

# Basic applications of Banach space ideas

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## 1. A good trick using uniform boundedness

The following sort of claim may seem nearly obviously true, but there is a missing key ingredient:

[1.1] **Claim:** Let  $b = (b_1, b_2, \dots)$  be a sequence of complex numbers such that  $\sum_n b_n c_n$  is convergent for every  $c = (c_1, c_2, \dots) \in \ell^2$ . Then  $b \in \ell^2$ .

*Proof:* Notably, the assumption that the indicated sums are finite (convergent) does not directly give enough information to conclude that the map  $\lambda(c) = \sum_n b_n c_n$  is a *continuous* linear functional on  $\ell^2$ . The uniform boundedness theorem is needed to reach this conclusion.

Namely, let  $\lambda_N(c) = \sum_{n \leq N} b_n c_n$ . These functionals are continuous on  $\ell^2$ . By uniform boundedness, either there is a uniform bound  $\beta < +\infty$  such that  $\sup_N |\lambda_N(c)| \leq \beta \cdot |c|$  for all  $c \in \ell^2$ , or there is a dense (hence, non-empty)  $G_\delta$  such that  $\sup_N |\lambda_N(c)|/|c| = +\infty$ . But the assumption is that all the latter sups are finite. Thus, there must be a *uniform* bound, so  $\lambda(c) = \sum_n b_n c_n$  is a continuous linear functional. By Riesz-Fréchet, it is given by an element of  $\ell^2$ . ///

[1.2] **Remark:** If we know that the dual of  $L^p$  is  $L^q$  for  $\sigma$ -finite measure spaces  $X$ , then the same sort of argument applies.

## 2. Fourier series of $C^o$ functions can diverge

The *density* of finite Fourier series in  $C^o(\mathbb{T})$  makes no claim about *which* finite Fourier series approach a given  $f \in C^o(\mathbb{T})$ . Indeed, the density proof given via the Féjer kernel uses finite Fourier series quite distinct from the finite partial sums of the Fourier series of  $f$  itself, namely,

$$N^{\text{th}} \text{ Féjer sum} = \frac{1}{N} \sum_{|n| \leq N} (N - |n|) \cdot \widehat{f}(n) \cdot e^{2\pi i n x}$$

The Banach-Steinhaus/uniform-boundedness theorem has a decisive corollary about convergence failure of Fourier series of  $C^o(\mathbb{T})$  functions:

[2.1] **Corollary:** There is  $f \in C^o(\mathbb{T})$  whose Fourier series

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{i n x} \quad \left( \text{with } \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i n x} f(x) dx \right)$$

*diverges* at 0. In fact, the divergence can be arranged for a dense  $G_\delta$  of continuous functions, and at any given countable set of points on  $\mathbb{T}$ .

*Proof:* To invoke Banach-Steinhaus, consider the functionals given by partial sums of the Fourier series of  $f$ , evaluated at 0:

$$\lambda_N(f) = \sum_{|n| \leq N} \hat{f}(n) = \sum_{|n| \leq N} \hat{f}(n) \cdot e^{2\pi i n \cdot 0}$$

There is an easy upper bound

$$|\lambda_N(f)| \leq \int_0^1 \left| \sum_{|n| \leq N} e^{-2\pi i n x} \right| \cdot |f(x)| dx \leq \|f\|_{C^0} \cdot \int_0^1 \left| \sum_{|n| \leq N} e^{-2\pi i n x} \right| dx = \|f\|_{C^0} \cdot \left\| \sum_{|n| \leq N} e^{-2\pi i n x} \right\|_{L^1(\mathbb{T})}$$

We will show that equality holds, namely, that

$$|\lambda_N| = \left\| \sum_{|n| \leq N} e^{-2\pi i n x} \right\|_{L^1}$$

and show that the latter  $L^1$ -norms go to  $\infty$  as  $N \rightarrow \infty$ .

Summing the finite geometric series and rearranging:

$$\sum_{|n| \leq N} e^{-2\pi i n x} = \frac{e^{-2\pi i N x} - e^{-2\pi i (-N-1)x}}{e^{-2\pi i x} - 1} = \frac{e^{2\pi i (N+\frac{1}{2})x} - e^{-2\pi i (N+\frac{1}{2})x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \frac{2\pi x}{2}}$$

The elementary inequality  $|\sin t| \leq |t|$  gives a lower bound

$$\begin{aligned} \int_0^1 \left| \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \frac{2\pi x}{2}} \right| dx &\geq \int_0^1 \left| \sin 2\pi(N+\frac{1}{2})x \right| \cdot \frac{2}{2\pi x} dx = \int_0^{2\pi(N+\frac{1}{2})} |\sin x| \cdot \frac{2}{2\pi x} dx \\ &\geq \sum_{\ell=1}^N \frac{1}{\pi \ell} \int_{2\pi(\ell-1)}^{2\pi \ell} |\sin x| dx \geq \sum_{\ell=1}^N \frac{1}{\pi \ell} \rightarrow +\infty \quad (\text{as } N \rightarrow \infty) \end{aligned}$$

Thus, the  $L^1$ -norms do go to  $\infty$ .

We claim that the norm of the *functional* is the  $L^1$ -norm of the *kernel*: let  $g(x)$  be the *sign* of the Dirichlet kernel

$$\sum_{|n| \leq N} e^{-2\pi i n x} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \frac{2\pi x}{2}}$$

Let  $g_j$  be a sequence of periodic continuous functions with  $|g_j| \leq 1$  and going to  $g$  pointwise. By *dominated convergence*

$$\lim_j \lambda_N(g_j) = \lim_j \int_0^1 g_j(x) \sum_{|n| \leq N} e^{-2\pi i n x} dx = \int_0^1 g(x) \sum_{|n| \leq N} e^{-2\pi i n x} dx = \int_0^1 \left| \sum_{|n| \leq N} e^{-2\pi i n x} \right| dx$$

By Banach-Steinhaus for the Banach space  $C^0(\mathbb{T})$ , since (as demonstrated above) there is *no* uniform bound  $|\lambda_N| \leq M$  for all  $N$ , there *exists*  $f$  in the unit ball of  $C^0(\mathbb{T})$  such that

$$\sup_N |\lambda_N v| = +\infty$$

In fact, the collection of such  $v$  is *dense* in the unit ball, and is an intersection of a *countable* collection of dense open sets (a  $G_\delta$ ). That is, the Fourier series of  $f$  does not converge at 0.

The result can be strengthened by using Baire's theorem again. For a dense countable set of points  $x_j$  in the interval, let  $\lambda_{j,N}$  be the continuous linear functionals on  $C^o(\mathbb{T})$  defined by evaluation of finite partial sums of the Fourier series at  $x_j$ 's:

$$\lambda_{j,N}(f) = \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x_j}$$

As in the previous, the set  $E_j$  of functions  $f$  where

$$\sup_N |\lambda_{j,N} f| = +\infty$$

is a dense  $G_\delta$ , so the intersection  $E = \bigcap_j E_j$  is a dense  $G_\delta$ , and, in particular, not empty. ///

### 3. Riemann-Lebesgue for $f \rightarrow \hat{f}$ on $L^1(\mathbb{T})$ and $L^1(\mathbb{R})$

The space  $c_o$  of two-sided sequences *vanishing at infinity* is

$$c_o = \{ \{a_n : n \in \mathbb{Z}\} : \lim_{|n| \rightarrow \infty} a_n = 0 \}$$

The space  $c_o$  is a Banach space with norm  $\|\{a_n\}\|_{c_o} = \sup_n |a_n|$ . Parametrizing the circle  $\mathbb{T}$  by the interval  $[0, 1]$  by the exponential map  $x \rightarrow e^{2\pi i x}$ , the Banach space  $L^1(\mathbb{T}) = L^1[0, 1]$  is measurable functions  $f$  on  $[0, 1]$  with finite integrals  $\int_0^1 |f|$  (modulo the equivalence relation of equality almost everywhere). The space  $L^1[0, 1]$  contains and is strictly larger than  $L^2[0, 1]$ . On  $L^2[0, 1]$ , Fourier transform is an isometry to  $\ell^2(\mathbb{Z})$ , by Parseval's theorem, and a relatively trivial form of a Riemann-Lebesgue lemma is that  $\hat{f} \in c_o$  for  $f \in L^2[0, 1]$ . The version for  $L^1$  is less trivial:

[3.1] **Lemma:** (*Riemann-Lebesgue*)  $\hat{f} \in c_o$  for  $f \in L^1(\mathbb{T})$ .

*Proof:* Finite linear combinations of exponentials are dense in  $C^o(\mathbb{T})$ , for example by Féjer's argument, and  $C^o(\mathbb{T})$  is dense in  $L^1(\mathbb{T})$ , essentially by the definition of integral and Urysohn's lemma. Thus, given  $f \in L^1$  there is  $g \in C^o(\mathbb{T})$  such that  $\|f - g\|_{L^1} < \varepsilon$  and a finite linear combination  $h$  of exponentials such that  $\|g - h\|_{C^o} < \varepsilon$ . Then  $\|f - h\|_{L^1} < 2\varepsilon$ .

Given such  $h$ , for large-enough  $n$  the Fourier coefficients are 0, by orthogonality of distinct exponentials. Thus,

$$|\hat{f}(n)| = \frac{1}{2\pi} \left| \int_0^{2\pi} (f(x) - h(x)) e^{-inx} dx \right| \leq \frac{\|f - h\|_{L^1}}{2\pi} < 2\varepsilon \quad (\text{for } n \text{ large, depending on } f)$$

This proves this Riemann-Lebesgue Lemma. ///

### 4. Non-surjection of $L^1[0, 1] \rightarrow c_o$ by $f \rightarrow \hat{f}$

Baire theorem and open mapping prove this.

[4.1] **Corollary:** (*of Baire and Open Mapping*) Not every sequence in  $c_o$  is the collection of Fourier coefficients of an  $L^1(\mathbb{T})$  function.

*Proof:* The Fourier-coefficient map

$$Tf = \{ \hat{f}(n) : n \in \mathbb{Z} \} \in c_o$$

does map  $L^1[0, 1] \rightarrow c_0$ , by Riemann-Lebesgue. The obvious inequality

$$|\widehat{f}(n)| = \left| \int_0^1 f(x) e^{-2\pi i n x} dx \right| \leq \int_0^1 |f(x)| dx = \|f\|_{L^1}$$

shows  $|T| \leq 1$ , so  $T$  is continuous. Taking  $f(x) = 1$  shows  $|T| = 1$ .

The density of finite Fourier series in  $C^0$  and density of  $C^0$  in  $L^1$ , as in the proof of the Riemann-Lebesgue lemma, shows that  $T$  is injective. If  $T$  were also *surjective*, then the open mapping theorem would guarantee  $\delta > 0$  such that for every  $L^1$  function  $f$

$$\|\widehat{f}\|_{\text{sup}} \geq \delta \cdot \|f\|_{L^1}$$

However, this is impossible: with

$$f_N(x) = \sum_{|n| \leq N} e^{-2\pi i n x}$$

the sup norm of  $\widehat{f}_N$  is certainly 1, yet the computation about divergence of Fourier series above shows that the  $L^1$  norm of  $f_N$  goes to  $\infty$  like  $\log N$  as  $N \rightarrow +\infty$ . Thus, there is no such  $\delta > 0$ . Thus,  $T$  cannot be surjective. ///

## 5. $C^\infty(\mathbb{T})$ is dense in $C^0(\mathbb{T})$

Féjer's argument proves that the Cesaro-summed finite partial sums of Fourier series of a continuous function converge to that function in the  $C^0(\mathbb{T})$  topology (that is, uniformly pointwise). These finite partial sums, as well as their Cesaro-summed forms, are in  $C^\infty(\mathbb{T})$ . Thus,

[5.1] Corollary:  $C^\infty(\mathbb{T})$  is dense in  $C^0(\mathbb{T})$ . ///

## 6. Typical $C^0$ functions are nowhere differentiable

[6.1] Claim: In  $C^0[a, b]$ , there is (at least) a dense  $G_\delta$  of functions which at every point fail to be differentiable.

*Proof:* Anticipating the application of Baire's theorem, we present everywhere-not-differentiable functions as a countable intersection of dense opens. First, for fixed large  $n > 0$  and small  $h \neq 0$ , let

$$X_{n,h} = \{f \in C^0[a, b] : |f(x+h) - f(x)| > n \cdot |h|, \text{ for all } x \in [a, b] \text{ such that } x+h \in [a, b]\}$$

To show that  $X_{n,h}$  is open, we observe that for a given  $f \in X_{n,h}$ , the function  $|f(x+h) - f(x)| - n \cdot |h|$  is continuous in  $x$ , and is positive. Thus, since the function is continuous on the compact interval  $[a, b]$ , its inf is strictly positive. Thus, for  $g$  with  $\|g - f\|_{C^0}$  sufficiently small,  $|g(x+h) - g(x)| - n \cdot |h|$  is still positive. That is,  $g \in X_{n,h}$ .

Next, each union

$$Y_{n,h} = \bigcup_{h' \neq 0, |h'| < |h|} X_{n,h'}$$

$$= \{f \in C^0[a, b] : \text{for every } x \in [a, b], \text{ there is } 0 < h' < h \text{ such that } |f(x+h') - f(x)| > n \cdot |h'|\}$$

(where implicitly  $x+h' \in [a, b]$ ) is a union of opens, so is open.

Density of  $Y_{n,h}$  in  $C^0[a, b]$  is that, for given  $f \in C^0[a, b]$ , there is  $g \in Y_{n,h}$  near  $f$ . To prove this, first approximate  $f$  to within  $\varepsilon > 0$  in sup norm by  $g \in C^1[a, b]$ . Among the several possible ways to do this,

we choose the following. First, adjust  $f$  by subtracting a polynomial to make  $f(a) = f(b)$ . Extending  $f$  by periodicity, Féjer's Cesaro-summed version of the finite partial sums of its Fourier series converge to it in sup norm. These finite approximations are all  $C^\infty$ , in fact, proving that we can approximate  $f$  to within  $\varepsilon > 0$  in sup norm by a  $C^1$  function  $g$ .

In particular, the derivative of  $g$  is a continuous function on  $[a, b]$ , so is *bounded* in absolute value, say by  $\beta$ .

Next, we use auxiliary piecewise- $C^1$  functions  $\varphi_{N,\varepsilon}$  in  $C^0[a, b]$  with sup norms less than a given  $\varepsilon > 0$ , but with absolute values of derivatives strictly greater than a given  $N$ , for any pair  $\varepsilon, N$ . For example, we can easily make piecewise-*linear* continuous functions  $\varphi_{N,\varepsilon}$  with slopes  $\pm(N + 1)$ , changing sign so often that they stay strictly between  $\pm\varepsilon$ . For  $N > \beta$ ,  $g + \varphi_{2N,\varepsilon}$  is in  $Y_{N,h}$  for all  $h > 0$ , and

$$|f - (g + \varphi_{2N,\varepsilon})|_{C^0} \leq |f - g|_{C^0} + |\varphi_{2N,\varepsilon}|_{C^0} < \varepsilon + \varepsilon$$

This proves the density of every open  $Y_{N,h}$  in  $C^0[a, b]$ .

By Baire's theorem, the countable intersection  $\bigcap_{n=1,2,\dots} Y_{n,\frac{1}{n}}$  of dense compacts is still dense. ///

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