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Introduction to Levi-Sobolev spaces

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1. *Prototypical Sobolev imbedding theorem*

The simplest case of a Levi-Sobolev *imbedding theorem* asserts that the +1-index Levi-Sobolev Hilbert space $H^1[a,b]$ described below is inside $C^o[a,b]$. This is a corollary of a Levi-Sobolev *inequality* asserting that the $C^o[a,b]$ norm is *dominated* by the $H^1[a,b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality. The point is that there is a large *Hilbert space* $H^1[a,b]$ inside the *Banach* space $C^o[a,b]$.

We will do much more with this idea subsequently.

We can think of $L^2[a,b]$ as

$$L^2[a,b] \;=\; \text{completion of } \; C^o[a,b] \text{ with respect to } |f|_{L^2} = \Big(\int_a^b |f(t)|^2\,dt\Big)^{1/2}$$

The +1-index *Levi-Sobolev* space [1] $H^1[a,b]$ is

$$H^1[a,b] \;=\; \text{completion of } \; C^1[a,b] \text{ with respect to } |f|_{H^1} = \Big(|f|^2_{L^2[a,b]} + |f'|^2_{L^2[a,b]}\Big)^{1/2}$$

[1.1] Theorem: *(Levi-Sobolev inequality)* On $C^1[a,b]$, the $H^1[a,b]$-norm *dominates* the $C^o[a,b]$-norm. That is, there is a constant C depending only on a,b such that $|f|_{C^o[a,b]} \le C\cdot|f|_{H^1[a,b]}$ for every $f\in C^1[a,b]$.

Proof: For $a\le x\le y\le b$, for $f\in C^1[a,b]$, the fundamental theorem of calculus gives

$$|f(y)-f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_x^y |f'(t)|\,dt \;\le\; \Big(\int_x^y |f'(t)|^2\,dt\Big)^{1/2}\cdot\Big(\int_x^y 1\,dt\Big)^{1/2}$$
$$\le\; |f'|_{L^2}\cdot|x-y|^{\frac12} \;\le\; |f'|_{L^2}\cdot|a-b|^{\frac12}$$

Using the continuity of $f\in C^1[a,b]$, let $y\in[a,b]$ be such that $|f(y)| = \min_x|f(x)|$. Using the previous inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x)-f(y)| \;\le\; \frac{\int_a^b|f(t)|\,dt}{|a-b|} \;+\; |f(x)-f(y)| \;\le\; \frac{\int_a^b|f|\cdot 1}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12}$$

$$\le\; \frac{|f|^{\frac12}_{L^2}\cdot|a-b|^{\frac12}}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12} \;=\; \frac{|f|^{\frac12}_{L^2}}{|a-b|^{\frac12}} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12} \;\le\; \Big(|f|_{L^2}+|f'|_{L^2}\Big)\cdot\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big)$$

$$\le\; 2\big(|f|^2+|f'|^2\big)^{1/2}\cdot\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big) \;=\; |f|_{H^1}\cdot 2\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big)$$

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[1] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to L^2, rather than L^p. Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906, giving a correct formulation of *Dirichlet's principle*. Sobolev's systematic development of these ideas was in the mid-1930's.

Thus, on $C^1[a,b]$ the H^1 norm dominates the C^o-norm. ///

[1.2] Corollary: *(Levi-Sobolev imbedding)* $H^1[a,b] \subset C^o[a,b]$.

Proof: Since $H^1[a,b]$ is the H^1-norm completion of $C^1[a,b]$, every $f \in H^1[a,b]$ is an H^1-limit of functions $f_n \in C^1[a,b]$. That is, $|f - f_n|_{H^1[a,b]} \to 0$. Since the H^1-norm dominates the C^o-norm, $|f - f_n|_{C^o[a,b]} \to 0$. A C^o limit of continuous functions is continuous, so f is continuous. ///

In fact, we have a stronger conclusion than continuity, namely, a *Lipschitz condition* with exponent $\frac{1}{2}$:

[1.3] Corollary: (of proof of theorem) $|f(x) - f(y)| \le |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$ for $f \in H^1[a,b]$. ///

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2. *Sobolev theorems on* $\mathbb{T}^n$

For $0 \le k \in \mathbb{Z}$ and $f \in C^\infty(\mathbb{T}^n)$, the k^{th} Sobolev norm can be defined in terms of L^2 norms of all its derivatives up through order k:

$$|f|^2_{H^k} \;=\; \sum_{|\alpha| \le k} |f^{(\alpha)}|_{L^2}$$

where as usual α is summed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers α_i, with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$f^{(\alpha)} \;=\; \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\, f$$

Then one way to define the k^{th} Sobolev space is

$$H^k(\mathbb{T}^n) \;=\; \text{completion of } C^\infty(\mathbb{T}^n) \text{ with respect to } |\cdot|_{H^k}$$

In this context, $H^{-k}(\mathbb{T}^n)$ for $-k < 0$ is defined to be the dual of $H^k(\mathbb{T}^n)$, with $H^0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism rather than $\mathbb{C}$-conjugate-linear). From the inclusion $H^{k+1} \to H^k$ for $0 \le k \in \mathbb{Z}$ dualizing gives a dual/adjoint map $H^{-k} \to H^{-k-1}$. Let

$$H^\infty(\mathbb{T}^n) \;=\; \bigcap_{k=0}^{\infty} H^k(\mathbb{T}^n) \;=\; \lim_k H^k(\mathbb{T}^n)$$

and

$$H^{-\infty}(\mathbb{T}^n) \;=\; \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{T}^n) \;=\; \mathrm{colim}_k H^{-k}(\mathbb{T}^n)$$

The picture is

$$\begin{tikzcd} H^\infty(\mathbb{T}^n) \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r] & H^1(\mathbb{T}^n) \arrow[r] & H^0(\mathbb{T}^n) \arrow[r] \arrow[rrr, bend left] & H^{-1}(\mathbb{T}^n) \arrow[r] \arrow[rr, bend left] & \ldots \arrow[r] & H^{-\infty}(\mathbb{T}^n) \end{tikzcd}$$

[2.1] Claim: All arrows are continuous injections with dense images.

Proof: [... iou ...] ///

A *spectral characterization* of Sobolev norms is often useful, and directly defines $H^s(\mathbb{T}^n)$ for all $s \in \mathbb{R}$: for $f \in C^\infty(\mathbb{T}^n)$, with Fourier coefficients $\widehat{f}(\xi)$,

$$|f|^2_{H^s} \;=\; \sum_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^s$$

and $H^s(\mathbb{T}^n)$ is the completion of $C^\infty(\mathbb{T}^n)$ with this norm.

[2.2] Claim: The spectral characterization gives the same topology on H^k as the characterization in terms of L^2 norms of derivatives, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

Sometimes it is convenient to give the derivative characterization slightly differently, as

$$|f|^2_{H^k} \;=\; \langle (1-\Delta)^k f,\, f\rangle_{L^2}$$

[2.3] Claim: The latter characterization gives the same topology on H^k as do the two previous characterizations, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[2.4] Theorem: *(Sobolev imbedding theorem)* $H^s(\mathbb{T}^n) \subset C^k(\mathbb{T}^n)$ for $s > \frac{n}{2}$.

Proof: [... iou ...] ///

[2.5] Corollary: $H^\infty(\mathbb{T}^n) \subset C^\infty(\mathbb{T}^n)$, and $H^{-\infty}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)^*$.

Proof: [... iou ...] ///

[2.6] Theorem: The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i \xi\cdot x}$,

$$\Big\langle \sum_\xi a_\xi \psi_\xi,\, \sum_\xi b_\xi \psi_\xi \Big\rangle_{H^s\times H^{-s}} \;=\; \sum_\xi a_\xi \cdot \overline{b_\xi}$$

Proof: [... iou ...] ///

That is, distributions on $\mathbb{T}^n$ admit Fourier expansions with coefficients of moderate growth, and evaluation of distributions on smooth functions can be done by a natural extension of Plancherel.

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3. *Sobolev theorems on* $\mathbb{R}^n$

The general shape of the discussion on $\mathbb{R}^n$ is similar to that on $\mathbb{T}^n$, with some unsurprising complications due to the non-compactness of $\mathbb{R}$. In particular, Fourier series are replaced by Fourier transforms and inversion.

For $0 \le k \in \mathbb{Z}$ and $f \in C_c^\infty(\mathbb{R}^n)$, the k^{th} Sobolev norm can be defined in terms of L^2 norms of all its derivatives up through order k:

$$|f|^2_{H^k} \;=\; \sum_{|\alpha|\le k} |f^{(\alpha)}|_{L^2}$$

One way to define the k^{th} Sobolev space is

$$H^k(\mathbb{R}^n) \;=\; \text{completion of } C_c^\infty(\mathbb{R}^n) \text{ with respect to } |\cdot|_{H^k}$$

In this context, $H^{-k}(\mathbb{R}^n)$ for $-k<0$ is defined to be the dual of $H^k(\mathbb{R}^n)$, with $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism rather than $\mathbb{C}$-conjugate-linear). From the inclusion $H^{k+1} \to H^k$ for $0 \le k \in \mathbb{Z}$ dualizing gives a dual/adjoint map $H^{-k} \to H^{-k-1}$. Let

$$H^\infty(\mathbb{R}^n) \;=\; \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^n) \;=\; \lim_k H^k(\mathbb{R}^n)$$

and

$$H^{-\infty}(\mathbb{R}^n) \;=\; \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{R}^n) \;=\; \mathrm{colim}_k H^{-k}(\mathbb{R}^n)$$

The picture is the same as for $\mathbb{T}^n$:

$$\begin{tikzcd}
H^\infty(\mathbb{R}^n) \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r] & H^1(\mathbb{R}^n) \arrow[r] & H^0(\mathbb{R}^n) \arrow[r] \arrow[rrr, bend left] & H^{-1}(\mathbb{R}^n) \arrow[r] \arrow[rr, bend left] & \ldots \arrow[r] & H^{-\infty}(\mathbb{R}^n)
\end{tikzcd}$$

[3.1] Claim: All arrows are continuous injections with dense images.

Proof: [... iou ...] ///

A *spectral characterization* of Sobolev norms is often useful, and directly defines $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$: for $f \in C_c^\infty(\mathbb{R}^n)$, with Fourier transform $\widehat{f}(\xi)$,

$$|f|^2_{H^s} \;=\; \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^s \; d\xi$$

and $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with this norm.

[3.2] Claim: The spectral characterization gives the same topology on H^k as the characterization in terms of L^2 norms of derivatives, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[3.3] Corollary: Distributions u in $H^{-\infty}(\mathbb{R}^n)$ have Fourier transforms that are in weighted L^2 spaces, with pointwise values almost everywhere.

Proof: [... iou ...] ///

Sometimes it is convenient to give the derivative characterization slightly differently, as

$$|f|^2_{H^k} \;=\; \langle (1-\Delta)^k f,\, f\rangle_{L^2}$$

[3.4] Claim: The latter characterization gives the same topology on H^k as do the two previous characterizations, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[3.5] Theorem: *(Sobolev imbedding theorem)* $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ for $s > \frac{n}{2}$.

Proof: [... iou ...] ///

Since $\mathbb{R}^n$ is non-compact, the conclusion of the following is weaker than for $\mathbb{T}^n$, since $C^\infty(\mathbb{R}^n)$ is not equal to $\mathscr{S}(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$:

[3.6] Corollary: $H^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ and $H^{-\infty}(\mathbb{R}^n) \supset C^\infty(\mathbb{R}^n)^*$.

Proof: [... iou ...] ///

[3.7] Corollary: If we know that $\mathcal{E}(\mathbb{R}^n)^* = C^\infty(\mathbb{R}^n)^*$ is exactly compactly-supported distributions, then we can conclude that $H^{-\infty}(\mathbb{R}^n)$ contains compactly-supported distributions. ///

[3.8] Theorem: The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$,

$$\langle f, F \rangle_{H^s \times H^{-s}} \;=\; \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot \overline{\widehat{F}(\xi)}\, d\xi$$

Proof: [... iou ...] ///

That is, evaluation of distributions in $H^{-\infty}(\mathbb{R}^n)$ on smooth functions in $H^\infty(\mathbb{R}^n)$ can be done by a natural extension of Plancherel.
