(*November 7, 2018*)

07c. $C^\infty(\mathbb{T})$ *is not normable*

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http://www.math.umn.edu/~garrett/m/real/notes_2018-19/07c_C-infinity_is_not_Banach.pdf]

Many natural function spaces, such as $C^\infty[a,b]$ and $C^\infty(\mathbb{T})$, are *not* Banach, nor even *norm-able* but still do have a metric topology and are complete: these are *Fréchet spaces*, appearing as countable (projective) *limits* of Banach spaces. It is reasonable to ask *why* these spaces are not Banach, and in fact not even *normable*, that is, their topologies cannot be given by a any norm, regardless of metric completeness.

In brief, in tangible terms, the root cause of this impossibility is that no estimates on the first k derivatives of a function on $\mathbb{T}$ give an estimate on the $(k+1)^{th}$ derivative, for any k. This is discussed precisely below, and abstracted somewhat.

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1. *Countable limits of Banach spaces*

We could take *countable limit of Banach spaces* as the definition of *Fréchet space*.

As earlier, $C^\infty(\mathbb{T})$ is a countable nested intersection, which is a countable (projective) *limit*:

$$C^\infty(\mathbb{T}) \;=\; \bigcap_{k\geq 0} C^k(\mathbb{T}) \;=\; \lim_k C^k(\mathbb{T})$$

From very general category-theory arguments, *there is at most one projective-limit topology* on $C^\infty(\mathbb{T})$, up to unique isomorphism. Existence of the topology on X satisfying the limit condition can be proven by identifying X as the diagonal *closed subspace* of the *topological product* of the *limitands* X_k: letting $p_{k,k-1}: X_k \to X_{k-1}$ be the transition maps,

$$X \;=\; \{\{x_k : x_k \in C^k[a,b]\} : p_{k,k-1}(x_k) = x_{k-1} \text{ for all } k\}$$

The subspace topology on X is the limit topology, seen as follows. The projection maps $p_k : \prod_j X_j \to X_k$ from the whole product to the factors X_k are continuous, so their restrictions to the diagonally imbedded X are continuous. Further, letting $i_k : X_k \to X_{k-1}$ be the transition map, on that diagonal copy of X we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, *any* family of maps $\varphi_k : Z \to X_k$ induces a map $\widetilde{\varphi} : Z \to \prod X_k$ such that $p_k \circ \widetilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\widetilde{\varphi}$ is inside the diagonal, that is, inside the copy of X. Thus, this construction does produce a limit.

A *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*. One of many topologically equivalent metrics is the usual

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

When the metric spaces X_k are *complete*, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so the diagonal X is complete metric.

Even in general, the topologies on vector spaces V are required to be *translation invariant*, meaning that for an open neighborhood U of 0, for any $x \in V$, the set $x + U = \{x + u : u \in U\}$ is an open neighborhood of

x, and vice-versa. [1] Thus, to specify the topology on a limit X of Banach spaces X_k, we need only give a local basis at 0. From the construction above, a local basis is given by all sets

$$U_{k,\delta} \;=\; \{x \in X : |p_k(x)|_{X_k} < \delta\} \qquad\qquad (\text{for } \delta > 0 \text{ and index } k)$$

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2. *Maps from limits of Banach spaces to normed spaces factor through limitands*

The assertion of the section title is *only* reliably true when the image of the limit in each limitand is *dense*. This hypothesis is unnecessary when the limitands are Hilbert spaces.

[2.1] Lemma: Given a continuous linear map T from $C^\infty(\mathbb{T})$ to a *normed space* Y, there is an index k such that when $C^\infty(\mathbb{T})$ is given the (weaker) C^k topology, $T : C^\infty(\mathbb{T}) \to Y$ is still continuous.

[2.2] Corollary: Every continuous linear map T from $C^\infty(\mathbb{T})$ to a Banach space Y factors through some limitand $C^k(\mathbb{T})$. That is, there is $T_k : C^k(\mathbb{T}) \to Y$ such that $T = T_k \circ i_k$, where $i_k : C^\infty(\mathbb{T}) \to C^k(\mathbb{T})$ is the inclusion.

Proof: (of Corollary) After applying the lemma, since the target space of T is *complete*, we can extend $T : C^\infty(\mathbb{T}) \to Y$ by continuity (in the C^k topology) to the C^k-completion of C^∞, which is C^k. ///

The lemma is a special case of the analogous lemma that has nothing to do with spaces of functions, but, rather, is true for more general reasons:

[2.3] Lemma: Let $X = \lim_k X_k$ be a limit of Banach spaces X_k, with projection maps $p_k : X \to X_k$. Suppose that $p_k(X)$ is *dense* in X_k. Then every continuous linear map $T : X \to Y$ to a *normed space* Y factors through some limitand X_k. That is, there is $T_k : X_k \to Y$ such that $T = T_k \circ p_k$.

Proof: Given $\varepsilon > 0$, by the description above of the topology on the limit, there are $\delta > 0$ and index k such that $T(U_{k,\delta})$ is inside the ε-ball at 0 in Y.

Then, given any other $\varepsilon' > 0$, we claim that T maps

$$\frac{\varepsilon'}{\varepsilon} \cdot U_{k,\delta} \;=\; U_{k,\delta\varepsilon'/\varepsilon}$$

to the open ε'-ball in Y. Indeed,

$$|T\big(\frac{\varepsilon'}{\varepsilon} \cdot U_{k,\delta}\big)|_Y \;=\; \frac{\varepsilon'}{\varepsilon} \cdot |T(U_{k,\delta})|_Y \;<\; \frac{\varepsilon'}{\varepsilon} \cdot \varepsilon \;=\; \varepsilon'$$

as claimed. Thus, $T : X \to Y$ is continuous when X is given the X_k topology, for the index k that makes this work. Thus, T extends by continuity to the $|\cdot|_{X_k}$-completion of X. By the density assumption, this is X_i. ///

[2.4] Remark: Finite Fourier series, which are in $C^\infty(\mathbb{T})$, are dense in every $C^k(\mathbb{T})$, so $C^\infty(\mathbb{T})$ is dense in every $C^k(\mathbb{T})$.

[2.5] Remark: In the case that $Y = \mathbb{C}$, the density assumption is unnecessary, since Hahn-Banach gives an extension. But for general Banach Y, without the density assumption, we can only conclude that T factors through the $|\cdot|_{X_k}$-completion of X, since not all closed subspaces of Banach spaces are *complemented*.

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[1] For Hilbert and Banach spaces, this translation-invariance is clear, since the topology is metric, and comes from a norm.

3. $C^\infty(\mathbb{T})$ *is not normable*

If $C^\infty(\mathbb{T})$ were *normable*, then the identity map $j : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ would be continuous when the source is given the C^k topology. In particular, for every $\varepsilon > 0$, there would be a sufficiently small $|\cdot|_{X_k}$-ball B whose image in $C^\infty(\mathbb{T})$ under the inclusion is inside the ε-ball in the $C^{k+1}(\mathbb{T})$ topology on $C^\infty(\mathbb{T})$. Specifically, for $\varepsilon = 1$, there should be a sufficiently small $\delta > 0$ such that the δ-ball in the C^k topology is inside the unit ball in the C^{k+1} topology.

However, it is easy-enough to construct C^∞ functions whose C^k norms are arbitrarily small, but whose C^{k+1} norm is 1, for example, e^{iNx}/N^{k+1}. Thus, we achieve a contradiction. ///