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08. Introduction to generalized functions (distributions)

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1. $\mathcal{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$

On $\mathbb{R}^n$ there are many useful spaces of functions. There are at least three distinct concepts of *nice functions*:

$$\begin{cases} \text{test functions} & = & \mathcal{D}(\mathbb{R}^n) & = & C_c^\infty(\mathbb{R}^n) \;=\; \{f \in C^\infty(\mathbb{R}^n) : f \text{ has compact support}\} \\ \\ \text{Schwartz functions} & = & \mathscr{S}(\mathbb{R}^n) & = & \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} (1+|x|)^m |f^\alpha(x)| < \infty \text{ for all } m, \alpha\} \\ \\ \text{smooth functions} & = & \mathcal{E}(\mathbb{R}^n) & = & C^\infty(\mathbb{R}^n) \end{cases}$$

As usual, description of the topologies on these spaces of functions is necessary, otherwise we have no idea what limits stay in the space, or fall outside. But, having acknowledged that debt, we can usefully continue.

2. *Extensions/duals*

Here, *dual* V^* of a topological vector space V means *continuous dual*, which means the collection of *continuous* linear maps from V to scalars.

The duals of the above basic spaces of nice functions have names:

$$\begin{cases} \text{distributions} & = & \mathcal{D}(\mathbb{R}^n)^* & = & C_c^\infty(\mathbb{R}^n)^* \\ \\ \text{tempered distributions} & = & \mathscr{S}(\mathbb{R}^n)^* \\ \\ \text{compactly-supported distributions} & = & \mathcal{E}(\mathbb{R}^n)^* & = & C^\infty(\mathbb{R}^n)^* \end{cases}$$

Distributions are also called *generalized functions*, since their utility mostly lies in the possibility of interpreting them as *extensions* of the notion of *function*, rather than as being in dual spaces.

In general there is no natural continuous inclusion of a topological vector space V into its dual V^*, so V^* is in no natural sense an *extension* of V. In contrast, for $\mathcal{D}$ and $\mathscr{S}$, we have inclusions $\mathcal{D} \subset \mathcal{D}^*$ and $\mathscr{S} \subset \mathscr{S}^*$ by $\varphi \to$ integrate-against-φ. That is, $\varphi \to u_\varphi$ with

$$u_\varphi(f) \;=\; \int_{\mathbb{R}^n} \varphi \cdot f$$

Via these inclusions, we *do* think of $\mathcal{D}^*$ and $\mathscr{S}^*$ as *extending* $\mathcal{D}$ and $\mathscr{S}$, respectively.

At some point, we will prove that $\mathcal{D}(\mathbb{R}^n)$ is *dense* in $\mathcal{D}(\mathbb{R}^n)^*$, and in every other space of distributions. The proof that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R})$ is not difficult: use smooth cut-offs with larger-and-larger compact support.

Since taking duals is inclusion-reversing, we have $\mathcal{E}^* \subset \mathscr{S}^* \subset \mathcal{D}^*$. The suggestion that $\mathcal{E}^*$ does literally consist of compactly-supported distributions, by calling it that, should be substantiated, which we do below.

Pictorially, with arrows being inclusions, on $\mathbb{R}^n$ we have

$$\begin{tikzcd}
\mathcal{D} \arrow[r] & \mathscr{S} \arrow[r] \arrow[dr] & \ldots \arrow[r] & L^2 \arrow[r] & \ldots \arrow[r] & \mathscr{S}^* \arrow[r] & \mathcal{D}^* \\
 & & \mathcal{E} & & \mathcal{E}^* \arrow[ur] & &
\end{tikzcd}$$

Unsurprisingly, all the arrows are *continuous* maps, as we will subsequently verify.

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3. *Topological details: families of seminorms*

Even though $\mathscr{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$ are complete metric spaces, their topologies are most coherently given *not* in terms of those metrics, but in terms of *countable families of seminorms*, as follows.

Fix a vectorspace V over $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* ν on V is a non-negative $\mathbb{R}$-valued functions on V with properties

$$\begin{cases} \nu(c \cdot v) = |c| \cdot \nu(v) & \text{(for scalar } c \text{ and } v \in V) \\ \nu(v+w) \le \nu(v) + \nu(w) & \text{(for } v, w \in V) \end{cases}$$

Unlike a genuine norm, we do *not* require the positive-definiteness property that $\nu(v) = 0$ implies $v = 0$. To compensate, we use *separating families* of seminorms: let X be a set of seminorms on V such that, for all $\neq v \in V$ there is $\nu \in X$ such that $\nu(v) \neq 0$. Since we only want Hausdorff topologies on vector space, we will only consider *separating* families of seminorms, so the modifier *separating* may be dropped.

To describe the topology on V corresponding to a family F of seminorms, first we tell all the (open) neighborhoods of $0 \in V$, and to do so we give a *local sub-basis* at 0: [1] sets

$$U_{\nu,\varepsilon} \;=\; \{v \in V : \nu(v) < \varepsilon\} \qquad \text{(for } \varepsilon > 0 \text{ and } \nu \in F)$$

Then make a local sub-basis at $v \in V$ by *translating* opens from 0: a set U containing v is open if and only if the translated set

$$U - v \;=\; \{u - v : u \in U\}$$

is an open containing 0. That is, open sets containing v are all of the form $U + v$ for opens containing 0, and vice-versa.

The topology on $\mathscr{S}(\mathbb{R}^n)$ is given by the natural countable family of (semi-) norms

$$\nu_{m,n}(f) \;=\; \sup_{|\alpha| \le m} \sup_x (1 + |x|)^n \, |f^{(\alpha)}(x)|$$

The topology on $\mathcal{E}(\mathbb{R}^n)$ is given by the natural countable family of seminorms

$$\nu_{N,n}(f) \;=\; \sup_{|\alpha| \le n} \sup_{|x| \le N} |f^{(\alpha)}(x)|$$

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[1] Recall that a *local sub-basis* at a point v_o in a topological space V is a set S of opens containing v_o, such that every open containing v_o contains a *finite intersection* of opens from S.

The topology on test functions is slightly more complicated, and we delay treatment of it. Of course, we cannot truly understand the spaces of (continuous!) linear functionals on any such space until we know the topology. Nevertheless, we proceed with an as-yet-unspecified topology on $\mathcal{D}(\mathbb{R}^n)$. [2]

A topology given by a *countable* (separating) family of seminorms $\{\nu_1, \nu_2, \ldots\}$ can also be given by a *metric*

$$d(v, w) \;=\; \sum_{n \geq 1} 2^{-n} \cdot \frac{\nu(v-w)}{1 + \nu(v-w)}$$

This metric still is *translation-invariant*, meaning that $d(v+x, w+x) = d(v, w)$, but it does *not* have any homogeneity properties, unlike the metrics $d(v, w) = |v - w|$ made from a single norm $|\cdot|$. For that matter, the constants 2^{-n} and the shape of the expression are not canonical. Thus, it's better to say that the topology is *metrizable*, rather than *metric*.

When a topology on V is given by a countable family of seminorms, and the associated metric as above is *complete* in the usual metric-space sense, V is a *Fréchet space*.

[3.1] Claim: Both $\mathscr{S}$ and $\mathcal{E}$ are Fréchet spaces. *[... iou ...]*

As in other contexts, continuity of linear maps is equivalent to continuity at 0: with V be a vector space with topology given by seminorms as above, [3]

[3.2] Claim: A linear map $\lambda : V \to \mathbb{C}$ is continuous if and only if it is continuous at 0.

Proof: *[... iou ...]* ///

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4. *Examples of generalized functions (distributions)*

Having specified topologies on $\mathscr{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}_n)$, we can illustrate continuity arguments that prove various natural functionals are in $\mathscr{S}(\mathbb{R}^n)^*$ and $\mathcal{E}(\mathbb{R}^n)$. Without having yet specified the topology on $\mathcal{D}(\mathbb{R}^n)$, granting that the inclusion $\mathcal{D}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is continuous allows proof that a given functional is in $\mathcal{D}(\mathbb{R}^n)^*$ by proving that it is in $\mathscr{S}(\mathbb{R}^n)^* \subset \mathcal{D}(\mathbb{R}^n)^*$.

[4.1] Claim: The Dirac delta functional $\delta(f) = f(0)$ is in $\mathcal{E}^*$, so is certainly in $\mathscr{S}^*$ (and in $\mathcal{D}^*$).

Proof: It suffices to use any of the seminorms $\nu_{N,0}$ on $\mathcal{E}$ as above, with $N > 0$. For example, with $\nu_{1,0}$,

$$|f(0)| \;\leq\; \sup_{|x| \leq 1} |f^{(0)}(x)| \;=\; \nu_{1,0}(f)$$

Thus, given $\varepsilon > 0$, for $f \in \mathcal{E}$ with $\nu_{1,0}(f) < \varepsilon$,

$$|f(0)| \;\leq\; \sup_{|x| \leq 1} |f^{(0)}(x)| \;=\; \nu_{1,0}(f) < \varepsilon$$

This proves the continuity. ///

[2] The topology of $\mathcal{D}(\mathbb{R}^n)$ *can* be described by a family of semi-norms, as is the case for *any* locally convex topological vector space. However, such a description does not make clear that $\mathcal{D}(\mathbb{R}^n)$ has suitable (non-metric!) *completeness*. For the latter, presenting $\mathcal{D}(\mathbb{R}^n)$ as a *strict colimit* is better, and we do this later.

[3] Again, all topological vector spaces of interest to us have topologies given by seminorms, since they are *locally convex*, meaning that there is a basis at 0 consisting of convex opens.

[4.2] Claim: For $\varphi \in L^1(\mathbb{R}^n)$ the integrate-against functional $u_\varphi(f) = \int_{\mathbb{R}} \varphi(x)\, f(x)\, dx$ is in $\mathscr{S}^*$.

Proof: Note the elementary inequalities

$$\Big|\int_{\mathbb{R}} \varphi(x)\, f(x)\, dx\Big| \;\le\; \int_{\mathbb{R}} |\varphi(x)|\cdot|f(x)|\, dx \;\le\; \int_{\mathbb{R}} |\varphi(x)|\, dx \cdot \sup_{x\in\mathbb{R}} |f(x)| \;=\; |\varphi|_{L^1}\cdot \nu_{0,0}(f)$$

with seminorm on $\mathscr{S}$ as above. Thus, it suffices to consider φ with $|\varphi|_{L^1} \neq 0$. Given $\varepsilon > 0$, for $\nu_{0,0}(f) < \varepsilon$,

$$\Big|\int_{\mathbb{R}} \varphi(x)\, f(x)\, dx\Big| \;\le\; |\varphi|_{L^1}\cdot \nu_{0,0}(f) \;<<\; |\varphi|_{L^1}\cdot\varepsilon$$

Since $|\varphi|_{L^1}$ is just a (non-zero) constant, this gives the continuity. ///

Recall that on $\mathbb{R}^n$ the *locally integrable* functions $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are the (measurable) functions f such that $\int_K |f| < \infty$ for every compact K. Since we do not yet have a description of the topology on $\mathcal{D}(\mathbb{R}^n)$, we cannot give a precise discussion of the following, but it should be recorded as eventually-provable fact:

[4.3] Claim: (Integration against) a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ gives a distribution, that is, an element of $\mathcal{D}(\mathbb{R}^n)^*$.

Proof: [... iou ...] ///

A pointwise-defined function φ is *of moderate/polynomial growth* when there is an exponent N such that

$$\sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \cdot |f(x)| \;<\; +\infty$$

[4.4] Claim: (Integration against) an L^1_{loc} function of moderate growth gives a tempered distribution.

Proof: [... iou ...] ///

[4.5] Corollary: The distribution given by integration-against $|x|^s$ on $\mathbb{R}^n$ for $\mathrm{Re}(s) > -n$ is *tempered*. ///

[4.6] Remark: In particular, as we see below, the distributions $|x|^s$ have *distributional/generalized* derivatives for $\mathrm{Re}(s) > -n$, even though for $\mathrm{Re}(s) < 0$ they are not classically differentiable at 0. Further, for $\mathrm{Re}(s)$ close to $-n$, their distributional derivatives are no longer in L^1_{loc}, but nevertheless give tempered distributions.

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5. *Weak dual topologies (weak $*$-topologies)*

Among other possibilities, we can give dual spaces V^* their *weak dual topologies* (also called *weak *-topologies*): for a topological vector space V with (continuous linear) dual V^*, the weak dual topology on V^* is given by the family of seminorms

$$\nu_v(\lambda) \;=\; |\lambda(v)| \qquad\qquad (\text{for } v\in V,\ \lambda\in V^*)$$

Thus, in the weak dual topology, $\lambda_n \to \lambda$ if and only if $|\lambda_n(v) - \lambda(v)| \to 0$ for all $v \in V$.

[5.1] Claim: Test functions $\mathcal{D}(\mathbb{R}^n)$ are (sequentially) *dense* in $\mathcal{E}(\mathbb{R}^n)^*$, in $\mathscr{S}(\mathbb{R}^n)^*$, and in $\mathcal{D}(\mathbb{R}^n)^*$, with their respective weak dual topologies. That is, the distributions u_φ given by integration against test functions φ are dense.

Proof: [... iou ...] ///

6. *Differentiation of generalized functions (distributions)*

A large part of the purpose of generalized functions is to be able to differentiate not-classically-differentiable functions in a way that nevertheless fits coherently with classical differentiation. That is, as it turns out, it is possible and useful to differentiate functions even when the (numerical) limits of difference quotients $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ do not exist.

Granting that $\mathcal{E}(\mathbb{R}^n)^* \subset \mathscr{S}(\mathbb{R}^n)^* \subset \mathcal{D}(\mathbb{R}^n)^*$, it suffices to define differentiation on $\mathcal{D}(\mathbb{R}^n)^*$:

By duality: For $u \in \mathcal{D}(\mathbb{R}^n)^*$, define

$$\Big(\frac{\partial}{\partial x_j} u\Big) \;=\; -u\Big(\frac{\partial}{\partial x_j} f\Big) \qquad\qquad \text{(for all } f \in \mathcal{D}(\mathbb{R}^n))$$

The sign flip is for compatibility with integration by parts, when u is integration-against a test function or Schwartz function. That is, the definition by duality is adjusted for consistency with the definition *by extension-by-continuity*:

By extension-by-continuity: Granting from above that test functions are (sequentially) dense in $\mathscr{S}^*$, $\mathcal{E}^*$, and $\mathcal{D}^*$, for a sequence $\{u_n\}$ of test functions that is Cauchy in the corresponding weak dual topology, define

$$\frac{\partial}{\partial x_j}(\lim_n u_n) \;=\; \lim_n \Big(\frac{\partial}{\partial x_j} u_n\Big)$$

where the left-hand side is defined by the right, where both limits are in the respective weak dual topologies, and the differentiations on the right-hand side are the classical limits of difference quotients.

In particular, letting u_f be the integration-against-f functional,

$$\frac{\partial}{\partial x_j} u_f \;=\; u_{\partial f/\partial x_j}$$

where the left side is distributional derivative, and $\partial f/\partial x_j$ is the classical limit-of-difference-quotients. That is, the distributional differentiation extended-by-continuity genuinely is an extension corresponding to the inclusions $\mathcal{D} \subset \mathcal{D}^*$, $\mathcal{D} \subset \mathscr{S} \subset \mathscr{S}^*$, and $\mathcal{D} \subset \mathcal{E}^*$.

[6.1] Claim: These two definitions give the same maps $\frac{\partial}{\partial x_j}$ of $\mathcal{D}^*$, $\mathscr{S}^*$, and $\mathcal{E}^*$ to themselves. These maps are continuous in the weak dual topologies.

Proof: [... iou ...] ///

[6.2] Claim: If $u \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the tempered distribution given by (integration against) u has distributional derivative equal to (integration against) the classical derivative of u.

Proof: [... iou ...] ///

The derivative δ' of the Dirac δ on $\mathbb{R}^n$: On one hand, we can directly define

$$\delta'(f) \;=\; -f'(0) \qquad\qquad \text{(with sign flip as suggested above)}$$

and verify that this is a compactly supported distribution, since

$$|\delta'(f)| \;=\; |-f'(0)| \;\le\; \sup_{|x|\le 1} |f'(x)|$$

and the right-hand side is one of the seminorms defining the topology on $\mathcal{E}(\mathbb{R})$. On the other hand, the previous claim systematically bundles-up such arguments, so we know in advance that δ' is a compactly-supported distribution.

The derivative of $1/\sqrt{|x|}$ on $\mathbb{R}$: *[... iou ...]*

Non-moderate growth tempered distributions: Although the idea that (integrate-against) moderate-growth functions gives tempered distributions is accurate, there are definitely more tempered distributions than functions with moderate pointwise growth. For example, $u(x) = e^{ie^x}$ is continuous and bounded, so locally L^1. Thus, it gives a tempered distribution. Unsurprisingly, its distributional derivative is correctly computed by taking classical/pointwise derivatives:

$$u' \;=\; \text{(integration against)} \quad ie^x\, e^{ie^x}$$

That function is not of moderate growth, but since u' is the derivative of a tempered distribution, it is a tempered distribution. Evidently, the extreme oscillation causes a great deal of cancellation.

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7. *Multiplication of generalized functions by smooth functions*

Another part of the purpose of generalized functions is to be able to multiply them by nicer functions in a way that cannot literally be *pointwise*, but nevertheless fits coherently with classical pointwise multiplication.

By duality: For $u \in \mathcal{D}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathcal{D}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication. Observe that the product $\varphi \cdot f$ is still in $\mathcal{D}(\mathbb{R}^n)$. Somewhat similarly, for $u \in \mathscr{S}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$ so that it and all its derivatives are *of moderate growth*, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathscr{S}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication, and $\varphi \cdot f \in \mathscr{S}(\mathbb{R}^n)$ by the hypothesis on φ. Last, for $u \in \mathcal{E}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathcal{E}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication, and $\varphi \cdot f \in \mathcal{E}(\mathbb{R}^n)^*$.

By extension-by-continuity: Granting from above that test functions are (sequentially) dense in $\mathscr{S}^*$, $\mathcal{E}^*$, and $\mathcal{D}^*$, for suitable $\varphi \in \mathcal{E}(\mathbb{R}^n)^*$ and a sequence $\{u_n\}$ of test functions Cauchy in the corresponding weak dual topology, define

$$\varphi \cdot (\lim_n u_n) \;=\; \lim_n (\varphi \cdot u_n)$$

where the left-hand side is defined by the right, where both limits are in the respective weak dual topologies, and the multiplications on the right-hand side are the pointwise multiplications.

In particular, letting u_f be the integration-against-f functional for nice functions f,

$$\varphi \cdot u_f \;=\; u_{\varphi \cdot f}$$

where the left-hand side is distributional multiplication, and on the right-hand side is classical pointwise multiplication. That is, the distributional multiplication extended-by-continuity is indeed an extension of classical pointwise multiplication of nice functions.

[7.1] Claim: These two definitions give the same maps of $\mathcal{D}^*$, $\mathscr{S}^*$, and $\mathcal{E}^*$ to themselves. These maps are continuous in the weak dual topologies.

Proof: [... iou ...] ///

Example: $x \cdot \delta = 0$ For f, by the duality definition,

$$(x \cdot \delta)(f) \;=\; \delta(x \cdot f) \;=\; (xf)(0) \;=\; 0 \cdot f(0) \;=\; 0$$

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8. *Fourier transforms of tempered distributions*

The Plancherel extension of the literal Fourier transform integral on $L^1(\mathbb{R}^n)$ is a precursor of the extension of Fourier transform to tempered distributions.

By duality: For $u \in \mathscr{S}(\mathbb{R}^n)^*$, ndefine

$$\widehat{u}(f) \;=\; u(\widehat{f}) \qquad\qquad (\text{for all } f \in \mathscr{S}(\mathbb{R}^n))$$

where $\widehat{f}\mathscr{S}(\mathbb{R}^n)$ is the literal Fourier transform integral.

By extension-by-continuity: Granting from above that Schwartz functions are (sequentially) dense in tempered distributions, for a sequence $\{u_n\}$ of Schwartz functions Cauchy in the weak dual topology on $\mathscr{S}(\mathbb{R}^n)^*$, define

$$(\lim_n u_n)\widehat{\ } \;=\; \lim_n(\widehat{u}_n)$$

with the left-hand side is defined by the right, where both limits are in the weak dual topology on $\mathscr{S}(\mathbb{R}^n)^*$, and the Fourier transforms on the right-hand side are the literal integrals.

In particular, letting u_f be the integration-against-f functional for nice functions f,

$$\widehat{u_f} \;=\; u_{\widehat{f}}$$

where the left-hand side is distributional Fourier transform, and on the right-hand side is the literal Fourier transform integral. That is, the distributional multiplication extended-by-continuity is indeed an extension of classical pointwise multiplication of nice functions.

[8.1] Claim: These two definitions give the same map of $\mathscr{S}^*$ to itself, and are continuous in the weak dual topology.

Proof: [... iou ...] ///

The following amounts to the assertion that the seemingly natural way to compute Fourier transforms of $u \in \mathcal{E}^* \subset \mathscr{S}^*$ is correct:

[8.2] Claim: For $u \in \mathcal{E}(\mathbb{R}^n)^*$, $\widehat{u}$ is a pointwise-valued function, and $\widehat{u}(\xi) = u(e^{-2\pi i \xi \cdot x})$, noting that the exponential functions are in $\mathcal{E}$.

Proof: [... iou ...] ///
