$$

where $|\lambda x|$ is the absolute value of the value $\lambda x \in \mathbb{C}$. We allow the value $+\infty$. Such a linear map λ is *bounded* if $|\lambda| < +\infty$.

As a special case of the corresponding general result:

[5.1] Corollary: For a k-linear map $\lambda : X \to k$ from a normed space X to k, the following conditions are equivalent:
- The map λ is *continuous.*
- The map λ is *continuous at one point.*
- The map λ is *bounded.*

Proof: These are special cases of the earlier proposition where the target was a general Banach space.
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The *dual space*

$$X^* \;=\; \mathrm{Hom}^o(X, \mathbb{C})$$

of X is the collection of *continuous* linear functionals on X. This dual space has a natural structure of vectorspace by

$$(\alpha\lambda)(x) \;=\; \alpha \cdot (\lambda x) \qquad \text{and} \qquad (\lambda + \mu)x \;=\; \lambda x + \mu x$$

for $\alpha \in \mathbb{C}$, $\lambda, \mu \in X^*$, and $x \in X$. It is easy to check that the norm

$$|\lambda| \;=\; \sup_{|x|\leq 1} |\lambda x|$$

really is a norm on X^*, in that it meets the conditions

- *Positivity:* $|\lambda| \geq 0$ with equality only if $\lambda = 0$.
- *Homogeneity:* $|\alpha\lambda| = |\alpha| \cdot |\lambda|$ for $\alpha \in k$ and $\lambda \in X^*$. As a special case of the discussion of the uniform norm on linear maps, we have

[5.2] Corollary: The dual space X^* of a normed space X, with the natural norm, is a Banach space. That is, with respect to the natural norm on continuous functionals, it is *complete*. ///

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6. *Baire's theorem*

Baire's theorem is not specifically about Banach spaces, as it applies more generally to complete metric spaces (and locally compact Hausdorff spaces), but it is the foundation for the subsequent basic non-trivial results on Banach spaces: uniform boundedness, open mapping, and closed graph theorems.

A set E in a topological space X is *nowhere dense* if its closure $\bar{E}$ contains no non-empty open set. A *countable union* of nowhere dense sets is said to be *of first category*, while every other subset (if any) is *of second category*. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric* spaces and *locally compact Hausdorff* spaces are *of second category*.

A G_δ set is a countable intersection of open sets. Concommitantly, an F_σ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense G_δ's is still a dense G_δ.*

[6.1] Theorem: *(Baire)* Let X be either a complete metric space or a locally compact Hausdorff topological space. The intersection of a *countable* collection $U_1, U_2, \ldots$ of *dense open subsets* U_i of X is still *dense* in X.

Proof: Let B_o be a non-empty open set in X, and show that $\bigcap_i U_i$ meets B_o. Suppose that we have inductively chosen an open ball B_{n-1}. By the denseness of U_n, there is an open ball B_n whose closure $\overline{B_n}$ satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take B_n to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take B_n to have compact closure.

Let

$$K \;=\; \bigcap_{n\geq 1} \overline{B_n} \;\subset\; B_o \cap \bigcap_{n\geq 1} U_n$$

For complete metric spaces, the centers of the nested balls B_n form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure* $\overline{B_n}$, so lies in the intersection. In particular, K is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so K is non-empty, and B_o necessarily meets the intersection of the U_n. ///

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7. *Banach-Steinhaus/uniform-boundedness theorem*

This result is *non-trivial* in the sense that it uses the *Baire category theorem.*

[7.1] Theorem: *(Banach-Steinhaus/uniform boundedness)* For a family of continuous linear maps $T_\alpha : X \to Y$ from a Banach space X to a normed space Y, *either* there is a uniform bound $M < \infty$ so that $|T_\alpha| \leq M$ for all α, *or* there is $x \in X$ such that

$$\sup_\alpha \frac{|T_\alpha x|}{|x|} \;=\; +\infty$$

In the latter case, in fact, there is a dense G_δ of such x.

Proof: Let $p(x) = \sup_\alpha |T_\alpha x|$. We allow the possibility that $p(x) = +\infty$. Being the sup of continuous functions, p is *lower semi-continuous*: for each integer n, the set $U_n = \{x : p(x) > n\}$ is *open*.

On one hand, if every U_n is dense in X, by Baire category the intersection is dense, so is *non-empty*. By definition, it is a dense G_δ. On that set p is $+\infty$.

On the other hand, if one of the U_n is *not* dense, then there is a ball B of radius $r > 0$ about a point x_o which does not meet U_n. For $|x - x_o| < r$ and for all α

$$|T_\alpha(x - x_o)| \;\le\; |T_\alpha x| + |T_\alpha x_o| \;\le\; 2n$$

As $x - x_o$ varies over the open ball of radius r the vector $x' = (x - x_o)/r$ varies over the open ball of radius 1, and

$$|T_\alpha x'| \;=\; \Big|T_\alpha \frac{(x - x_o)}{r}\Big| \;\le\; 2n/r$$

Thus, $|T_\alpha| \le 2n/r$, which is the uniform boundedness. ///

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8. *Open mapping theorem*

The open mapping theorem is non-trivial, since it invokes the Baire category theorem.

[8.1] Theorem: *(open mapping)* For a continuous linear *surjection* $T : X \to Y$ of Banach spaces, there is $\delta > 0$ such that for all $y \in Y$ with $|y| < \delta$ there is $x \in X$ with $|x| \le 1$ such that $Tx = y$. In particular, T is an *open map*.

[8.2] Corollary: A *bijective* continuous linear map of Banach spaces is an *isomorphism*. ///

Proof: In the corollary the non-trivial point is that T is *open*, which is the point of the theorem. The linearity of the inverse is easy.

For every $y \in Y$ there is $x \in X$ so that $Tx = y$. For some integer n we have $n > |x|$, so Y is the union of the sets $TB(n)$, with usual open balls

$$B(n) \;=\; \{x \in X : |x| < n\}$$

By Baire category, the *closure* of some one of the sets $TB(n)$ contains a non-empty open ball

$$V \;=\; \{y \in Y : |y - y_o| < r\}$$

for some $r > 0$ and $y_o \in Y$. Since we are in a metric space, the conclusion is that every point of V occurs as the limit of a Cauchy sequence consisting of elements from $TB(n)$.

Certainly

$$\{y \in Y : |y| < r\} \;\subset\; \{y_1 - y_2 : y_1, y_2 \in V\}$$

Thus, every point in the ball B'_r of radius r centered at 0 in Y is the sum of two limits of Cauchy sequences from $TB(n)$. Thus, surely every point in B'_r is the limit of a single Cauchy sequence from the image $TB(2n)$ of the open ball $B(2n)$ of twice the radius. That is, the *closure* of $TB(2n)$ contains the ball $B'(r)$.

Using the linearity of T, the *closure* of $TB(\rho)$ contains the ball $B'(r\rho/2n)$ in Y.

Given $|y| < 1$, choose $x_1 \in B(2n/r)$ so that $|y - Tx_1| < \varepsilon$. Choose $x_2 \in B(\varepsilon \cdot \frac{2n}{r})$ so that

$$|(y - Tx_1) - Tx_2| \;<\; \varepsilon/2$$

Choose $x_3 \in B(\frac{\varepsilon}{2} \cdot \frac{2n}{r})$ so that

$$|(y - Tx_1 - Tx_2) - Tx_3| \ < \ \varepsilon/2^2$$

Choose $x_4 \in B(\frac{\varepsilon}{2^2} \cdot \frac{2n}{r})$ so that

$$|(y - Tx_1 - Tx_2 - Tx_3) - Tx_4| \ < \ \varepsilon/2^3$$

and so on. The sequence

$$x_1, \ x_1 + x_2, \ x_1 + x_2 + x_3, \ \ldots$$

is Cauchy in X. Since X is complete, the limit x of this sequence exists in X, and $Tx = y$. We find that

$$x \ \in \ B(\frac{2n}{r}) + B(\varepsilon\frac{2n}{r}) + B(\frac{\varepsilon}{2} \cdot \frac{2n}{r}) + B(\frac{\varepsilon}{2^2} \cdot \frac{2n}{r}) + \ldots \subset B((1+2\varepsilon)\frac{2n}{r})$$

Thus,

$$TB((1+\varepsilon)\frac{2n}{r}) \ \supset \ \{y \in Y : |y| < 1\}$$

This proves open-ness at 0. ///

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9. *Closed graph theorem*

The closed graph theorem uses the open mapping theorem, so invokes Baire category, so is non-trivial.

It is straightforward to show [4] that a *continuous* map $f : X \to Y$ of *Hausdorff* topological spaces has *closed graph*

$$\Gamma_f \ = \ \{(x,y) \ : \ f(x) = y\} \ \subset \ X \times Y$$

Similarly, a topological space X is Hausdorff if and only if the diagonal $X^\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$. [5]

[9.1] Theorem: A linear map $T : V \to W$ of Banach spaces is continuous if it has closed graph

$$\Gamma \ = \ \Gamma_T \ = \ \{(v,w) \ : \ Tv = w\}$$

Proof: It is routine to check that $V \times W$ with norm $|v \times w| = |v| \cdot |w|$ is a Banach space. Since Γ is a closed subspace of $V \times W$, it is a Banach space itself with the restriction of this norm.

The projection $\pi_V : V \times W \to V$ is a continuous linear map. The restriction $\pi_V|_\Gamma$ of π_V to Γ is still continuous, and still *surjective*, because it T is an everywhere-defined function on V. By the open mapping theorem, $\pi_V|_\Gamma$ is *open*. Thus, the bijection $\pi_V|_\Gamma$ is a *homeomorphism*. Letting $\pi_W : V \times W \to W$ be the projection to W,

$$T \ = \ \pi_W \circ \big(\pi_V|\Gamma\big)^{-1} \ : \ V \longrightarrow W$$

expresses T as a composite of continuous functions. ///

[4] To show that a continuous map $f : X \to Y$ of topological spaces with Y Hausdorff has closed graph Γ_f, show the complement is open. Take $(x,y) \notin \Gamma_f$. Let V_1 be a neighborhood of $f(x)$ and V_2 a neighborhood of y such that $V_1 \cap V_2 = \phi$, using Hausdorff-ness. By continuity of f, for x' in a suitable neighborhood U of x, the image $f(x')$ is inside V_1. Thus, the neighborhood $U \times V_2$ of (x,y) does not meet Γ_f.

[5] To show that closed-ness of the diagonal X^Δ in $X \times X$ implies X is Hausdorff, let $x_1 \neq x_2$ be points in X. Then there is a neighborhood $U_1 \times U_2$ of (x_1, x_2), with U_i a neighborhood of x_i, not meeting the diagonal. That is, $(x,x') \in U_1 \times U_2$ implies $x \neq x'$. That is, $U_1 \cap U_2 = \phi$.

[9.2] Remark: The proof introduced two readily verifiable, useful ideas: a product of Banach spaces is a Banach space, and a closed vector subspace of a Banach space is a Banach space.

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10. *Hahn-Banach Theorem*

Hahn-Banach does *not* use completeness, much less Baire category. The salient feature is *convexity*, and the scalars must be $\mathbb{R}$ or $\mathbb{C}$. Indeed, the Hahn-Banach theorem seems to be a result about *real* vectorspaces. Note that a $\mathbb{C}$-vectorspace may immediately be considered as a $\mathbb{R}$-vectorspace simply by forgetting some of the structure.

For Y a vector subspace of X, and for $S : Y \to Z$ a linear map to another vectorspace Z, a linear map $T : X \to Z$ is an *extension* of S to X when the restriction $T|_Y$ of T to Y is S.

[10.1] Theorem: *(Hahn-Banach)* Let X be a normed vectorspace with scalars $\mathbb{R}$ or $\mathbb{C}$, Y be a subspace, and λ be a continuous linear functional on Y. Then there is an extension Λ of λ to X such that

$$|\Lambda| \;=\; |\lambda|$$

[10.2] Corollary: Given $x \neq y$ in a normed space X, neither a scalar multiple of the other, there is a continuous linear functional λ on X so that $\lambda x = 1$ while $\lambda y = 0$. ///

[10.3] Corollary: Let Y be a closed subspace of a normed space X, and $x_o \notin Y$. Then there is a continuous linear functional λ on X which is 0 on Y, has $|\lambda| = 1$, and $\lambda(x_o) = |x_o|$. ///

Proof: We treat the case that the scalars are $\mathbb{R}$, and reduce the complex case to this.

The critical part is to extend a linear functional by just one dimension. That is, for given $x_o \notin Y$ make an extension λ' of λ to $Y' = Y + \mathbb{R}x_o$. Every vector in Y' has a unique expression as $y + cx_o$ with $c \in \mathbb{R}$, so define functionals by

$$\mu(y + cx_o) \;=\; \lambda y + c\ell \qquad\qquad (\text{for arbitrary } \ell \in \mathbb{R})$$

The issue is to choose ℓ so that $|\mu| = |\lambda|$.

Certainly $\lambda = 0$ is extendable by $\Lambda = 0$, so we consider the case that $|\lambda| \neq 0$. We can divide by $|\lambda|$ to suppose that $|\lambda| = 1$.

The condition $|\mu| = |\lambda|$ is a condition on ℓ:

$$|\lambda y + c\ell| \;\le\; |y + cx_o| \qquad\qquad (\text{for every } y \in Y)$$

We have simplified to the situation that we know this *does* hold for $c = 0$. So for $c \neq 0$, divide through by $|c|$ and replace $y \in Y$ by cy, so that the condition becomes

$$|\lambda y + \ell| \;\le\; |y + x_o| \qquad\qquad (\text{for every } y \in Y)$$

Replacing y by $-y$, the condition on ℓ is that

$$|\ell - \lambda y| \;\le\; |y - x_o| \qquad\qquad (\text{for every } y \in Y)$$

For a single $y \in Y$, the condition on ℓ is that

$$\lambda y - |y - x_o| \;\le\; \ell \le \lambda y + |y - x_o|$$

To have a common solution ℓ, it is exactly necessary that every *lower* bound be less than every *upper* bound. To see that this is so, start from

$$\lambda y_1 - \lambda y_2 \;=\; \lambda(y_1 - y_2) \;\le\; |\lambda(y_1 - y_2)| \;\le\; |y_1 - y_2| \;\le\; |y_1 - x_o| + |y_2 - x_o|$$

by the triangle inequality. Subtracting $|y_1 - x_o|$ from both sides and adding λy_2 to both sides,

$$\lambda y_1 - |y_1 - x_o| \;\le\; \lambda y_2 + |y_2 - x_o|$$

as desired. That is, we have proven the existence of at least one extension from Y to $Y' = Y + \mathbb{R}x_o$ with the same norm.

An equivalent of the Axiom of Choice will extend to the *whole* space while preserving the norm, as follows. Consider the set of pairs (Z, ζ) where Z is a subspace containing Y and ζ is a continuous linear functional on Z extending λ and with $|\zeta| \le 1$. *Order* these by

$$(Z, \zeta) \;\le\; (Z', \zeta')$$

when $Z \subset Z'$ and ζ' extends ζ. For a totally ordered collection (Z_α, ζ_α) of such,

$$Z' \;=\; \bigcup_\alpha Z_\alpha$$

is a subspace of X. In general, of course, the union of a family of subspaces would not be a subspace, but these are *nested*.

We obtain a continuous linear functional ζ' on this union Z', extending λ and with $|\zeta'| \le 1$, as follows. Any *finite* batch of elements already occur inside some Z_α. Given $z \in Z'$, let α be any index large enough so that $z \in Z_\alpha$, and put

$$\zeta'(z) \;=\; \zeta_\alpha(z)$$

The family is totally ordered, so the choice of α does not matter so long as it is sufficiently large. Certainly for $c \in R$

$$\zeta'(cz) \;=\; \zeta_\alpha(cz) \;=\; c\zeta_\alpha(z) \;=\; c\zeta'(z)$$

For z_1 and z_2 and α large enough so that *both* z_1 and z_2 are in Z_α,

$$\zeta'(z_1 + z_2) \;=\; \zeta_\alpha(z_1 + z_2) \;=\; \zeta_\alpha(z_1) + \zeta_\alpha(z_2) \;=\; \zeta'(z_1) + \zeta'(z_2)$$

proving linearity. Thus, there is a *maximal* pair (Z', ζ'). The earlier argument shows that Z' must be all of X, since otherwise we could construct a further extension, contradicting the maximality. *This completes the proof for the case that the scalars are the real numbers.*

To reduce the complex case to the real case, the main trick is that, for λ_o a *real*-linear *real*-valued functional, the functional

$$\lambda x \;=\; \lambda_o(x) - i\lambda(ix)$$

is *complex*-linear, and has the *same norm* as λ_o. In particular, when

$$\lambda_o(x) \;=\; \mathrm{Re}\,\lambda(x) \;=\; \frac{\lambda x + \overline{\lambda x}}{2}$$

is the real part of λ we recover λ itself by this formula.

Granting this, given λ on a complex subspace, take its real part λ_o, a real-linear functional, and extend λ_o to a real-linear functional Λ_o with the same norm. Then the desired extension of λ is

$$\Lambda x \;=\; \Lambda_o(x) - i\Lambda(ix)$$

proving the theorem in the complex case.

Consider the construction

$$\lambda x \;=\; \lambda_o(x) - i\lambda(ix)$$

Since $\lambda_o(x+y) = \lambda_o x + \lambda_o y$ it follows that λ also has this additivity property. For a, b real,

$$\lambda((a+bi)x) \;=\; \lambda_o((a+bi)x) - i\lambda_o(i(a+bi)x) \;=\; \lambda_o(ax) + \lambda_o(ibx) - i\lambda_o(iax) - i\lambda_o(-bx)$$

$$=\; a\lambda_o x + b\lambda_o(ix) - ia\lambda_o(ix) + ib\lambda_o x = (a+bi)\lambda_o x - i(a+bi)\lambda_o(ix) = (a+bi)\lambda(x)$$

This gives the linearity.

Regarding the norm: since λ_o is real-valued, always

$$|\lambda_o(x)| \;\leq\; \sqrt{\lambda_o(x)^2 + \lambda_o(ix)^2} \;=\; |\lambda x|$$

On the other hand, given x there is a complex number μ of absolute value 1 so that $\mu\lambda(x) = |\lambda x|$. And

$$\lambda_o(x) \;=\; \lambda(x) + \overline{\lambda(x)}$$

Then

$$|\lambda(x)| \;=\; \mu\lambda(x) \;=\; \lambda(\mu x) \;=\; \lambda_o(\mu x) - i\lambda_o(i\mu x)$$

Since the left-hand side is real, and since λ_o is real-valued, $\lambda_o(\mu x) = 0$. Thus,

$$|\lambda(x)| \;=\; \lambda_o(\mu x)$$

Since $|\mu x| = |x|$, we have equality of norms of the functionals λ_o and λ. This completes the justification of the reduction of the complex case to the real case. ///