(*April 3, 2019*)

09a. Operators on Hilbert spaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

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1. Boundedness, continuity, operator norms

A linear (not necessarily continuous) map $T : X \to Y$ from one Hilbert space to another is *bounded* if, for all $\varepsilon > 0$, there is $\delta > 0$ such that $|Tx|_Y < \varepsilon$ for all $x \in X$ with $|x|_X < \delta$.

[1.1] Proposition: For a linear, not necessarily continuous, map $T : X \to Y$ of Hilbert spaces, the following three conditions are equivalent:
(i) T is continuous
(ii) T is continuous at 0
(iii) T is bounded

Proof: For T continuous as 0, given $\varepsilon > 0$ and $x \in X$, there is small enough $\delta > 0$ such that $|Tx' - 0|_Y < \varepsilon$ for $|x' - 0|_X < \delta$. For $|x'' - x|_X < \delta$, using the linearity,

$$|Tx'' - Tx|_X \;=\; |T(x'' - x) - 0|_X \;<\; \delta$$

That is, continuity at 0 implies continuity.

Since $|x| = |x - 0|$, continuity at 0 is immediately equivalent to boundedness. ///

[1.2] Definition: The *kernel* $\ker T$ of a linear (not necessarily continuous) linear map $T : X \to Y$ from one Hilbert space to another is

$$\ker T \;=\; \{x \in X : Tx = 0 \in Y\}$$

[1.3] Proposition: The kernel of a continuous linear map $T : X \to Y$ is closed.

Proof: For T continuous

$$\ker T = T^{-1}\{0\} \;=\; X - T^{-1}(Y - \{0\}) \;=\; X - T^{-1}(\text{open}) \;=\; X - \text{open} \;=\; \text{closed}$$

since the inverse images of open sets by a continuous map are open. ///

[1.4] Definition: The *operator norm* $|T|$ of a linear map $T : X \to Y$ is

$$\text{operator norm } T \;=\; |T| \;=\; \sup_{x \in X \,:\, |x|_X \le 1} |Tx|_Y$$

[1.5] Corollary: A linear map $T : X \to Y$ is continuous if and only if its operator norm is finite. ///

2. *Adjoints*

An *adjoint* T^* of a continuous linear map $T : X \to Y$ from a pre-Hilbert space X to a pre-Hilbert space Y (if T^* exists) is a continuous linear map $T^* : Y \to X$ such that

$$\langle Tx, y\rangle_Y \;=\; \langle x, T^*y\rangle_X$$

[2.1] Remark: When a pre-Hilbert space X is not complete, that is, is not a Hilbert space, an operator $T : X \to Y$ may fail to have an adjoint.

[2.2] Theorem: A continuous linear map $T : X \to Y$ from a *Hilbert* space X to a Hilbert space Y has a unique adjoint T^*.

[2.3] Remark: In fact, the target space of T need not be a Hilbert space, that is, need not be complete, but we will not use this.

Proof: For each $y \in Y$, the map

$$\lambda_y : X \longrightarrow \mathbb{C}$$

given by

$$\lambda_y(x) \;=\; \langle Tx, y\rangle$$

is a continuous linear functional on X. By Riesz-Fréchet, there is a unique $x_y \in X$ so that

$$\langle Tx, y\rangle \;=\; \lambda_y(x) \;=\; \langle x, x_y\rangle$$

Try to define T^* by $T^*y = x_y$. This is a well-defined map from Y to X, and has the adjoint property $\langle Tx, y\rangle_Y = \langle x, T^*y\rangle_X$.

To prove that T^* is continuous, prove that it is bounded. From Cauchy-Schwarz-Bunyakowsky

$$|T^*y|^2 \;=\; |\langle T^*y, T^*y\rangle_X| \;=\; |\langle y, TT^*y\rangle_Y| \le |y|\cdot|TT^*y| \;\le\; |y|\cdot|T|\cdot|T^*y|$$

where $|T|$ is the operator norm. For $T^*y \neq 0$, divide by it to find

$$|T^*y| \;\le\; |y|\cdot|T|$$

Thus, $|T^*| \le |T|$. In particular, T^* is bounded. Since $(T^*)^* = T$, by symmetry $|T| = |T^*|$. Linearity of T^* is easy. ///

[2.4] Corollary: For a continuous linear map $T : X \to Y$ of Hilbert spaces, $T^{**} = T$. ///

An operator $T \in \mathrm{End}(X)$ commuting with its adjoint is *normal*, that is,

$$TT^* \;=\; T^*T$$

This only applies to operators from a Hilbert space *to itself*. An operator T is *self-adjoint* or *hermitian* if $T = T^*$. That is, T is hermitian when

$$\langle Tx, y\rangle \;=\; \langle x, Ty\rangle \qquad \text{(for all } x, y \in X)$$

An operator T is *unitary* when

$$TT^* \;=\; \text{identity map on } Y \qquad T^*T \;=\; \text{identity map on } X$$

There are simple examples in infinite-dimensional spaces where $TT^* = 1$ does not imply $T^*T = 1$, and vice-versa. Thus, it does *not* suffice to check something like $\langle Tx, Tx\rangle = \langle x, x\rangle$ to prove unitariness. Obviously hermitian operators are normal, as are unitary operators, using the more careful definition.

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3. *Stable subspaces and complements*

Let $T : X \to X$ be a continuous linear operator on a Hilbert space X. A vector subspace Y is T-*stable* or T-*invariant* if $Ty \in Y$ for all $y \in Y$. Often one is most interested in the case that the subspace be *closed* in addition to being *invariant*.

[3.1] Proposition: For $T : X \to X$ a continuous linear operator on a Hilbert space X, and Y a T-stable subspace, $Y^\perp$ is T^*-stable.

Proof: For $z \in Y^\perp$ and $y \in Y$,

$$\langle T^*z, y\rangle \;=\; \langle z, T^{**}y\rangle \;=\; \langle z, Ty\rangle$$

since $T^{**} = T$, from above. Since Y is T-stable, $Ty \in Y$, and this inner product is 0, and $T^*z \in Y^\perp$. ///

[3.2] Corollary: For continuous *self-adjoint* T on a Hilbert space X, and Y a T-stable subspace, both Y and $Y^\perp$ are T-stable. ///

[3.3] Remark: *Normality* of $T : X \to X$ is insufficient to assure the conclusion of the corollary, in general. For example, with the two-sided ℓ^2 space

$$X \;=\; \{\{c_n : n \in \mathbb{Z}\} : \sum_{n\in\mathbb{Z}} |c_n|^2 \;<\; \infty\}$$

the right-shift operator

$$(Tc)_n \;=\; c_{n-1} \qquad\qquad (\text{for } n \in \mathbb{Z})$$

has adjoint the *left* shift operator

$$(T^*c)_n \;=\; c_{n+1} \qquad\qquad (\text{for } n \in \mathbb{Z})$$

and

$$T^*T \;=\; TT^* \;=\; 1_X$$

So this T is not merely *normal*, but *unitary*. However, the T-stable subspace

$$Y \;=\; \{\{c_n\} \in X : c_k = 0 \text{ for } k < 0\}$$

is not T^*-stable, nor is its orthogonal complement T-stable.

On the other hand, adjoint-stable *collections* of operators have a good stability result:

[3.4] Proposition: Suppose for every T in a set A of bounded linear operators on a Hilbert space V the adjoint T^* is also in A. Then, for an A-stable subspace W of V, the orthogonal complement $W^\perp$ is also A-stable.

Proof: For y in $W^\perp$ and $T \in A$, for $x \in W$,

$$\langle x, Ty\rangle \;=\; \langle T^*x, y\rangle \in \langle W, y\rangle \;=\; \{0\}$$

since $T^* \in A$. ///

4. *Spectrum, eigenvalues*

For a continuous linear operator $T \in \mathrm{End}(X)$, the λ-*eigenspace* of T is

$$X_\lambda = \{x \in X : Tx = \lambda x\}$$

If this space is not simply $\{0\}$, then λ is an *eigenvalue*.

[4.1] Proposition: An eigenspace X_λ for a continuous linear operator T on X is a *closed* and T-stable subspace of X. For *normal* T the adjoint T^* acts by the scalar $\overline{\lambda}$ on X_λ.

Proof: The λ-eigenspace is the kernel of the continuous linear map $T - \lambda$, so is closed. The stability is clear, since the operator restricted to the eigenspace is a scalar operator. For $v \in X_\lambda$, using normality,

$$(T - \lambda)T^*v = T^*(T - \lambda)v = T^* \cdot 0 = 0$$

Thus, X_λ is T^*-stable. For $x, y \in X_\lambda$,

$$\langle (T^* - \overline{\lambda})x, y\rangle = \langle x, (T - \lambda)y\rangle = \langle x, 0\rangle$$

Thus, $(T^* - \overline{\lambda})x = 0$. ///

[4.2] Proposition: For T *normal*, for $\lambda \neq \mu$, and for $x \in X_\lambda, y \in X_\mu$, always $\langle x, y\rangle = 0$. For T *self-adjoint*, if $X_\lambda \neq 0$ then $\lambda \in \mathbb{R}$. For T *unitary*, if $X_\lambda \neq 0$ then $|\lambda| = 1$.

Proof: Let $x \in X_\lambda$, $y \in X_\mu$, with $\mu \neq \lambda$. Then

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle$$

invoking the previous result. Thus,

$$(\lambda - \mu)\langle x, y\rangle = 0$$

giving the result. For T self-adjoint and x a non-zero λ-eigenvector,

$$\lambda\langle x, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$$

Thus, $(\lambda - \overline{\lambda})\langle x, x\rangle = 0$. Since x is non-zero, the result follows. For T unitary and x a non-zero λ-eigenvector,

$$\langle x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = |\lambda|^2 \cdot \langle x, x\rangle$$

///

In what follows, for a complex scalar λ write simply λ for scalar multiplication by λ on a Hilbert space X.

[4.3] Definition: The *spectrum* $\sigma(T)$ of a continuous linear operator $T : X \to X$ on a Hilbert space X is the collection of complex numbers λ such that $T - \lambda$ *does not have* a continuous linear inverse.

[4.4] Definition: The *discrete spectrum* $\sigma_{\mathrm{disc}}(T)$ is the collection of complex numbers λ such that $T - \lambda$ fails to be *injective*. In other words, the discrete spectrum is the collection of eigenvalues.

[4.5] Definition: The *continuous spectrum* $\sigma_{\mathrm{cont}}(T)$ is the collection of complex numbers λ such that $T - \lambda \cdot 1_X$ *is* injective, *does* have dense image, but fails to be *surjective*.

[4.6] Definition: The *residual spectrum* $\sigma_{\mathrm{res}}(T)$ is everything else: neither discrete nor continuous spectrum. That is, the residual spectrum of T is the collection of complex numbers λ such that $T - \lambda \cdot 1_X$ *is* injective, and *fails* to have dense image (so is certainly not surjective).

[4.7] Remark: To see that there are no *other* possibilities, note that the *Closed Graph Theorem* implies that a bijective, continuous, linear map $T : X \to Y$ of Banach spaces has continuous inverse. Indeed, granting that the inverse exists as a linear map, its graph is

$$\text{graph of } T^{-1} \;=\; \{(y,x) \in Y \times X \;:\; (x,y) \text{ in the graph of } T \;\subset X \times Y\}$$

Since the graph of T is closed, the graph of T^{-1} is closed, and by the Closed Graph Theorem T^{-1} is continuous.

The potential confusion of *residual* spectrum does not occur in many situations of interest"

[4.8] Proposition: A normal operator $T : X \to X$ has *empty* residual spectrum.

Proof: The adjoint of $T - \lambda$ is $T^* - \overline{\lambda}$, so consider $\lambda = 0$ to lighten the notation. Suppose that T does *not* have dense image. Then there is non-zero z such that

$$0 \;=\; \langle z, Tx\rangle \;=\; \langle T^* z, x\rangle \qquad\qquad (\text{for every } x \in X)$$

Therefore $T^* z = 0$, and the 0-eigenspace Z of T^* is non-zero. Since $T^*(Tz) = T(T^* z) = T(0) = 0$ for $z \in Z$, T^* stabilizes Z. That is, Z is both T and T^*-stable. Therefore, $T = (T^*)*$ acts on Z by (the complex conjugate of) 0, and T has non-trivial 0-eigenvectors, contradiction. ///

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5. *Generalities on spectra*

It is convenient to know that spectra of continuous operators are *non-empty*, *compact* subsets of $\mathbb{C}$.

Knowing this, *every* non-empty compact subset of $\mathbb{C}$ is easily made to appear as the spectrum of a continuous operator, even *normal* ones, as below.

[5.1] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator $T : V \to V$ on a Hilbert space V is *bounded* by the operator norm $|T|_{\text{op}}$.

Proof: For $|\lambda| > |T|_{\text{op}}$, an obvious heuristic suggests an expression for the *resolvent* $R_\lambda = (T - \lambda)^{-1}$:

$$(T-\lambda)^{-1} \;=\; -\lambda^{-1} \cdot \big(1 - \frac{T}{\lambda}\big)^{-1} \;=\; -\lambda^{-1} \cdot \Big(1 + \frac{T}{\lambda} + \big(\frac{T}{\lambda}\big)^2 + \ldots\Big)$$

The infinite series converges in operator norm for $|T/\lambda|_{\text{op}} < 1$, that is, for $|\lambda| > |T|_{\text{op}}$. Then

$$(T-\lambda) \cdot (-\lambda^{-1}) \cdot \Big(1 + \frac{T}{\lambda} + \big(\frac{T}{\lambda}\big)^2 + \ldots\Big) \;=\; 1$$

giving a continuous inverse $(T-\lambda)^{-1}$, so $\lambda \notin \sigma(T)$. ///

[5.2] Remark: The same argument applied to T^n shows that $\sigma(T^n)$ is inside the closed ball of radius $|T^n|_{\text{op}}$. By the elementary identity

$$T^n - \lambda^n \;=\; (T - \lambda) \cdot (T^{n-1} + T^{n-2}\lambda + \ldots + T\lambda^{n-2} + \lambda^{n-2})$$

$(T-\lambda)^{-1}$ exists for $|\lambda^n| > |T^n|_{\text{op}}$, that is, for $|\lambda| > |T^n|_{\text{op}}^{1/n}$. That is, $\sigma(T)$ is inside the closed ball of radius $\inf_{n\ge 1} |T^n|_{\text{op}}^{1/n}$. The latter expression is the *spectral radius* of T. This notion is relevant to *non-normal* operators, such as the *Volterra operator*, whose spectral radius is 0, while its operator norm is much larger.

[5.3] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator $T : V \to V$ on a Hilbert space V is *compact*.

Proof: That $\lambda \notin \sigma(T)$ is that there is a continuous linear operator $(T-\lambda)^{-1}$. We claim that for μ sufficiently close to λ, $(T-\mu)^{-1}$ exists. Indeed, a heuristic suggests an expression for $(T-\mu)^{-1}$ in terms of $(T-\lambda)^{-1}$. The algebra is helpfully simplified by replacing T by $T+\lambda$, so that $\lambda = 0$. With μ near 0 and granting existence of T^{-1}, the heuristic is

$$(T-\mu)^{-1} \;=\; (1-\mu T^{-1})^{-1} \cdot T^{-1} \;=\; \Big(1 + \mu T^{-1} + (\mu T^{-1})^2 + \ldots\Big) \cdot T^{-1}$$

The geometric series converges in operator norm for $|\mu T^{-1}|_{\mathrm{op}} < 1$, that is, for $|\mu| < |T^{-1}|_{\mathrm{op}}^{-1}$. Having found the obvious candidate for an inverse,

$$(1-\mu T^{-1}) \cdot \Big(1 + \mu T^{-1} + (\mu T^{-1})^2 + \ldots\Big) \;=\; 1$$

and

$$(T-\mu) \cdot \Big(1 + \mu T^{-1} + (\mu T^{-1})^2 + \ldots\Big) \cdot T^{-1} \;=\; 1$$

so there is a continuous linear operator $(T-\mu)^{-1}$, and $\mu \notin \sigma(T)$. Having already proven that $\sigma(T)$ is *bounded*, it is *compact*. ///

[5.4] Proposition: The spectrum $\sigma(T)$ of a continuous linear operator on a Hilbert space $V \neq \{0\}$ is *non-empty.*

Proof: The argument reduces the issue to Liouville's theorem from complex analysis, that a *bounded* entire (holomorphic) function is *constant.* Further, an entire function that goes to 0 at ∞ is identically 0.

Suppose the resolvent $R_\lambda = (T-\lambda)^{-1}$ is a continuous linear operator for all $\lambda \in \mathbb{C}$. The operator norm is readily estimated for large λ:

$$|R_\lambda|_{\mathrm{op}} \;=\; |\lambda|^{-1} \cdot \Big|1 + \frac{T}{\lambda} + \big(\frac{T}{\lambda}\big)^2 + \ldots\Big|_{\mathrm{op}}$$

$$\le\; |\lambda|^{-1} \cdot \Big(1 + \big|\frac{T}{\lambda}\big|_{\mathrm{op}} + \big|\frac{T}{\lambda}\big|_{\mathrm{op}}^2 + \ldots\Big) \;=\; \frac{1}{|\lambda|} \cdot \frac{1}{1 - \dfrac{|T|_{\mathrm{op}}}{|\lambda|}}$$

This goes to 0 as $|\lambda| \to \infty$. *Hilbert's identity* asserts the complex differentiability as operator-valued function:

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} \;=\; R_\lambda \cdot \frac{(T-\mu)-(T-\lambda)}{\lambda-\mu} \cdot R_\mu \;=\; R_\lambda \cdot R_\mu \;\longrightarrow\; R_\lambda^2 \qquad\qquad (\text{as } \mu \to \lambda)$$

since $\mu \to R_\mu$ is continuous for large μ, by the same identity:

$$|R_\lambda - R_\mu|_{\mathrm{op}} \;\le\; |\lambda-\mu| \cdot |R_\mu \cdot R_\lambda|_{\mathrm{op}}$$

Thus, the scalar-valued functions $\lambda \to \langle R_\lambda v, w\rangle$ for $v, w \in V$ are complex-differentiable, and satisfy

$$|\langle R_\lambda v, w\rangle| \;\le\; |R_\lambda v| \cdot |w| \;\le\; |R_\lambda|_{\mathrm{op}} \cdot |v| \cdot |w| \;\le\; \frac{1}{|\lambda|} \cdot \frac{1}{1 - \dfrac{|T|_{\mathrm{op}}}{|\lambda|}} \cdot |v| \cdot |w|$$

By Liouville, $\langle R_\lambda v, w\rangle = 0$ for all $v, w \in V$, which is impossible. Thus, the spectrum is not empty. ///

[5.5] Proposition: The entire spectrum, both point-spectrum and continuous-spectrum, of a *self-adjoint* operator is a non-empty, compact subset of $\mathbb{R}$. The entire spectrum of a *unitary* operator is a non-empty, compact subset of the unit circle.

Proof: For self-adjoint T, we claim that the imaginary part of $\langle (T-\mu)v, v\rangle$ is at least $\langle v, v\rangle$ times the imaginary part of μ. Indeed, $\langle Tv, v\rangle$ is *real*, since

$$\langle Tv, v\rangle \;=\; \langle v, T^* v\rangle \;=\; \langle v, Tv\rangle \;=\; \overline{\langle Tv, v\rangle}$$

so

$$\langle (T-\mu)v, v\rangle \;=\; \langle Tv, v\rangle - \mu \cdot \langle v, v\rangle$$

and

$$|\mathrm{Im}\,\langle (T-\mu)v, v\rangle| \;=\; |\mathrm{Im}\,\mu| \cdot \langle v, v\rangle$$

and by Cauchy-Schwarz-Bunyakowsky

$$|(T-\mu)v| \cdot |v| \;\ge\; |\langle (T-\mu)v, v\rangle| \;\ge\; |\mathrm{Im}\,\mu| \cdot \langle v, v\rangle \;=\; |\mathrm{Im}\,\mu| \cdot |v|^2$$

Dividing by $|v|$,

$$|(T-\mu)v| \;\ge\; |\mathrm{Im}\,\mu| \cdot |v|$$

This inequality shows more than the injectivity of $T-\mu$. Namely, the inequality gives a bound on the operator norm of the inverse $(T-\mu)^{-1}$ *defined on the image of* $T-\mu$. The image is *dense* since μ is not an eigenvalue and there is no residual spectrum for normal operators T. Thus, the inverse extends by continuity to a continuous linear map defined on the whole Hilbert space. Thus, $T-\mu$ has a continuous linear inverse, and μ is not in the spectrum of T.

For T unitary, $|Tv| = |v|$ for all v implies $T|_{\mathrm{op}} = 1$. Thus, $\sigma(T)$ is contained in the unit disk, by the general bound on spectra in terms of operator norms. From $(T-\lambda)^* = T^* - \overline{\lambda}$, the spectrum of T^* is obtained by complex-conjugating the spectrum of T. Thus, for unitary T, the spectrum of $T^{-1} = T^*$ is also contained in the unit disk. At the same time, the natural

$$T - \lambda \;=\; -T \cdot (T^{-1} - \lambda^{-1}) \cdot \lambda$$

gives

$$(T-\lambda)^{-1} \;=\; -\lambda^{-1} \cdot (T^{-1} - \lambda^{-1})^{-1} \cdot T^{-1}$$

so $\lambda^{-1} \in \sigma(T^{-1})$ exactly when $\lambda \in \sigma(T)$. Thus, the spectra of both T and $T^* = T^{-1}$ are inside the unit circle. ///

[5.6] Remark:

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6. *Positive examples*

Let ℓ^2 be the usual space of square-summable sequences $(a_1, a_2, \ldots)$, with standard orthonormal basis

$$e_j \;=\; \underbrace{(0, \ldots, 0, 1, 0, \ldots)}_{1 \text{ at } jth \text{ position}}$$

[6.1] Multiplication operators with specified eigenvalues Given a countable, bounded list of complex numbers λ_j, the operator $T : \ell^2 \to \ell^2$ by

$$T \;:\; (a_1, a_2, \ldots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \ldots)$$

has λ_j-eigenvector the standard basis element e_j. Clearly

$$T^* \;:\; (a_1, a_2, a_3, \ldots) \longrightarrow (\overline{\lambda}_1 \cdot a_1, \overline{\lambda}_2 \cdot a_2, \overline{\lambda}_3 \cdot a_3, \ldots)$$

so T is *normal*, in the sense that $TT^* = T^*T$. To see that there are no *other* eigenvalues, suppose $Tv = \mu \cdot v$ with μ not among the λ_j. Then

$$\mu \cdot \langle v, e_j \rangle \;=\; \langle Tv, e_j \rangle \;=\; \langle v, T^* e_j \rangle \;=\; \langle v, \overline{\lambda}_j e_j \rangle \;=\; \lambda_j \cdot \langle v, e_j \rangle$$

Thus, $(\mu - \lambda_j) \cdot \langle v, e_j \rangle = 0$, and $\langle v, e_j \rangle = 0$ for all j. Since e_j form an orthonormal basis, $v = 0$. ///

[6.2] Every compact subset of $\mathbb{C}$ is the spectrum of an operator Grant for the moment a countable dense subset $\{\lambda_j\}$ of a non-empty compact subset [1] C of $\mathbb{C}$, and as above let

$$T \;:\; (a_1, a_2, a_3, \ldots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \lambda_3 \cdot a_3, \ldots)$$

We saw that there are no further eigenvalues. Since spectra are *closed*, the closure C of $\{\lambda_j\}$ is *contained* in $\sigma(T)$.

It remains to show that the continuous spectrum is no larger than the closure C of the eigenvalues, *in this example*. That is, for $\mu \notin C$, exhibit a continuous linear $(T - \mu)^{-1}$.

For $\mu \notin C$, there is a uniform lower bound $0 < \delta \le |\mu - \lambda_j|$. That is, $\sup_j |\mu - \lambda_j|^{-1} \le \delta^{-1}$. Thus, the naturally suggested map

$$(a_1, a_2, \ldots) \;\longrightarrow\; \Big((\lambda_1 - \mu)^{-1} \cdot a_1,\, (\lambda_2 - \mu)^{-1} \cdot a_2,\, \ldots \Big)$$

is a bounded linear map, and gives $(T - \mu)^{-1}$.

[6.3] Two-sided shift has no eigenvalues Let V be the Hilbert space of *two-sided* sequences $(\ldots, a_{-1}, a_0, a_1, \ldots)$ with natural inner product

$$\langle (\ldots, a_{-1}, a_0, a_1, \ldots),\, (\ldots, b_{-1}, b_0, b_1, \ldots) \rangle \;=\; \ldots + a_{-1} b_{-1} + a_0 b_0 + a_1 b_1 + \ldots$$

The right and left *two-sided* shift operators are

$$(R \cdot a)_n \;=\; a_{n-1} \qquad\qquad (L \cdot a)_n \;=\; a_{n+1}$$

These operators are mutual adjoints, mutual inverses, so are unitary. Being unitary, their operator norms are 1, so their spectra are non-empty compact subsets of the unit circle.

They have no eigenvalues: indeed, for $Rv = \lambda \cdot v$, if there is any index n with $v_n \ne 0$, then the relation $Rv = \lambda \cdot v$ gives $v_{n+k+1} = \lambda \cdot v_{n+k}$ for $k = 0, 1, 2, \ldots$. Since $|\lambda| = 1$, such a vector is not in ℓ^2.

Nevertheless, we claim that $\lambda \in \sigma(L)$ for every λ with $|\lambda| = 1$, and similarly for R. Indeed, for λ *not* in the spectrum, there is a continuous linear operator $(L - \lambda)^{-1}$, so $|(L - \lambda)v| \ge \delta \cdot |v|$ for some $\delta > 0$. It is easy to make *approximate* eigenvectors for L for any $|\lambda| = 1$: let

$$v^{(\ell)} \;=\; (\ldots, 0, \ldots, 0, 1, \lambda, \lambda^2, \lambda^3, \ldots, \lambda^\ell, 0, 0, \ldots)$$

Obviously it doesn't matter where the non-zero entries begin. From

$$(L - \lambda) v^{(\ell)} \;=\; (\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \lambda^{\ell+1}, 0, 0, \ldots)$$

$|(L - \lambda) v^{(\ell)}| = \sqrt{1 + 1}$, while $|v^{(\ell)}| = \sqrt{\ell + 1}$. Thus, $|(L - \lambda) v^{(\ell)}| / |v^{(\ell)}| \longrightarrow 0$, and there can be no $(L - \lambda)^{-1}$. Thus, every λ on the unit circle is in $\sigma(R)$.

[1] To make a countable dense subset of C, for $n = 1, 2, \ldots$ cover C by finitely-many disks of radius $1/n$, each meeting C, and in each choose a point of C. The union over $n = 1, 2, \ldots$ of these finite sets is countable and dense in C.

[6.4] Compact multiplication operators on ℓ^2 For a sequence of complex numbers $\lambda_n \to 0$, we claim that the multiplication operator

$$T \;:\; (a_1, a_2, \ldots) \longrightarrow (\lambda_1 \cdot a_1, \lambda_2 \cdot a_2, \ldots)$$

is *compact*. We already showed that it has eigenvalues exactly $\lambda_1, \lambda_2, \ldots$, and spectrum the *closure* of $\{\lambda_j\}$. Thus, the spectrum includes 0, but 0 is an *eigenvalue* only when it appears among the λ_j, which may range from 0 times to infinitely-many times.

To prove that the operator is compact, we prove that the image of the unit ball is pre-compact, by showing that it is *totally bounded*. Given $\varepsilon > 0$, take k such that $|\lambda_i| < \varepsilon$ for $i > k$. The ball in $\mathbb{C}^k$ of radius $\max\{|\lambda_j| : j \le k\}$ is precompact, so has a finite cover by ε-balls, centered at points $v^1, \ldots, v^N$. For $v = (v_1, v_2, \ldots)$ with $|v| \le 1$,

$$Tv \;=\; (\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_k v_k, 0, 0, \ldots) + (0, \ldots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \ldots)$$

With v^j the closest of the $v^1, \ldots, v^N$ to $(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_k v_k, 0, 0, \ldots)$,

$$|Tv - v^j| \;<\; \varepsilon + |(0, \ldots, 0, \lambda_{k+1} v_{k+1}, \lambda_{k+2} v_{k+2}, \ldots)| \;<\; \varepsilon + \varepsilon \cdot |(0, \ldots, 0, v_{k+1}, v_{k+2}, \ldots)| \;\le\; \varepsilon + \varepsilon \cdot |v| \;\le\; 2\varepsilon$$

Thus, the image of the unit ball under T is covered by finitely-many 2ε-balls. ///

[6.5] Multiplication operators on $L^2[a,b]$ For $\varphi \in C^o[a,b]$, we claim that the multiplication operator

$$M_\varphi \;:\; L^2[a,b] \;\longrightarrow\; L^2[a,b]$$

by

$$M_\varphi f(x) \;=\; \varphi(x) \cdot f(x)$$

is *normal*, and has spectrum the image $\varphi[a,b]$ of φ. The eigenvalues are λ such that $\varphi(x) = \lambda$ on a subset of $[a,b]$ of positive measure. The normality is clear, so, beyond eigenvalues, we need only examine continuous spectrum, not residual.

On one hand, if $\varphi(x) = \lambda$ on a set of positive measure, there is an infinite-dimensional sub-space of $L^2[0,1]$ of functions supported there, and all these are eigenvectors. On the other hand, if $f \ne 0$ in $L^2[0,1]$ and $\varphi(x) \cdot f(x) = \lambda \cdot f(x)$, even if f is altered on a set of measure 0, it must be that $\varphi(x) = \lambda$ on a set of positive measure.

To understand the continuous spectrum, for $\varphi(x_o) = \lambda$ make *approximate eigenvectors* by taking L^2 functions f supported on $[x_o - \delta, x_o + \delta]$, where $\delta > 0$ is small enough so that $|\varphi(x) - \varphi(x_o)| < \varepsilon$ for $|x - x_o| < \delta$. Then

$$|(M_\varphi - \lambda) f|^2_{L^2} \;=\; \int |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 \, dx \;\le\; \varepsilon^2 \cdot |f|^2_{L^2}$$

Thus, $\inf_{f \ne 0} |(M_\varphi - \lambda) f|_{L^2} / |f|_{L^2} = 0$, so $M_\varphi - \lambda$ is not invertible. If λ is not an eigenvalue, it is in the continuous spectrum. On the other hand, if $\varphi(x) \ne \lambda$, then there is some $\delta > 0$ such that $|\varphi(x) - \lambda| \ge \delta$ for all $x \in [0,1]$, by the compactness of $[0,1]$. Then

$$|(M_\varphi - \lambda) f|^2_{L^2} \;=\; \int_0^1 |\varphi(x) - \lambda|^2 \cdot |f(x)|^2 \, dx \;\ge\; \int_0^1 \delta^2 \cdot |f(x)|^2 \, dx \;=\; \delta^2 \cdot |f|^2_{L^2}$$

Thus, there is a continuous inverse $(M_\varphi - \lambda)$, and λ is *not* in the spectrum.

7. *Cautionary examples: non-normal operators*

[7.1] Shift operators on one-sided ℓ^2 We claim the following: The right-shift

$$R : (a_1, a_2, \ldots) \longrightarrow (0, a_1, a_2, \ldots)$$

and the left-shift

$$L : (a_1, a_2, a_3, \ldots) \longrightarrow (a_2, \ldots)$$

are mutual adjoints. These operators are not normal, since $L \circ R = 1_{\ell^2}$ but

$$R \circ L : (a_1, a_2, \ldots) \longrightarrow (0, a_2, \ldots)$$

The eigenvalues of the left-shift L are all complex numbers in the open unit disk in $\mathbb{C}$. In particular, there is a *continuum* of eigenvalues and eigenvectors, so they *cannot be mutually orthogonal*. The spectrum $\sigma(L)$ is the closed unit disk.

The right-shift R has *no* eigenvalues, has continuous spectrum the unit circle, and residual spectrum the open unit disk with 0 removed.

Indeed, suppose

$$(0, a_1, a_2, \ldots) = R(a_1, a_2, \ldots) = \lambda \cdot (a_1, a_2, \ldots)$$

With n the lowest index such that $a_n \neq 0$, the n^{th} component in the eigenvector relation gives $0 = a_{n-1} = \lambda \cdot a_n$, so $\lambda = 0$. Then, the $(n+1)^{th}$ component gives $a_n = \lambda \cdot a_{n+1} = 0$, contradiction. This proves that R has *no* eigenvalues.

Oppositely, for $|\lambda| < 1$,

$$L(1, \lambda, \lambda^2, \ldots) = (\lambda, \lambda^2, \ldots) = \lambda \cdot (1, \lambda, \lambda^2, \ldots)$$

so every such λ is an eigenvector for L. On the other hand, for $|\lambda| = 1$, in an eigenvector relation

$$(a_2, \ldots) = L(a_1, a_2, \ldots) = \lambda \cdot (a_1, a_2, \ldots)$$

let n be the smallest index n with $a_n \neq 0$. Then $a_{n+1} = \lambda \cdot a_n$, $a_{n+2} = \lambda \cdot a_{n+1}$, ..., so

$$(a_1, a_2, \ldots) = (0, \ldots, 0, a_n, \lambda a_n, \lambda^2 a_n, \ldots)$$

But this is not in ℓ^2 for $|\lambda| = 1$ and $a_n \neq 0$, so λ on the unit circle is *not* an eigenvalue.

For $|\lambda| = 1$, we can make *approximate λ-eigenvectors* for L by

$$v^{[N]} = (1, \lambda, \lambda^2, \ldots, \lambda^N, 0, 0, \ldots)$$

since

$$(L - \lambda)\, v^{[N]} = (\lambda, \lambda^2, \ldots, \lambda^N, 0, 0, 0, \ldots) - \lambda \cdot (1, \lambda, \lambda^2, \ldots, \lambda^N, 0, 0, \ldots) = (0, 0, \ldots, 0, 0, \lambda^{N+1}, 0, 0, \ldots)$$

Since

$$\frac{|(L-\lambda)v^{[N]}|}{|v^{[N]}|} = \frac{|\lambda|^{N+1}}{(1 + |\lambda|^2 + \ldots + |\lambda|^{2N})^{1/2}} = \frac{1}{\sqrt{N+1}} \longrightarrow 0$$

there can be no continuous $(L - \lambda)^{-1}$. Thus, λ on the unit circle is in the spectrum, but not in the point spectrum.

That the unit circle is in the spectrum also follows from the observation above that all λ with $|\lambda| < 1$ are eigenvalues, and the fact that the spectrum is *closed*.

The spectrum of L is bounded by the operator norm $|L|_{\rm op}$, and $|L|_{\rm op}$ is visibly 1, so is nothing *else* in the spectrum.

To see that the unit circle is the *continuous* spectrum of L, as opposed to *residual*, we show that $L - \lambda$ has dense image for $|\lambda| = 1$. Indeed, for w such that, for all $v \in \ell^2$,

$$0 \;=\; \langle (L-\lambda)v, w\rangle \;=\; \langle v, (L^* - \overline{\lambda})w\rangle \;=\; \langle v, (R-\overline{\lambda})w\rangle$$

we would have $(R - \overline{\lambda})w = 0$. However, we have seen that R has no eigenvalues. Thus, $L - \lambda$ always has dense image, and the unit circle is continuous spectrum for L.

Reversing that discussion, every λ with $|\lambda| < 1$ is in the residual spectrum of R, because such λ is not an eigenvalue, and $R - \lambda$ does *not* have dense image: for w a $\overline{\lambda}$-eigenvector for L,

$$\langle (R-\lambda)v, w\rangle \;=\; \langle v, (R^* - \overline{\lambda})w\rangle \;=\; \langle v, (L-\overline{\lambda})w\rangle \;=\; \langle v, 0\rangle \;=\; 0$$

That is, the image $(R - \lambda)\ell^2$ is in the orthogonal complement to the eigenvector w. The same computation shows that the unit circle is *continuous* spectrum for R, because it is *not* eigenvalues for L.

[7.2] Volterra operator We will show that the Volterra operator $Vf(x) = \int_0^x f(t)\,dt$ on $L^2[0,1]$ is not self-adjoint, that its spectrum is $\{0\}$, and that it has no eigenvalues.

As in the next chapter, since the Volterra operator is given by an L^2 integral kernel, it is *Hilbert-Schmidt*, hence *compact*.

A relation $Tf = \lambda \cdot f$ for $f \in L^2$ and $\lambda \neq 0$ implies f is *continuous*:

$$|\lambda| \cdot |f(x+h) - f(x)| \;=\; |Tf(x+h) - Tf(x)| \;\leq\; \int_x^{x+h} 1 \cdot |f(t)|\,dt \;=\; \leq\; |h|^{\frac{1}{2}} \cdot |f|_{L^2}$$

The fundamental theorem of calculus would imply f is continuously differentiable and $\lambda \cdot f' = (Tf)' = f$. Thus, f would be a constant multiple of $e^{x/\lambda}$, by the mean value theorem. However, by Cauchy-Schwarz-Bunyakowsky, for a λ-eigenfunction

$$|\lambda| \cdot |f(x)| \;\leq\; |x|^{\frac{1}{2}} \cdot |f|_{L^2}$$

No non-zero multiple of the exponential satisfies this. Thus, there are no eigenvectors for *non-zero* eigenvalues.

For $f \in L^2[0,1]$ and $Tf = 0 \in L^2[0,1]$, Tf is almost everywhere 0. Since $x \to Tg(x)$ is unavoidably *continuous*, $Tf(x)$ is 0 for all x. Thus, for all x, y in the interval,

$$0 \;=\; 0 - 0 \;=\; Tf(y) - Tf(x) \;=\; \int_x^y f(t)\,dt$$

That is, $x \to Tf(x)$ is orthogonal in $L^2[0,1]$ to all characteristic functions of intervals. Finite linear combinations of these are dense in $C^o[0,1]$ in the L^2 topology, and $C^o[0,1]$ is dense in $L^2[0,1]$. Thus $f = 0$, and there are no eigenvectors for the Volterra operator.

To see that the spectrum is *at most* $\{0\}$, show that the *spectral radius* is 0:

$$T^n f(x) \;=\; \int_0^x \int_0^{x_{n-1}} \ldots \int_0^{x_2} \int_0^{x_1} f(t)\,dt\,dx_1 \ldots dx_{n-1} \;=\; \int_0^x f(t)\Big(\int_t^x \int_t^{x_{n-1}} \ldots \int_t^{x_2} dx_1 \ldots dx_{n-1}\Big)\,dt$$

$$=\; \int_0^x f(t) \cdot \frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

From this, $|T^n|_{\mathrm{op}} \le \frac{1}{n!}$, and

$$\log \lim_{2n} \Big(\frac{1}{(2n)!}\Big)^{1/2n} \;=\; -\lim_{2n} \frac{1}{2n}\cdot \log(2n)! \;=\; -\lim_{2n} \frac{1}{2n} \sum_{1\le k\le 2n} \log k$$

$$=\; -\lim_{2n} \frac{1}{2n} \sum_{1\le k\le n} (\log k + \log(2n-k+1)) \;\le\; -\lim_{2n} \frac{1}{2n} \sum_{1\le k\le \frac{n}{k}} (\log k + \log(2n-k+1))$$

$$\le\; -\lim_{2n} \frac{1}{2n} \sum_{1\le k\le \frac{n}{k}} \log 2n \;=\; -\lim_{2n} \frac{\log 2n}{2} \;=\; -\infty$$

since $k(2n-k)\ge 2n$ for $1\le k\le n$, noting the sign. That is, $\lim_n |T^n|_{\mathrm{op}}^{1/n} = 0$, so the spectral radius is 0. Since the spectrum is non-empty, it must be exactly $\{0\}$.

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8. *Weyl's criterion for continuous spectrum*

H. Weyl gave a criterion for *continuous* spectrum analogous to the definition of *discrete* spectrum. This criterion is decisive for *normal* operators.

For $\lambda\in\mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) in the Hilbert space V such that $(T-\lambda)v_n \to 0$ as $n\to+\infty$ is an *approximate eigenvector* for λ.

[8.1] Theorem: For λ *not* an eigenvalue for T, and for $(T-\lambda)V$ *not closed*, λ is in the spectrum of T if and only if λ has an approximate eigenvector.

[8.2] Remark: This criterion is *not* uniformly reliable for detecting *residual* spectrum, which is why we must impose a further condition. [2] For example, we have seen that, for $T: V\to V$ a *norma* linear operator, for λ in the spectrum but not an eigenvalue, $(T-\lambda)V$ is *dense* in V but is not all of V. Thus, the hypothesis of the theorem is met for normal T. We give an example of failure to detect residual spectrum after the proof.

Proof: Certainly if λ is an eigenvector, with non-zero eigenvalue v, the constant sequence $v,v,v,\ldots$ fits the requirement.

For general spectrum, let $S=T-\lambda$. For $v_1,v_2,\ldots$ with $|v_n|=1$ and $Sv_n\to 0$, any alleged (continuous [3]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 \;=\; S^{-1}(\lim_n Sv_n) \;=\; \lim_n S^{-1}Sv_n \;=\; \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T-\lambda$ implies that $T-\lambda$ is not invertible.

Conversely, for $S=T-\lambda$ not invertible, but λ not an eigenvector, then S is *injective* but not *surjective*. We further assume that the image of S is *not closed*. [4] In that case, S is injective, not surjective, and

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[2] Recall that *residual* spectrum of T is λ such that $T-\lambda$ is *injective*, but does *not* have dense image.

[3] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S, necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta>0$ such that $|Sv|\ge\delta\cdot|v|$ for all v. This exactly asserts the boundedness of S^{-1}, so S^{-1} is *continuous*.

[4] The image is not closed, for example, when T (hence S) has no residual spectrum, which is the case when T (hence S) is *normal*, or *self-adjoint*.

by non-closedness of the image there is v_o (with $|v_o| = 1$) not in the image of S, and $v_1, v_2, \ldots$ such that $Sv_1, Sv_2, \ldots \to v_o$. If $\{v_n\}$ were a Cauchy sequence, then it would have a limit, and by continuity of S

$$v_o \;=\; \lim_n Sv_n \;=\; S(\lim_n v_n)$$

and v_o would be in the image of S, contradicting our assumption. Thus, $\{v_n\}$ is *not* Cauchy. In particular, we can replace $\{v_n\}$ by a subsequence such that there is $\delta > 0$ such that $|v_m - v_n| \geq \delta$ for all $m \neq n$. Then $w_n = v_n - v_{n+1}$ forms an approximate 0-eigenvector, since their lengths are bounded away from 0, and

$$Sw_n \;=\; S(v_n - v_{n+1}) \;=\; Sv_n - Sv_{n+1} \;\longrightarrow\; v_o - v_o \;=\; 0$$

as desired. ///

[8.3] Remark: As noted, the case that λ is not an eigenvector, $T - \lambda$ is not surjective, and/but the image of $S = T - \lambda$ is *closed*, can only occur for non-normal T. For example, $T : \ell^2 \to \ell^2$ by

$$T(c_1, c_2, \ldots) \;=\; (c_1, 0, c_2, 0, c_3, 0, \ldots)$$

is injective, not surjective, and has closed image. It is not invertible, but there is no approximate eigenvector for 0, so the criterion fails in this (non-normal) example.
