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01. *Metrics and topologies on vector spaces*

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1. *Euclidean spaces*

Let $\mathbb{R}^n$ be the usual Euclidean n-space, that is, ordered n-tuples $x = (x_1, \ldots, x_n)$ of real numbers. In addition to vector addition (termwise) and scalar multiplication, we have the usual *distance function* on $\mathbb{R}^n$, in coordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, defined by

$$d(x, y) \;=\; \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

Of course there is visible *symmetry* $d(x, y) = d(y, x)$, and *positivity*: $d(x, y) = 0$ only for $x = y$. The *triangle inequality*

$$d(x, z) \;\le\; d(x, y) + d(y, z)$$

is not trivial to prove. In the one-dimensional case, the triangle inequality is an inequality on absolute values, and can be proven case-by-case. In $\mathbb{R}^n$, it is best to use the following set-up. The usual *inner product* (or *dot-product*) on $\mathbb{R}^n$ is

$$x \cdot y \;=\; \langle x, y\rangle \;=\; \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle \;=\; x_1 y_1 + \ldots + x_n y_n$$

and the *length* or *norm* of x is $|x| = \langle x, x\rangle^{\frac{1}{2}}$. *Context* distinguishes the norm $|x|$ of $x \in \mathbb{R}^n$ from the usual absolute value $|c|$ on real or complex numbers c. Distance from x to y is

$$d(x, y) \;=\; |x - y|$$

The inner product $\langle x, y\rangle$ is *linear* in both arguments: in the first argument

$$\langle x + x', y\rangle \;=\; \langle x, y\rangle + \langle x', y\rangle \qquad \langle cx, y\rangle \;=\; c \cdot \langle x, y\rangle \qquad (\text{for } x, x', y \in \mathbb{R}^n \text{ and scalar } c)$$

and similarly for the second argument. The triangle inequality will be a corollary of the following universally-useful inequality:

[1.1] Claim: *(Cauchy-Schwarz-Bunyakowsky inequality)* For $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \;\le\; |x| \cdot |y|$$

Assuming that neither x nor y is 0, *strict* inequality holds *unless* x and y are scalar multiples of each other.

Proof: If $|y| = 0$, the assertions are trivially true. Thus, take $y \neq 0$. With real t, consider the quadratic polynomial function

$$f(t) \;=\; |x - ty|^2 \;=\; |x|^2 - 2t\langle x, y\rangle + t^2|y|^2$$

Certainly $f(t) \ge 0$ for all $t \in \mathbb{R}$, since $|x - ty| \ge 0$. Its minimum occurs where $f'(t) = 0$, namely, where $-2\langle x, y\rangle + 2t|y|^2 = 0$. This is where $t = \langle x, y\rangle/|y|^2$. Thus,

$$0 \;\le\; (\text{minimum}) \;\le\; f(\langle x, y\rangle/|y|^2) \;=\; |x|^2 - 2\frac{\langle x, y\rangle}{|y|^2}\langle x, y\rangle + \Big(\frac{\langle x, y\rangle}{|y|^2}\Big)^2 \cdot |y|^2 \;=\; |x|^2 - \Big(\frac{\langle x, y\rangle}{|y|^2}\Big)^2 \cdot |y|^2$$

Multiplying out by $|y|^2$,

$$0 \;\le\; |x|^2 \cdot |y|^2 - \langle x, y\rangle^2$$

which gives the inequality. Further, for the inequality to be an *equality*, it must be that $|x - ty| = 0$, so x is a multiple of y. ///

[1.2] Remark: We did not use properties of $\mathbb{R}^n$, only of the inner product!

[1.3] Corollary: *(Triangle inequality)* For $x, y, z \in \mathbb{R}^n$,

$$|x + y| \;\le\; |x| + |y|$$

Therefore,

$$d(x, z) \;=\; |x - z| \;=\; |(x - y) - (z - y)| \;\le\; |x - y| + |z - y| \;=\; d(x, y) + d(y, z)$$

Proof: With the Cauchy-Schwarz-Bunyakowsky inequality in hand, this is a direct computation:

$$|x + y|^2 \;=\; \langle x + y, x + y\rangle \;=\; \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \;=\; |x|^2 + 2\langle x, y\rangle + |y|^2 \;\le\; |x|^2 + 2|\langle x, y|\rangle + |y|^2$$

$$\le\; |x|^2 + 2|x| \cdot |y| + |y|^2 \;=\; (|x| + |y|)^2$$

Taking positive square roots gives the result. ///

As usual, the *open ball* of radius $r > 0$ centered at a point y is

$$\text{open ball} \;=\; \{x \in \mathbb{R}^n \,:\, d(x, y) < r\}$$

The *closed ball* of radius $r > 0$ centered at a point y is

$$\text{closed ball} \;=\; \{x \in \mathbb{R}^n \,:\, d(x, y) \le r\}$$

Obviously in many regards the two are barely different from each other. However, the fact that the *closed* ball includes its *boundary* (in both an intuitive an technical sense as below), namely, the sphere

$$S^{n-1} \;=\; \{x \in \mathbb{R}^n \,:\, d(x, y) = r\}$$

while the *open* ball does *not*. A different distinction is what we'll exploit most directly:

[1.4] Corollary: For any point x in an open ball B in $\mathbb{R}^n$, for sufficiently small radius $\varepsilon > 0$ the open ball of radius ε centered at x is contained in B.

Proof: This is essentially the triangle inequality. Let B be the open ball of radius r centered at y. Then $x \in B$ if and only if $|x - y| < r$. Thus, we can take $\varepsilon > 0$ such that $|x - y| + \varepsilon < r$. For $|z - x| < \varepsilon$, by the triangle inequality

$$|z - y| \;\le\; |z - x| + |x - y| \;<\; \varepsilon + |x - y| \;<\; r$$

That is, the open ball of radius ε at x is inside B. ///

An *open set* in $\mathbb{R}^n$ is any set with the property observed in the latter corollary, namely a set U in $\mathbb{R}^n$ is *open* if for every x in U there is an open ball centered at x contained in U.

This definition allows us to rewrite the epsilon-delta definition of *continuity* in a useful form:

[1.5] Claim: A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is *continuous* if and only if the *inverse image*

$$f^{-1}(U) \;=\; \{x \in \mathbb{R}^m \;:\; f(x) \in U\}$$

of every *open* set U in $\mathbb{R}^n$ is *open* in $\mathbb{R}^m$. *(We prove this below for general metric spaces.)* ///

Some properties of open sets in $\mathbb{R}^n$ that will be abstracted:

[1.6] Claim: The union of an *arbitrary* set of open subsets of $\mathbb{R}^n$ is open. The intersection of a *finite* set of open subsets of $\mathbb{R}^n$ is open.

Proof: A point $x \in \mathbb{R}^n$ is in the union U of an arbitrary set $\{U_\alpha : \alpha \in A\}$ of open subsets of $\mathbb{R}^n$ exactly when there is some U_α so that $x \in U_\alpha$. Then a small-enough open ball B centered at x is inside U_α, so $B \subset U_\alpha \subset U$.

For x in the intersection $I = U_1 \cap \ldots \cap U_m$ of a finite number of opens, let $\varepsilon_j > 0$ such that the open ε_j-ball at x is contained in U_j. Let ε be the minimum of the ε_j. The minimum of a *finite* set of (strictly) positive real numbers is still (strictly) positive, so $\varepsilon > 0$, and the ε-ball at x is contained inside every ε_j-ball at x, so is contained in the intersection. ///

One of many equivalent ways to say that a set E in $\mathbb{R}^n$ is *bounded* is that it is contained in some (sufficiently large) *ball*. [1] At various technical points in advanced calculus, we find ourselves caring about *closed and bounded* sets, and perhaps proving the *Heine-Borel property* or *Bolzano-Weierstraß property*[2]

[1.7] Theorem: A set E in $\mathbb{R}^n$ is closed and bounded *if and only if* every sequence of points in E has a *convergent subsequence*. ///

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2. *Metric spaces*

By design, the previous discussion of Euclidean spaces made minimal use of particular features of Euclidean space. This allows *abstraction* in a manner using intuition about finite-dimensional Euclidean spaces to suggest things about less familiar spaces. The process of abstraction has several stopping places, and this section looks at one of the first.

[1] A few moments' thought show that it does not matter where the ball is centered, nor whether the ball is closed or open.

[2] This property is not at all trivial to prove, especially from an elementary viewpoint.

We can abstract the *distance function* on $\mathbb{R}^n$ usefully, as follows. For a set X be a set, a non-negative-real-valued function

$$d : X \times X \longrightarrow \mathbb{R}$$

is a *distance function* if it satisfies the conditions

$$\begin{cases} d(x,y) \geq 0 & \text{(with equality only for } x = y) \quad \textit{(positivity)} \\ d(x,y) = d(y,x) & \textit{(symmetry)} \\ d(x,z) \leq d(x,y) + d(y,z) & \textit{(triangle inequality)} \end{cases}$$

for all points $x, y, z \in X$. Such a distance function is also called a *metric*. The set X with the metric d is a *metric space*.

In analogy with the situation for $\mathbb{R}$ and $\mathbb{R}^n$, a sequence $\{x_n\}$ in a metric space X is *convergent* to $x \in X$ when, for every $\varepsilon > 0$, there is n_o such that, for all $n \geq n_o$, $|x_n - x| < \varepsilon$. Likewise, a sequence $\{x_n\}$ in X is a *Cauchy* sequence when, for all $\varepsilon > 0$, there is n_o such that for all $m, n \geq n_o$, $|x_m - x_n| < \varepsilon$. A metric space is *complete* if every Cauchy sequence is convergent.

The following standard lemma makes a bit of intuition explicit:

[2.1] Lemma: Let $\{x_i\}$ be a Cauchy sequence in a metric space X, d converging to x in X. Given $\varepsilon > 0$, let N be sufficiently large such $d(x_i, x_j) < \varepsilon$ for $i, j \geq N$. Then $d(x_i, x) \leq \varepsilon$ for $i \geq N$.

Proof: Let $\delta > 0$ and take $j \geq N$ also large enough such that $d(x_j, x) < \delta$. Then for $i \geq N$ by the triangle inequality

$$d(x_i, x) \;\leq\; d(x_i, x_j) + d(x_j, x) \;<\; \varepsilon + \delta$$

Since this holds for every $\delta > 0$ we have the result. ///

[2.2] Example: Variants of the usual Euclidean metric on $\mathbb{R}^n$ also make sense:

$$d_1(x,y) \;=\; |x_1 - y_1| + \ldots + |x_n - y_n| \qquad\qquad d_\infty(x,y) \;=\; \max_i |x_i - y_i|$$

In fact, the triangle inequality for these metrics are easy to prove, needing just the triangle inequality for the absolute value on $\mathbb{R}$. Later, we will see [3] that

$$d_p(x,y) \;=\; \Big(|x_1 - y_1|^p + \ldots + |x_n - y_n|^p\Big)^{1/p} \qquad (\text{for } 1 \leq p < \infty)$$

also gives a metric.

[2.3] Example: A *discrete set* or *discrete metric space* X is one in which (roughly) no two distinct points are close to each other. That is, for each $x \in X$ there should be a bound $\delta_x > 0$ such that $d(x,y) \geq \delta_x$ for all $y \neq x$ in X. For example, the set $\mathbb{Z}$ of integers, with the natural distance

$$d(x,y) \;=\; |x - y| \qquad (\text{with usual absolute value})$$

has the property that $|x - y| \geq 1$ for distinct integers. Every discrete metric space is complete.

[2.4] Example: *Any* set X can be made into a *discrete* metric space by defining

$$d(x,y) \;=\; \begin{cases} 1 & (\text{for } x \neq y) \\ 0 & (\text{for } x = y) \end{cases}$$

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[3] The triangle inequality for such metrics is an instance of the *Hölder inequality*.

This is obviously positive and symmetric, and satisfies the triangle inequality condition for silly reasons. Little is learned from this example except that it is possible to do such things.

[2.5] Example: The collection $C^o[a,b]$ of continuous functions [4]) on an interval $[a,b]$ on the real line can be given the metric

$$d(f,g) \;=\; \sup_{x\in[a,b]} |f(x)-g(x)|$$

Positivity and symmetry are easy, and the triangle inequality is not hard, either. This metric space is *complete*, because a Cauchy sequence is a *uniformly pointwise convergent* sequence of continuous functions.

[2.6] Example: The collection $C^o(\mathbb{R})$ of continuous functions [5] on the *whole* real line does *not* have an obvious candidate for a metric, since the *sup* metric of the previous example may give infinite values. Yet there is the metric

$$d(f,g) \;=\; \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{|x|\le n} |f(x)-g(x)|}{1+\sup_{|x|\le n} |f(x)-g(x)|}$$

This metric space is complete, for similar reasons as $C^o[a,b]$.

[2.7] Example: A sort of infinite-dimensional analogue of the standard metric on $\mathbb{R}^n$ is the space ℓ^2, the collection of all sequences $\alpha=(\alpha_1,\alpha_2,\ldots)$ of complex numbers such that $\sum_{n\ge1}|\alpha_n|^2<+\infty$. The metric is

$$d(\alpha,\beta) \;=\; \sqrt{\sum_{n\ge1}|\alpha_n-\beta_n|^2}$$

In fact, ℓ^2 is a vector space, being closed under addition and under scalar multiplication, with *inner product*

$$\langle\alpha,\beta\rangle \;=\; \sum_{n\ge1}\alpha_n\cdot\overline{\beta}_n$$

Cauchy-Schwarz-Bunyakowsky shows that the latter sum converges absolutely. The associated *norm* is $|\alpha|=\langle\alpha,\alpha\rangle^{\frac{1}{2}}$, and $d(\alpha,\beta)=|\alpha-\beta|$. The Cauchy-Schwarz-Bunyakowsky holds for ℓ^2, by the same proof as given earlier, and proves the triangle inequality. This metric space is *complete*, as we show a little later.

[2.8] Example: For $1\le p<\infty$, the sequence space ℓ^p is

$$\ell^p \;=\; \{x=(x_1,x_2,\ldots)\;:\;\sum_{i=1}^{\infty}|x_i|^p<\infty\}$$

with metric

$$d_p(x,y) \;=\; \Big(\sum_{i=1}^{\infty}|x_i-y_i|^p\Big)^{1/p}$$

Proof of the triangle inequality needs *Hölder's inequality*, and we will take care of this later. These metric spaces are complete, as we see later. Unlike the case of varying metrics on $\mathbb{R}^n$, the underlying sets ℓ^p are not the same, and the topologies are all different. For example, ℓ^2 is strictly larger than ℓ^1.

[2.9] Example: Even before having a modern notion of measure and integral, an analogue of ℓ^2 can be formulated: on $C^o[a,b]$, form an inner product

$$\langle f,g\rangle \;=\; \int_a^b f(x)\,\overline{g(x)}\,dx$$

[4] Throughout discussion of these examples, it doesn't matter much whether we think of real-valued functions or complex-valued functions.

[5] Real-valued or complex-valued, for example.

It is easy to check that this does give a hermitian inner product. The L^2 norm is $|f|_{L^2} = \langle f,f\rangle^{\frac{1}{2}}$, and the distance function is $d(f,g) = |f-g|$. The basic properties of a metric are immediate, except that the triangle inequality needs the integral form of the Cauchy-Schwarz-Bunyakowsky inequality, whose proof is the same as that given earlier. This metric space is *not* complete, because there are sequences of continuous functions that are Cauchy in this L^2 metric (but not in the $C^o[a,b]$ metric) and do not converge to a continuous function. For example, we can make piecewise-linear continuous functions approaching the discontinuous function that is 0 on $[a, \frac{a+b}{2}]$ and 1 on $[\frac{a+b}{2}, b]$, by

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } a \le x \le \frac{a+b}{2} - \frac{1}{n}) \\ \frac{n}{2}\cdot(x - (\frac{a+b}{2} - \frac{1}{n})) & (\text{for } \frac{a+b}{2} - \frac{1}{n} \le x \le \frac{a+b}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{a+b}{2} + \frac{1}{n} \le x \le b) \end{cases}$$

(Draw a picture.) The pointwise limit is 0 to the left of the midpoint, and 1 to the right. Despite the fact that the pointwise limit does not exist at the midpoint,

$$d_2(f_i,f_j)^2 \;\le\; \int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} 1\,dx \;\le\; \frac{2}{n} \qquad\qquad (\text{for } i,j \ge n)$$

which goes to 0 as $n \to \infty$. That is, $\{f_n\}$ is Cauchy in the L^2 metric, but does not converge to a continuous function.

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3. *Topologies of metric spaces*

The notion of metric space allows a useful generalization of the notion of *continuous function* via the obvious analogue of the epsilon-delta definition:

A function or map $f: X \to Y$ from one metric space (X, d_X) to another metric space (Y, d_Y) is *continuous* at a point $x_o \in X$ when, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x,x_o) < \delta \quad\Longrightarrow\quad d_Y(f(x), f(x_o)) < \varepsilon$$

In a metric space (X,d), the *open ball* of radius $r > 0$ centered at a point y is

$$\{x \in X \,:\, d(x,y) < r\}$$

The *closed ball* of radius $r > 0$ centered at a point y is

$$\{x \in X \,:\, d(x,y) \le r\}$$

As in $\mathbb{R}^n$, in many regards the two are barely different from each other. However, the *closed* ball includes the *sphere*

$$\{x \in X \,:\, d(x,y) = r\}$$

while the *open* ball does *not*. A different distinction is what we'll exploit most directly:

[3.1] Claim: For any point x in an open ball B in X, for sufficiently small radius $\varepsilon > 0$ the open ball of radius ε centered at x is contained in B. *(As for $\mathbb{R}^n$, this follows immediately by use of the triangle inequality.* ///

An *open set* in X is any set with the property observed in this proposition. That is, a set U in X is *open* if for every x in U there is an open ball centered at x contained in U.

This definition allows us to rewrite the epsilon-delta definition of *continuity* in a form that will apply in more general topological spaces:

[3.2] Claim: A function $f : X \to Y$ from one metric space to another is *continuous* in the ε-δ sense if and only if the inverse image

$$f^{-1}(U) \;=\; \{x \in \mathbb{R}^m \,:\, f(x) \in U\}$$

of every *open* set U in Y is *open* in X.

Proof: On one hand, suppose f is continuous in the ε-δ sense. For U open in Y and $x \in f^{-1}(U)$, with $f(x) = y$, let $\varepsilon > 0$ be small enough so that the ε-ball at y is inside U. Take $\delta > 0$ small enough so that, by the ε-δ definition of continuity, the δ-ball B at x has image $f(B)$ inside the ε-ball at y. Then $x \in B \subset f^{-1}(U)$. This holds for every $x \in f^{-1}(U)$, so $f^{-1}(U)$ is open.

On the other hand, suppose $f^{-1}(U)$ is open for every open $U \subset Y$. Given $x \in X$ and $\varepsilon > 0$, let U be the ε-ball at $f(x)$. Since $f^{-1}(U)$ is open, there is an open ball B at x contained in $f^{-1}(U)$. Let $\delta > 0$ be the radius of B. ///

A set E in a metric space X is *closed* if and only its *complement*

$$E^c \;=\; X - E \;=\; \{x \in X \,:\, x \notin E\}$$

is *open*.

A set E in a metric space X is *bounded* when it is contained in some (sufficiently large) *ball*. This makes sense in general metric spaces, but does not have the same implications.

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4. *Vector spaces with inner products*

Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). An *inner product* or *scalar product* or *dot product* on V is a $\mathbb{C}$-valued function $\langle , \rangle : V \times V \to \mathbb{C}$ which is *linear* in its first argument:

$$\langle ax + bx', y\rangle \;=\; a\langle x, y\rangle + b\langle x', y\rangle \qquad \text{(for scalars } a, b \text{ and } x, x', y \in V)$$

hermitian in the sense that

$$\langle y, x\rangle \;=\; \overline{\langle x, y\rangle}$$

and *positive* in the sense that $\langle x, x\rangle \geq 0$, with equality only for $x = 0$. This entails that the inner product is *conjugate-linear* in its second argument:

$$\langle x, ay + by'\rangle \;=\; \overline{a}\langle x, y\rangle + \overline{b}\langle x, y'\rangle \qquad \text{(for scalars } a, b \text{ and } x, y, y' \in V)$$

We say that V is an *inner-product space* or *pre-Hilbert space.*

An inner product gives rise to a *norm* $|x| = \langle x, x\rangle^{\frac{1}{2}}$, and a *metric* by $d(x, y) = |x - y|$. The norm is *homogeneous* in the sense that

$$|cx| \;=\; \langle cx, cx\rangle^{\frac{1}{2}} \;=\; \Big(c\overline{c} \cdot \langle x, x\rangle\Big) \;=\; |c|_{\mathbb{C}} \cdot |x|_v$$

and *positive* in the sense that $|x| \geq 0$, with $|x| = 0$ only for $x = 0$.

The proof of Cauchy-Schwarz-Bunyakowsky given earlier in the context of $\mathbb{R}^n$ applies in arbitrary inner-product spaces (with minor technical accommodation of complex scalars):

$$|\langle x, y\rangle| \; le \; |x| \cdot |y| \qquad \text{(for all } x, y \in V)$$

with equality only for x, y collinear. Then the *triangle inequality* for this norm follows, as well:

$$|x+y|^2 \;=\; |x|^2 + \langle x,y\rangle + \langle y,x\rangle + |y|^2 \;=\; |x|^2 + 2\mathrm{Re}\,\langle x,y\rangle + |y|^2 \;\le\; |x|^2 + 2|\langle x,y\rangle| + |y|^2$$

$$\le\; |x|^2 + 2|x||y| + |y|^2 \;=\; (|x|+|y|)^2$$

When V is *complete* with respect to the metric, V is called a *Hilbert space*. Finite-dimensional inner-product spaces are *always* complete. The simplest infinite-dimensional Hilbert space is ℓ^2. The completeness requires proof.

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5. *Vector spaces with norms*

Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A *norm* on V is an $\mathbb{R}$-valued function $|\cdot| = |\cdot|_V : V \to \mathbb{R}$ with *homogeneity*

$$|cx| \;=\; |c|_{\mathbb{C}} \cdot |x|_V \qquad\qquad (\text{for } c \in \mathbb{C} \text{ and } x \in V)$$

satisfying the *triangle inequality*

$$|x+y| \;\le\; |x| + |y|$$

and and *positive* in the sense that $|x| \ge 0$, with equality only for $x = 0$. We may say that V is a *normed space* or *pre-Banach space.*

When V is *complete* with respect to this metric, it is called a *Banach space.* Finite-dimensional normed spaces are *always* complete. A simple infinite-dimensional Banach space is $C^o[a,b]$. The completeness requires proof.

The normed spaces ℓ^p with $1 \le p < \infty$ are complete, so are Banach spaces. Proving that the triangle inequality is met requires proof, as does the completeness.

Most norms do *not* arise from inner products. If $|x| = \langle x,x\rangle^{\frac{1}{2}}$ *does* arise from an inner product $\langle,\rangle$, then the inner product can be recovered from the norm via the *polarization/parallelogram* identity: for simplicity, suppose the scalars are $\mathbb{R}$ rather than $\mathbb{C}$, to dodge complex conjugation for a moment:

$$|x+y|^2 - |x-y|^2 \;=\; \langle x+y, x+y\rangle - \langle x-y, x-y\rangle$$

$$=\; \Big(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\Big) - \Big(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle\Big) \;=\; 4\langle x,y\rangle$$

For complex scalars, we use $x\backslash pmy$ and $x \pm iy$, in a slightly more complicated identity, to recover the inner product from the norm.

In particular, if the function $|x+y|^2 - |x-y|^2$ for a given norm *does not* have the properties of an inner product, then that norm cannot arise from any inner product.

Metrics on vector spaces arising from norms are *translation-invariant*:

$$d(x+z, y+z) \;=\; |(x+z)-(y+z)| \;=\; |x-y| \;=\; d(x,y)$$

and *homogeneous*:

$$d(cx, cy) \;=\; |cx - cy| \;=\; |c|_{\mathbb{C}} \cdot |x-y|_V \;=\; |c| \cdot d(x,y)$$

6. *Product topologies and metrics*

The *product topology* on the Cartesian product $\prod_{\alpha\in A} X_\alpha$ for topological spaces X_α is usually described as having sub-basis of sets $\prod_{\alpha\in A} U_\alpha$ where for all indices except a single one α_o, we have $U_\alpha = X_\alpha$, and for that single α_o the set U_{α_o} is any open set in X_{α_o}.

For infinite index sets A, this topology is coarser than may be expected. In particular, for infinite A, the product topology is usually *not* the *box topology* on $\prod_\alpha X_\alpha$, which has basis of sets $\prod_\alpha U_\alpha$ where each U_α is an open in X_α.

An appendix explains the coarseness of the product topology, and, in particular, why it is correct, and why it is uniquely determined.

[6.1] Theorem: *Countable* products of *metric* spaces are metric spaces. If all the factors are complete, then the product is complete.

Proof: Let $X = \prod_n X_n$, where X_n has metric d_n. On X consider the function

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) \;=\; \sum_n 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \qquad \text{(for } x_n, y_n \in X_n \text{ for } n = 1, 2, \ldots)$$

One trick here is that $\frac{t}{1+t} \le 1$ for all non-negative reals t, so the sum converges. In fact, there are many other similar expressions that achieve the same efffect, and give the same topology.

There is some work to be done to prove that this expression gives a metric, and that the topology attached to that metric is the product topology... ///

7. *Topological vector spaces*

There are some important, translation-invariant metrics on vector spaces which do *not* come from *norms* by the recipe $d(x, y) = |x - y|$. In fact, noting that many different metrics can give the same topology, there are some *topologies* on vector spaces which cannot come from norms.

More precisely, a *topological vector space* V is a (probably real or complex) vector space with a topology, such that scalar multiplication and vector addition are continuous, and that points are closed sets. (The latter condition excludes some pathologies, and in fact implies that the topology is *Hausdorff*.) Inner product spaces and normed vector spaces all fit into this category.

[7.1] Claim: A topological vector space topology on V is *translation-invariant* in the sense that for open $U \subset V$, and for $v \in V$ the translate $U + v$ is open.

Proof: Fix $v \in V$. The addition and subtraction maps A_v, S_v defined by $A_v(w) = w + v$ and $S_v(w) = w - v$ are continuous, by assumption, and are mutual inverses. Thus, they are both homeomorphisms. Thus, $A_v(U) = U + v$ is open. ///

[7.2] Corollary: To describe the topology on a topological vector space, it suffices to give a *local basis* at 0. ///

The fundamental example of a vector space whose natural topology *can* be given by a complete, translation-invariant metric, but whose topology *cannot* arise from a norm, is

$$C^\infty[a, b] \;=\; \bigcap_{k\ge 0} C^k[a, b]$$

where each space $C^k[a,b]$ of k-fold continuously differentiable functions *does* have (complete, metric) topology given by the norm

$$|f|_{C^k} \;=\; \sum_{i=0}^{k} |f^{(i)}|_{C^o}$$

The completeness of each $C^k[a,b]$ does require some proof, and for $k > 0$ seems to need the fundamental theorem of calculus. The usual metric on $C^\infty[a,b]$ is

$$d(f,g) \;=\; \sum_{n\geq 0} 2^{-n} \frac{|f^{(n)} - g^{(n)}|_{C^o}}{1 + |f^{(n)} - g^{(n)}|_{C^o}}$$

The completeness requires some work, similar to proving that a countable product of complete metric spaces is complete. Thus, $C^\infty[a,b]$ is a *Fréchet space.* [6] Much as there is an argument for the correctness of the product topology, there is an argument for the correctness of this topology on $C^\infty[a,b]$. The metric itself is non-unique, but the correct topology is unique. The following example shows that the class of Banach spaces is not broad enough.

[7.3] Claim: There is no norm on $C^\infty[a,b]$ giving the natural (complete metric) topology above.

[Proof later.] Basically, the point is that sup-norm bounds on the first k derivatives do not give any bound on the $(k+1)^{th}$ derivative. For example, $\sin(nx)/n^{k+1}$.

[7.4] Remark: Also, as remarked earlier, the correct topology on the space $C^\infty_c(\mathbb{R})$ of *test functions* cannot be given by *any* metric. Likewise *weak dual topologies* are rarely given by metrics.

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8. *Completions of metric spaces*

Again, a metric space X, d is *complete* when every Cauchy sequence is convergent. [7] Completeness is a convenient, desirable feature, allowing (sequential) limits without leaving the space. As in the example of $C^o[a,b]$ with the L^2 metric, we might want to imbed a non-complete metric space in a complete one in an optimal and universal way.

A traditional notion of the *completion* of a metric space X is a *construction* of a complete metric space $\widetilde{X}$ with a distance-preserving injection $j : X \to \widetilde{X}$ so that $j(X)$ is *dense* in $\widetilde{X}$, in the sense that every point of $\widetilde{X}$ is the limit of a Cauchy sequence in $j(X)$.

The intention is that every Cauchy sequence in X has a limit in $\widetilde{X}$, so we should (somehow) *adjoin* points as needed for these limits. However, different Cauchy sequences may happen to have the same limit.

Thus, we want an equivalence relation on Cauchy sequences that says they should have the same limit, even without knowing the limit exists or having somehow constructed or adjoined the limit point.

A more *external* characterization of a completion $j : X \to \widetilde{X}$ of X is that every map $f : X \to Y$ with Y complete metric, meeting some further conditions, factors through $\widetilde{X}$, in the sense that there is a unique

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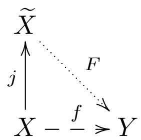
[6] There is another requirement for a vector space to be a Fréchet space, namely, that open balls are *convex.* This *local convexity* requirement can be checked for $C^\infty[a,b]$.

[7] In topological spaces whose topologies are not given by metrics, this notion of completeness, *sequential* completness, is inadequate to capture relevant phenomena. One version of the necessary extension of *sequence* is *net*, which is a set of points in the space indexed by a *directed set* S, which is a *poset* (partially ordered set) with the property that for all $x, y \in S$ there is $z \in S$ such that $x \leq z$ and $y \leq z$. The most general form of completeness is that every Cauchy net has a limit in the space. For metric spaces, this provably reduces to sequential completeness. In general, this fullest notion of completeness is too much to ask for, and refinements are needed.

map $F : \widetilde{X} \to Y$, meeting some further conditions, giving a commutative diagram



For example, to uniquely characterize the completion it suffices to require that every *isometry* (to its image) f extends to a unique *isometry* (to its image) F. Another possibility is to require that every *uniformly* continuous f extends to a unique *uniformly* continuous F. Or we might want to prove the latter stronger condition from the former. We carry this out in the following section.

Define a pseudo-metric p on the set C of Cauchy sequences in X, by

$$p(\{x_s\}, \{y_t\}) \;=\; \lim_s d(x_s, y_s)$$

Define an equivalence relation $\sim$ on C by $x \sim y$ if and only if $p(x, y) = 0$. Attempt to define a metric on the set $C/\sim$ of equivalence classes by

$$d(\{x_s\}, \{y_t\}) \;=\; \lim_s \, d(x_s, y_s)$$

We will verify that this is well-defined on the quotient $C/\sim$ and gives a metric. We have an injection $j : X \to C/\sim$ by

$$x \to \{x, x, x, \ldots\} \bmod \;\sim$$

[8.1] Claim: $j : X \to C/\sim$ is a completion of X.

Proof: We prove that the apparent metric on $\widetilde{X}$ truly is a metric, and is complete. First, the limit in the attempted definition

$$d(\{x_s\}, \{y_t\}) \;=\; \lim_s \, d(x_s, y_s)$$

does exist: given $\varepsilon > 0$, take N large enough so that $d(x_i, x_j) < \varepsilon$ and $d(y_i, y_j) < \varepsilon$ for $i, j \geq N$. By the triangle inequality,

$$d(x_i, y_i) \;\leq\; d(x_i, x_N) + d(x_N, y_N) + d(y_N, y_i) \;<\; \varepsilon + d(x_N, y_N) + \varepsilon$$

Similarly,

$$d(x_i, y_i) \;\geq\; -d(x_i, x_N) + d(x_N, y_N) - d(y_N, y_i) \;>\; -\varepsilon + d(x_N, y_N) - \varepsilon$$

Thus, unsurprisingly,

$$\Big| d(x_i, y_i) - d(x_N, y_N) \Big| \;<\; 2\varepsilon$$

and the sequence of real numbers $d(x_i, y_i)$ is Cauchy, so convergent.

Similarly, when $\lim_i d(x_i, y_i) = 0$, then $\lim_i d(x_i, z_i) = \lim_i d(y_i, z_i)$ for any other Cauchy sequence z_i, so the distance function is *well-defined* on $C/\sim$.

The positivity and symmetry for the alleged metric on $C/\sim$ are immediate. For the triangle inequality, given $\{x_s\}, \{y_s\}, \{z_s\}$ and $\varepsilon > 0$, let N be large enough so that $d(x_i, x_j) < \varepsilon$, $d(y_i, y_j) < \varepsilon$, and $d(z_i, z_j) < \varepsilon$ for $i, j \geq N$. As just above,

$$\Big| d(\{x_s\}, \{y_s\}) - d(x_i, y_i) \Big| \;<\; 2\varepsilon$$

Thus,

$$d(\{x_s\}, \{y_s\}) \;\leq\; 2\varepsilon + d(x_N, y_N) \;\leq\; 2\varepsilon + d(x_N, z_N) + d(z_N, y_N) \;\leq\; 2\varepsilon + d(\{x_s\}, \{z_s\}) + 2\varepsilon + d(\{z_s\}, \{y_s\}) + 2\varepsilon$$

This holds for all $\varepsilon > 0$, so we have the triangle inequality.

Finally, perhaps anticlimactically, the completeness of $C/\sim$. There is not a unique line of argument here.

We take the following approach. Given a Cauchy sequence $\{x_n\}$, we construct another Cauchy sequence $\{y_n\}$ in the same equivalence class, but with better convergence properties. In particular, we take $\{y_n\}$ to be a subsequence of $\{x_n\}$. Let $y_1 = x_{n_1}$ where n_1 is sufficiently large such that for all $m, n \geq n_1$, $d(x_m, x_n) \leq 2^{-1}$. Let $y_2 = x_{n_2}$ where $n_2 > n_1$ is sufficiently large such that for all $m, n \geq n_2$, $d(x_m, x_n) \leq 2^{-2}$. Generally, let $y_{k+1} = x_{n_{k+1}}$ where $n_{k+1} > n_k$ is sufficiently large such that for all $m, n \geq n_{k+1}$, $d(x_m, x_n) \leq 2^{-(k+1)}$.

We claim that $p(\{y_n\}, \{x_n\}) = 0$. Indeed, given $\varepsilon > 0$, let n_o be large enough so that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_o$. Then

$$d(x_k, y_k) \;=\; d(x_k, x_{n_k}) \;<\; \varepsilon \qquad\qquad (\text{since } n_k \geq k)$$

Given a sequence $\{c_s : s = 1, 2, \ldots\}$ of Cauchy sequences $c_s = \{x_{sj}\}$ in X, such that $\{c_s\}$ is Cauchy in $C/\sim$, for $n = 1, 2, 3, \ldots$, replace each c_s by the subsequence $b_s = \{y_{sj}\}$ as above. We claim that the diagonal sequence z given by $z_s = y_{ss}$ is Cauchy, and that $\lim_s p(b_s, z) = 0$.

By construction, $d(z_k, z_{k+1}) \leq 2^{-k}$ for all k, so $d(z_k, z_\ell) \leq 2 \cdot 2^{-k}$ for all $\ell \geq k$, and z is Cauchy.

Since $p(c_s, b_s) = 0$ for all s, and since p is a pseudo-metric on the set C of Cauchy sequences in X, for each k and for $s \geq k$,

$$p(c_s, z) \;=\; p(c_s, b_s) + p(b_s, z) \;=\; 0 + p(b_s, z) \;=\; \lim_n d(y_{sn}, z_n) \;=\; \lim_n d(y_{sn}, y_{nn})$$

$$\leq\; \sup n \geq k d(y_{sn}, y_{nn}) \;\leq\; 2 \cdot 2^{-k}$$

This goes to 0 as $k \to +\infty$, so $c_s \to z$ in $C/\sim$. ///

[8.2] Remark: When a metric space X is a vector space, we would want to check that its completion $\widetilde{X}$ is also a vector space, with natural (continuous) scalar multiplication and (continuous) vector addition extending those of X. Indeed, attempt to define vector addition by

$$\lim x_n + \lim x_n' \;=\; \lim_n (x_n + x_n')$$

and prove well-definedness and continuity by devices similar to those in the previous proof.

Many natural metric spaces are complete without any need to complete them. The historically notable exception was $\mathbb{Q}$ itself, completed to $\mathbb{R}$. A slightly more recent example:

[8.3] Example: One description of the space $L^2[a, b]$ is as the completion of $C^o[a, b]$ with respect to the L^2 norm above. The more common description depends on notions of *measurable function* and *Lebesgue integral*, and presents the space as equivalence classes of functions, having somewhat ambiguous pointwise values.

9. *Extension by continuity*

[9.1] Theorem: Let X be a metric space, $\widetilde{X}$ its completion, and $f : X \to Y$ a *uniformly* continuous map to a complete metric space Y. Then f extends to a unique uniformly continuous map $\widetilde{f} : \widetilde{X} \to Y$ given by

$$\widetilde{f}(\widetilde{X}-\lim x_n) \;=\; Y-\lim f(x_n)$$

Proof: To simplify notation, identify X with a dense subset of $\widetilde{X}$. First, we prove that $\{x_n\}$ Cauchy implies $\{f(x_n)\}$ Cauchy. Given $\varepsilon > 0$, by uniform continuity there is $\delta > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$. Since $\{x_n\}$ is Cauchy, there exists n_o such that $m, n \geq n_o$ implies $d(x_m, x_n) < \delta$, and then $d(f(x_m), f(x_n)) < \varepsilon$. Thus, $\{f(x_n)\}$ is Cauchy.

To prove that $\widetilde{f}$ is well-defined, let $\{x_n\}$ and $\{x'_n\}$ be two Cauchy sequences in X with the same limit z in $\widetilde{X}$. That is, $\lim_n d(x_n, x'_n) = 0$. Given $\delta > 0$, let n_o be large enough so that, for all $n \geq n_o$, $d_X(x_n, z) < \delta/2$ and $d_X(x'_n, z) < \delta/2$. Then $d_X(x_n, x'_n) < delta$, and $d_Y(f(x_n), f(x'_n)) < \varepsilon$. This holds for all $\varepsilon > 0$, so

$$Y-\lim_n f(x_n) \;=\; Y-\lim_n f(x'_n)$$

and $\widetilde{f}$ is well-defined.

For continuity of $\widetilde{f}$, let $z, z' \in \widetilde{X}$, with Cauchy sequences x_t and x'_t approaching z and z'. Given $\varepsilon > 0$, by uniform continuity of f, there is N large enough such that $d_Y(f(x_r), f(x_s)) < \varepsilon$ and $d_Y(f(x'_r), f(x'_s)) < \varepsilon$ for $r, s \geq N$. For such r, even in the limit, the strict inequalities are at worst non-strict inequalities:

$$d_Y(f(x_r), \widetilde{f}(z)) \;\leq\; \varepsilon \qquad \text{and} \qquad d_Y(f(x'_r), \widetilde{f}(z')) \;\leq\; \varepsilon$$

By the triangle inequality, since $f : X \to Y$ is continuous, we can increase r to have $d_X(x_r, x'_r)$ small enough so that $d_Y(f(x_r), f(x'_r)) < \varepsilon$, and then

$$d_Y(F(z), F(z')) \;\leq\; d_Y(F(z), f(x_r)) + d_Y(f(x_r), f(x'_r)) + d_Y(f(x'_r), \widetilde{f}(z')) \;\leq\; \varepsilon + \varepsilon + \varepsilon$$

Then

$$d_X(x_r, x'_r) \;=\; d_{\widetilde{X}}(x_r, x'_r) \;\leq\; d_{\widetilde{X}}(x_r, z) + d_{\widetilde{X}}(z, z') + d_{\widetilde{X}}(x'_r, z')$$

so

$$d_X(x_r, x'_r) \;\leq\; d_{\widetilde{X}}(z, z') \;+\; 2\varepsilon$$

Thus,

$$d_Y(\widetilde{f}(z), \widetilde{f}(z')) \;\leq\; d_{\widetilde{X}}(z, z') + 4\varepsilon \qquad \text{(for all } \varepsilon > 0)$$

Thus, $\widetilde{f}$ is continuous.

Every element of $\widetilde{X}$ is a limit of a Cauchy sequence $j(x_k)$ for x_k in X, and *any* continuous $\widetilde{X} \to Y$ respects limits, so $\widetilde{f}$ is the only possible *continuous* extension of f to $\widetilde{X}$. ///

Continuous linear maps among topological vector spaces have a similar extension property without an explicit hypothesis of uniform continuity, because they are *always* uniformly continuous, even when not metrizable. That is, uniform continuity of continuous linear maps is just a paraphrase of the point that, for a linear map, continuity is equivalent to continuity at 0. Later, with a more appropriate notion of completeness for not-metric spaces, we can prove a more general extension about extension-by-continuity.

10. *Topologies more general than metric topologies*

Many of the ideas and bits of terminology for metric spaces make sense and usefully extend to more general situations. Some do not.

[10.1] A topology on a set X is a collection τ of subsets of X, called the *open sets*, such that X itself and the empty set ϕ are in τ, *arbitrary unions* of elements of τ are in τ, and *finite intersections* of elements of τ are in τ. A set X with an explicitly or implicitly specified topology is a *topological space*.

[10.2] Finer/coarser, stronger/weaker topologies From thinking of metric spaces, the relevance of the following ideas/definitions is surely not clear. Nevertheless, eventually, it is convenient to have names for these phenomena.

Given a set X and topologies σ, τ (subsets of the power set of X) on X, we say that σ is *weaker* or *coarser* than τ when $\sigma \subset \tau$. And τ is *stronger* or *finer* than σ.

That is, topology σ is weaker/coarser when every open of σ is an open of τ. Equivalently, τ is stronger/finer.

[10.3] A continuous map $f : X \to Y$ for topological spaces X, Y is a set-map so that inverse images $f^{-1}(U)$ of opens U in Y are open in X.

Uniform continuity of functions or maps has no natural formulation in general topological spaces, in effect because we have no device by which to compare the topology at varying points, unlike the case of metric spaces, where there is a common notion of distance that does allow such comparisons.

[10.4] Closed sets in a topological space are exactly the complements of open sets. Arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

[10.5] A basis for a topology is a collection of (open) subsets so that any open set is a union of the (open) sets in the basis. In a metric space, the open balls of all possible sizes, at all points, are a natural basis.

[10.6] A neighborhood of a point is any set containing an open set containing the point. Often, one considers only *open* neighborhoods, to avoid irrelevant misunderstandings.

[10.7] A local basis at a point x in a space X is a collection of open neighborhoods of x such that every neighborhood of x *contains* a neighborhood from the collection. In a metric space, the collection of open balls at a given point with *rational radius* is a countable local basis at that point.

[10.8] The closure of a set E (in a topological space X), sometimes denoted $\overline{E}$, is the intersection of all closed sets containing E. It is a closed set. Equivalently, it is the set of $x \in X$ such that every neighborhood of x meets[8] E. The closure of E contains E.

[10.9] The interior of a set E (in a topological space X) is the union of all open sets contained in it. It is open. Equivalently, it is the set of $x \in X$ such that there is a neighborhood of x inside E. The interior of E is a subset of E.

[10.10] The boundary of a set E (in a topological space X), often denoted ∂E, is the intersection of the

[8] A set X *meets* another set Y if $X \cap Y \neq \phi$.

closure of E and the closure of the complement of E. Equivalently, it is the set of $x \in X$ such that every neighborhood of x meets both E and the complement of E.

[10.11] A Hausdorff topology is one in which any two points x, y have neighborhoods $U \ni x$ and $V \ni y$ which are disjoint: $U \cap V = \phi$. This is a reasonable condition to impose on a space on which functions should live.

[10.12] Claim: Metric spaces are Hausdorff.

Proof: Given $x \neq y$ in a metric space, let B_1 be the open ball of radius $d(x,y)/2$, and let B_2 the open ball of radius $d(x,y)/2$ at y. For any $z \in B_1 \cap B_2$, by the triangle inequality,

$$d(x,y) \;\le\; d(x,z) + d(z,y) \;<\; \frac{d(x,y)}{2} + \frac{d(x,y)}{2} \;=\; d(x,y)$$

which is impossible. Thus, there is no z in the intersection of these two open neighborhoods of x and y. ///

[10.13] Claim: In Hausdorff spaces, singleton sets $\{x\}$ are closed.

Proof: Fixing x, for $y \neq x$ let U_y be an open neighborhood of y not containing x. (We do not use the open neighborhood of x not meeting U_y.) Then $E = \bigcup_{y \neq x} U_y$ is open, does not contain x, and contains every other point in the space. Thus, E is the complement of the singleton set $\{x\}$ and is open, so $\{x\}$ is closed. ///

[10.14] Convergence of sequences: In a topological space X, a sequence $x_1, x_2, \ldots$ converges to $x_\infty \in X$, written $\lim_n x_n = x_\infty$, if, for every neighborhood U of x_∞, there is an index m such that for all $n \ge m$, $x_n \in U$.

In more general, non-Hausdorff spaces, it is easily possible to have a sequence converge to more than one point, which is fairly contrary to our intention for the notion of *convergence*.

In a *metric* space, the notion of *Cauchy* sequence has a sense, and in a *complete* metric space, the notions of Cauchy sequence and convergent sequence are identical, and there is a unique limit to which such a sequence converges.

In more general, non-Hausdorff spaces, and not-locally-countably-based spaces, things can go haywire in several different ways, which are mostly irrelevant to the situations we care about. Still, one should be aware that not all spaces are Haudorff, and may fail to be countably locally based.

[10.15] Sequentially compact sets E in a topological space X are those such that every sequence has a convergent subsequence (with limit in E).

Although the definition of *convergent* does not directly mention potential difficulties and ambiguities, there are indeed problems in non-Haudorff spaces, and in spaces that fail to have countable local bases.

[10.16] Accumulation points of a subset E of a topological space X are points $x \in X$ such that every neighborhood of x contains infinitely-many elements of E. Every accumulation point of E lies in the *closure* of E, but not vice-versa.

[10.17] Claim: A closed set E is sequentially compact if and only if every sequence in E either has an accumulation point in E, or contains only finitely-many distinct points.

Proof: First, the technicality: if a sequence contains only finitely-many distinct points, it cannot have any accumulation points, but certainly contains convergent subsequences. For a sequence $x_1, x_2, \ldots$ including infinitely-many distinct points, drop any repeated points, so that $x_i \neq x_j$ for all $i \neq j$. For E sequentially compact, there is a subsequence with limit x_∞ in E. Relabel if necessary so that the subsequence is still denoted $x_1, x_2, \ldots$. The subsequence still consists of mutually distinct points. Since $\lim_n x_n = x_\infty$, given a neighborhood U of x_∞, there is m such that $x_n \in U$ for all $n \geq m$. Since $x_m, x_{m+1}, \ldots$ is an infinite set of distinct points, x_∞ is an accumulation point of the subsequence, hence, of the original sequence.

Conversely, if a sequence has an accumulation point, it has a subsequence converging to that accumulation point. ///

[10.18] Compact sets in topological spaces are subsets such that *every open cover has a finite subcover*. That is, K is compact when, for any collection of open sets $\{U_\alpha : \alpha \in A\}$ such that $K \subset \bigcup_{\alpha \in A} U_\alpha$, there is a finite collection $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $K \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

[10.19] Claim: For $f : X \to Y$ continuous and K compact in X, the image $f(K)$ is compact in Y.

Proof: Given an open cover $\{U_\alpha : \alpha \in A\}$ of $f(K)$, the inverse images $f^{-1}(U_\alpha)$ give an open cover of K. Thus, there is a finite subcover $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a (finite) cover of $f(K)$. ///

Since singleton sets $\{x\}$ are certainly compact, the following generalizes the earlier claim about closedness of singleton sets in Hausdorff spaces:

[10.20] Claim: In Hausdorff spaces, compact sets are *closed*.

Proof: Let E be a compact subset of X. For $y \notin E$, for each $x \in E$, let $U_x \ni y$ be open and $V_x \ni x$ open so that $U_x \cap V_x = \phi$. Then $\{V_x : x \in E\}$ is an open cover of E, with finite subcover $E \subset V_{x_1} \cup \ldots \cup V_{x_n}$. The finite intersection $W_y = U_{x_1} \cap \ldots \cap U_{x_n}$ is open, and disjoint from $V_{x_1} \cup \ldots \cup V_{x_n}$, so is disjoint from E. Thus, W_y is open and contains y. The union $W = \bigcup_{y \notin E} W_y$ is open, and contains every $y \notin E$. Thus E is the complement of an open set, so is closed. ///

[10.21] Claim: In Hausdorff spaces, a *nested* collection of compact sets has non-empty intersection.

Proof: Let X be the ambient space, and K_α the compacts, with index set A *totally ordered*, in the sense A has an order relation $<$ such that for every distinct $\alpha, \beta \in A$, either $\alpha < \beta$ or $\beta < \alpha$. The *nested* condition is that if $\alpha < \beta$ then $K_\alpha \supset K_\beta$. (It can equally well be the opposite direction of containment.) We claim that $\bigcap_\alpha K_\alpha$ is compact.

From above, each K_α is closed, so the complements $U_\alpha = X - K_\alpha$ are open. If $\bigcap_\alpha K_\alpha = \phi$, then $\bigcup_\alpha U_\alpha = X$. In particular, $\bigcup_\alpha U_\alpha \supset K_\beta$ for all indices β. For fixed index α_o, let $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ be a finite subcover of K_{α_o}, so certainly a cover of $K_{\alpha'}$ for all $\alpha' > \alpha$. Because of the nested-ness, for $\beta = \max\{\alpha_1, \ldots, \alpha_n\}$, $U_\beta = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. But U_β is the complement of K_β, so certainly cannot cover it, contradiction. ///

[10.22] A locally compact topology is one in which every point has a neighborhood with compact closure. This is a reasonable condition to impose on a space on which functions will live. $\mathbb{R}^n$ is locally compact, but the metric space ℓ^2 is *not*. Later, we will see that *no* infinite-dimensional Hilbert space or Banach space is locally compact. That is, *natural spaces of functions* are not usually locally compact, but the physical spaces on which the functions live usually *are* locally compact.

[10.23] Separable topological spaces are those with countable dense subsets. For example, the countable set $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Nearly all topological spaces arising in practice are separable, but most basic results do not directly use this property.

[10.24] Countably-based topological spaces are those with a countable basis. Sometimes such spaces are called *second-countable*. Perhaps counter-intuitively, *first-countable* spaces are those in which every point has a countable *local* basis. Many topological spaces arising in practice are countably-based, but most basic results do not directly use this property.

[10.25] Claim: Separable metric spaces are countably-based. Specifically, for countable dense subset S of metric space X, open balls of *rational* radius centered at points of S form a basis.

Proof: Since there are only countably-many $s \in S$ and only countably many rational radiuses, the set of such open balls is indeed countable.

Fix an open $U \subset X$. Given $x \in U$, let $r > 0$ be sufficiently small so that the open ball at x of radius r is inside U. Let $s_x \in S$ be such that $d(x, s) < r/2$. By density of rational numbers in $\mathbb{R}$, there is a rational number q_x such that $d(x, s) < q_x < r/2$. Thus, by the triangle inequality, the ball B_x at s_x of radius q_x contains x and lies inside the open ball at x of radius r, so $B_x \subset U$.

The union of all B_x over $x \in U$ is a subset of U containing all $x \in U$, so is U itself. ///

[10.26] Test functions $C_c^\infty(\mathbb{R}^n)$ The space $C_c^\infty(\mathbb{R}^n)$ of *test functions* on $\mathbb{R}^n$ is the space of smooth, compactly-supported (complex-valued) functions. As usual, we would like to be able to take *suitable* limits and stay in side this space. The question is, what limits *are* suitable? That is, what *topology* makes the space of test functions (suitably) *complete*? It turns out that the appropriate complete topology is *not* metric, because any remotely reasonable topology on $C_c^\infty(\mathbb{R}^n)$ violates the conclusion of the Baire category theorem (below). This important example illustrates the practical need for topologies more general than metric topologies. This topology will be considered in detail later.

[10.27] Weak topologies Another type of topology that is not metric is the *weak dual* or *weak* $*$ topology on the *dual space*

$$V^* = \{\text{continuous linear maps } V \to \mathbb{C}\}$$

for vector spaces V with topologies. This sort of weak topology is much *coarser* than the Hilbert-space dual topology on duals of Hilbert spaces, and much *coarser* than the Banach space dual topology on the duals of Banach spaces. Weak dual topologies will be considered in detail later.

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11. *Compactness versus sequential compactness*

In general topological spaces, *compactness* is a stronger condition than *sequential compactness*. First, without any further hypotheses on the spaces, however noting the point that sequential compactness easily fails to be what we anticipate in topological spaces that are not necessarily Hausdorff or locally countably-based:

[11.1] Claim: Compact sets are sequentially compact.

Proof: Given a sequence, if some $y \in E$ is an accumulation point, then there is a subsequence converging to y, and we are done. If no $y \in E$ is an accumulation point of the given sequence, then each $y \in E$ has an open neighborhood U_y such that U_y meets the sequence in only finitely-many points. The sets U_y cover E. For E compact, there is a finite subcover $U_{y_1}, \ldots, U_{y_n}$. Each U_{y_i} contains only finitely-many points of the sequence, so the sequence contains only finitely-many distinct points, so certainly has a convergent subsequence. ///

[11.2] Claim: In a countably-based topological space X, sequentially compact sets are compact.

Proof: Let $E \subset X$ be sequentially compact. The opens in an arbitrary cover of E are (necessarily countable) unions of some of the countably-many opens in the countable basis for X. Thus, it suffices to show that a *countable* cover $E \subset U_1 \cup U_2 \cup \ldots$ admits a finite subcover.

If *no* finite collection of the U_n covers E, then for each $n = 1, 2, \ldots$ there is $e_n \in E$ such that $e_n \notin U_1 \cup \ldots \cup U_n$. Since every e_n does lie in *some* U_i, we can replace $\{e_n\}$ by a subsequence so that $e_i \neq e_j$ for all $i \neq j$, and still $e_n \notin U_1 \cup \ldots \cup U_n$.

By sequential compactness, $e_1, e_2, \ldots$ has a convergent subsequence, with limit $e_\infty \in E$. The point e_∞ lies in some U_m. Thus, there would be infinitely-many indices n such that $e_n \in U_m$. This is impossible, since $e_n \notin U_1 \cup \ldots \cup U_n$. Thus, there must be a finite subcover. ///

The argument for the previous claim can be improved, to show

[11.3] Claim: In complete metric spaces, sequentially compact sets are compact.

Proof: Let $\{U_\alpha : \alpha \in A\}$ be an open cover of a subset E of a complete metric space X, admitting no finite subcover. Using an equivalent of the Axiom of Choice, we can arrange to have a *minimal* subcover, that is, so that no U_β can be removed an still cover E. We do this at the end of the argument.

Granting this, without loss of generality the open cover is *minimal*, and not finite. Using the minimality (and again using the Axiom of Choice), for each index $\beta \in A$, let x_β be a point in E that is *not* in $\bigcup_{\alpha \neq \beta} U_\alpha$. Since the cover is minimal, these x_β's must be *distinct*. Since the cover is not finite, there are infinitely-many (distinct) x_β's. Since the are distinct, any countable subset of $\{x_\beta : \beta \in A\}$ gives a sequence $y_1, y_2, \ldots$ of distinct points. By sequential compactness, this sequence has at least one accumulation point $y_\infty \in E$.

Let U_{α_o} be an open in the cover containing y_∞. Since $\lim_n y_n = y_\infty$, there is n_o such that for all $n \geq n_o$ we have $y_n \in U_{\alpha_o}$. All those y_n's are among the x_β's, but the only x_β in U_{α_o} is x_{α_o}. That is, there cannot be infinitely-many distinct x_β's in U_{n_o}. Thus, assuming that a minimal cover is infinite leads to a contradiction.

To obtain a minimal subcover from a given cover $\{U_\alpha : \alpha \in A\}$, *well-order* the index set A. We choose a minimal subcover by transfinite induction, as follows. The idea is to ask, in the order chosen for A, cumulatively, whether or not U_α can be removed from the current subcover while still having a cover of the given set. That is, we inductively define a subset B of the index set A by transfinite induction: initially, $B = A$. At the α^{th} stage, remove α from B if U_α is unnecessary for maintaining the cover property. That is, remove α if

$$E \;\subset\; \bigcup_{\beta<\alpha,\; \beta\in B} U_\beta \;\cup\; \bigcup_{\beta>\alpha} U_\beta$$

otherwise keep α in B. By transfinite induction, B is an index set for a subcover of $\{U_\alpha : \alpha \in A\}$, and that subcover is *minimal* in the sense that no open can be removed without the result failing to be a cover. ///

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12. *Total-boundedness criterion for pre-compactness*

In general metric spaces, closed and bounded sets need not be compact (nor sequentially compact). For example, the closed unit ball

$$B \;=\; \{v \in B : \Big(\sum_n |v_n|^2\Big)^{\frac{1}{2}} \leq 1\} \;\subset\; \ell^2$$

is *not* sequentially compact: the vectors

$$\begin{aligned} e_1 &= (1,0,0,\ldots) \\ e_2 &= (0,1,0,\ldots) \\ e_3 &= (0,0,1,0,\ldots) \\ \ldots \end{aligned}$$

have distance $\sqrt{2}$ from each other, so cannot have a convergent subsequence, since elements of a convergent (sub-) sequence get closer and closer together. More is required, as follows.

A set E in a metric space is *totally bounded* if, given $\varepsilon > 0$, there are finitely-many open balls of radius ε covering E. The property of *total boundedness* in a metric space is generally stronger than mere *boundedness*. It is immediate that any subset of a totally bounded set is totally bounded.

[12.1] Theorem: A set E in a metric space X has compact closure *if and only if* it is totally bounded.

[12.2] Remark: Sometimes a set with compact closure is said to be *pre-compact*.

Proof: Certainly if a set has compact closure then it admits a finite covering by open balls of arbitrarily small (positive) radius, by the compactness.

On the other hand, suppose that a set E is totally bounded in a complete metric space X. To show that E has compact closure it suffices to show *sequential* compactness, namely, that any sequence $\{x_i\}$ in E has a convergent subsequence.

We choose such a subsequence as follows. Cover E by finitely-many open balls of radius 1, invoking the total boundedness. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball B_1, and let i_1 be the smallest index so that x_{i_1} lies in this ball.

The set $E \cap B_1$ is still totally bounded (and contains infinitely-many elements from the sequence). Cover it by finitely-many open balls of radius $1/2$, and choose a ball B_2 with infinitely-many elements of the sequence lying in $E \cap B_1 \cap B_2$. Choose the index i_2 to be the smallest one so that both $i_2 > i_1$ and so that x_{i_2} lies inside $E \cap B_1 \cap B_2$.

Proceeding inductively, suppose that indices $i_1 < \ldots < i_n$ have been chosen, and balls B_i of radius $1/i$, so that

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it B_{n+1}, containing infinitely-many elements of the sequence. Let i_{n+1} be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \;\in\; E \cap B_1 \cap \ldots \cap B_{n+1}$$

Then for $m < n$ we have $d(x_{i_m}, x_{i_n}) \;\leq\; \frac{1}{m}$ so this subsequence is Cauchy. ///

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13. *Baire's theorem*

This standard result is both indispensable and mysterious.

A set E in a topological space X is *nowhere dense* if its closure $\bar{E}$ contains no non-empty open set. A *countable union* of nowhere dense sets is said to be *of first category*, while every other subset (if any) is *of second category*. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric* spaces and *locally compact Hausdorff* spaces are *of second category*.

A G_δ set is a countable intersection of open sets. Concommitantly, an F_σ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense G_δ's is still a dense G_δ.*

[13.1] Theorem: *(Baire)* Let X be either a complete metric space or a locally compact Hausdorff topological space. The intersection of a *countable* collection $U_1, U_2, \ldots$ of *dense open subsets* U_i of X is still *dense* in X.

Proof: Let B_o be a non-empty open set in X, and show that $\bigcap_i U_i$ meets B_o. Suppose that we have inductively chosen an open ball B_{n-1}. By the denseness of U_n, there is an open ball B_n whose closure $\overline{B_n}$

satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take B_n to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take B_n to have compact closure.

Let

$$K \;=\; \bigcap_{n\geq 1} \overline{B_n} \;\subset\; B_o \cap \bigcap_{n\geq 1} U_n$$

For complete metric spaces, the centers of the nested balls B_n form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure* $\overline{B_n}$, so lies in the intersection. In particular, K is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so K is non-empty, and B_o necessarily meets the intersection of the U_n. ///

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14. *Urysohn's lemma*

Urysohn's lemma proves existence of sufficiently many functions on reasonable topological spaces.

[14.1] Theorem: *(Urysohn)* In a locally compact Hausdorff topological space X, given a compact subset K contained in an open set U, there is a continuous function $0 \leq f \leq 1$ which is 1 on K and 0 off U.

Proof: First, we prove that there is an open set V such that

$$K \;\subset\; V \;\subset\; \overline{V} \;\subset\; U$$

For each $x \in K$ let V_x be an open neighborhood of x with compact closure. By compactness of K, some finite subcollection $V_{x_1}, \ldots, V_{x_n}$ of these V_x cover K, so K is contained in the open set $W = \bigcup_i V_{x_i}$ which has compact closure $\bigcup_i \overline{V}_{x_i}$ since the union is *finite*.

Using the compactness again in a similar fashion, for each x in the closed set $X - U$ there is an open W_x containing K and a neighborhood U_x of x such that $W_x \cap U_x = \phi$.

Then

$$\bigcap_{x\in X-U} (X-U) \cap \overline{W} \cap \overline{W}_x \;=\; \phi$$

These are compact subsets in a Hausdorff space, so (again from compactness) some *finite* subcollection has empty intersection, say

$$(X-U) \cap \big(\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_n}\big) \;=\; \phi$$

That is,

$$\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_n} \;\subset\; U$$

Thus, the open set

$$V \;=\; W \cap W_{x_1} \cap \ldots \cap W_{x_n}$$

meets the requirements.

Using the possibility of inserting an open subset and its closure between any $K \subset U$ with K compact and U open, we inductively create opens V_r (with compact closures) indexed by rational numbers r in the interval $0 \leq r \leq 1$ such that, for $r > s$,

$$K \;\subset\; V_r \;\subset\; \overline{V}_r \;\subset\; V_s \;\subset\; \overline{V}_s \;\subset\; U$$

From any such configuration of opens we construct the desired continuous function f by

$$f(x) \;=\; \sup\{r \text{ rational in } [0,1] :\; x \in V_r,\; \} \;=\; \inf\{r \text{ rational in } [0,1] :\; x \in \overline{V}_r,\; \}$$

It is not immediate that this sup and inf are the same, but if we *grant* their equality then we can prove the *continuity* of this function $f(x)$. Indeed, the sup description expresses f as the supremum of characteristic functions of open sets, so f is at least *lower semi-continuous.* [9] The inf description expresses f as an infimum of characteristic functions of closed sets so is *upper* semi-continuous. Thus, f would be continuous.

To finish the argument, we must construct the sets V_r and prove equality of the inf and sup descriptions of the function f.

To construct the sets V_i, start by finding V_0 and V_1 such that

$$K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset U$$

Fix a well-ordering $r_1, r_2, \ldots$ of the rationals in the open interval $(0,1)$. Supposing that $V_{r_1}, \ldots, v_{r_n}$ have been chosen. let i, j be indices in the range $1, \ldots, n$ such that

$$r_j > r_{n+1} > r_i$$

and r_j is the *smallest* among $r_1, \ldots, r_n$ *above* r_{n+1}, while r_i is the *largest* among $r_1, \ldots, r_n$ *below* r_{n+1}. Using the first observation of this argument, find $V_{r_{n+1}}$ such that

$$V_{r_j} \subset \overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i} \subset \overline{V}_{r_i}$$

This constructs the nested family of opens.

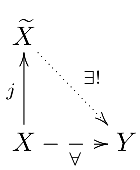
Let $f(x)$ be the sup and $g(x)$ the inf of the characteristic functions above. If $f(x) > g(x)$ then there are $r > s$ such that $x \in V_r$ and $x \notin \overline{V}_s$. But $r > s$ implies that $V_r \subset \overline{V}_s$, so this cannot happen. If $g(x) > f(x)$, then there are rationals $r > s$ such that

$$g(x) > r > s > f(x)$$

Then $s > f(x)$ implies that $x \notin V_s$, and $r < g(x)$ implies $x \in \overline{V}_r$. But $V_r \subset \overline{V}_s$, contradiction. Thus, $f(x) = g(x)$. ///

15. *Appendix: mapping-property characterization of completion*

Our *intention* is that, when a metric space X is not complete, there should be a *complete* metric space $\widetilde{X}$ and an *isometry* (distance-preserving) $j : X \to \widetilde{X}$, such that every isometry $f : X \to Y$ to *complete* metric space Y *factors through* j uniquely. That is, there are *commutative diagrams*[10] of continuous maps



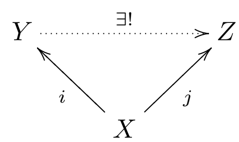
(for every isometry $X \to Y$

[9] A (real-valued) function f is *lower* semi-continuous when for all bounds B the set $\{x : f(x) > B\}$ is open. The function f is *upper* semi-continuous when for all bounds B the set $\{x : f(x) < B\}$ is open. It is easy to show that a sup of lower semi-continuous functions is lower semi-continuous, and an inf of upper semi-continuous functions is upper semi-continuous. As expected, a function both upper and lower semi-continuous is continuous.

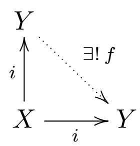
[10] A diagram of maps is *commutative* when the composite map from one object to another within the diagram does not depend on the route taken within the diagram.

Without describing any *constructions* of completions, we can prove some things about the behavior of *any possible* completion. In particular, we prove that any two completions are *naturally isometrically isomorphic* to each other. *Thus, the outcome will be independent of construction.*

[15.1] Claim: *(Uniqueness)* Let $i : X \to Y$ and $j : X \to Z$ be two completions of a metric space X. Then there is a unique isometric homeomorphism $h : Y \to Z$ such that $j = h \circ i$. That is, we have a commutative diagram

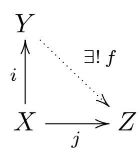


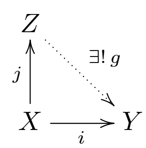
Proof: First, take $Y = Z$ and $f : X \to Y$ to be the inclusion i, in the characterization of $i : X \to Y$. The characterization of $i : X \to Y$ shows that there is *unique* isometry $f : Y \to Y$ fitting into a commutative diagram



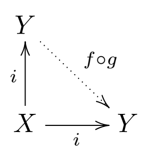
Since the *identity* map $Y \to Y$ certainly fits into this diagram, the *only* map f fitting into the diagram is the identity on Y.

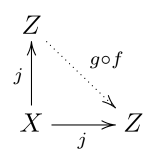
Next, applying the characterizations of both $i : X \to Y$ and $j : X \to Z$, we have unique $f : Y \to Z$ and $g : Z \to Y$ fitting into





Then $f \circ g : Y \to Y$ and $g \circ f : Z \to Z$ fit into





By the first observation, this means that $f \circ g$ is the identity on Y, and $g \circ f$ is the identity on Z, so f and g are mutual inverses, and Y and Z are *homeomorphic.* ///

[15.2] Remark: A virtue of the characterization of completion is that it does not refer to the *internals* of any completion.

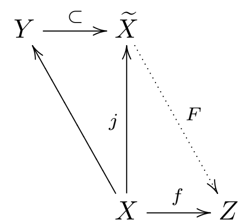
Next, we see that the mapping-property characterization of a completion does not introduce more points than absolutely necessary:

[15.3] Claim: Every point in a completion $\widetilde{X}$ of X is the limit of a Cauchy sequence in X. That is, X is *dense* in $\widetilde{X}$.

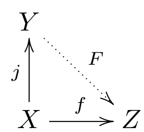
Proof: Write $d(,)$ for both the metric on X and its extension to $\widetilde{X}$. Let $Y \subset \widetilde{X}$ be the collection of limits of Cauchy sequences of points in X. We claim that Y itself is *complete*. Indeed, given a Cauchy sequence $\{y_i\}$ in Y with limit $z \in \widetilde{X}$, let $x_i \in X$ such that $d(x_i, y_i) < 2^{-i}$. It will suffice to show that $\{x_i\}$ is Cauchy with limit z. Indeed, given $\varepsilon > 0$, take N large enough so that $d(y_i, z) < \varepsilon/2$ for all $i \geq N$, and increase

N if necessary so that $2^{-i} < \varepsilon/2$. Then, by the triangle inequality, $d(x_i, z) < \varepsilon$ for all $i \geq N$. Thus, Y is complete.

By the defining property of $\widetilde{X}$, every isometry $f : X \to Z$ to complete Z has a unique extension to an isometry $F : \widetilde{X} \to Z$ fitting into



Since Y is already complete and $j(X) \subset Y$, the restriction of F to Y gives a diagram



That is, Y fits the characterization of a completion of X. By uniqueness, $Y \subset \widetilde{X}$ is a homeomorphism, so $Y = \widetilde{X}$. ///

16. *Appendix: why the product topology is so coarse*

Often it is not the *internal structure* of a thing that is interesting, but its *interactions* with *other objects*. That is, often we have little long-term interest in the details of a *construction* of the thing, but care more about how it *behaves*. Thus, to express our genuine intentions, we should *not* first *construct* the thing, and only gradually admit that it does what we had planned all along. Instead, we should tell what *external interactions* we demand or expect, and worry about internal details later.

Happily, often the characterizations of an object in terms of maps to and from other objects of the same sort succeed in uniquely determining the thing. Even more surprisingly, often this uniqueness follows merely from the shape of the diagrams of the maps, not from any subtler or internal features of the maps or objects.

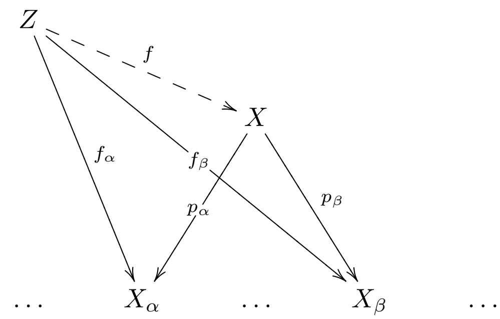
For example, the nature of the *product topology* on products of topological spaces is illuminated by this approach. In particular, one might have (at some point) wondered *why the product topology is so coarse*. That is, on infinite products the product topology is strictly coarser than the *box topology*. The answer is that *the product topology is what it has to be*. That is, there is no genuine *choice* in the construction. Of course, this needs explanation.

[16.1] Definition and uniqueness of products

A *product* of non-empty topological spaces X_α for α in an index set A is a topological space X with (*projection*) maps $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from some other topological space *factors through* the p_α *uniquely*, in the sense that there is a unique $f : Z \to X$ such that

$$f_\alpha \;=\; p_\alpha \circ f \qquad \text{(for all } \alpha)$$

Pictorially, one says that *all triangles commute* in the diagram [11]

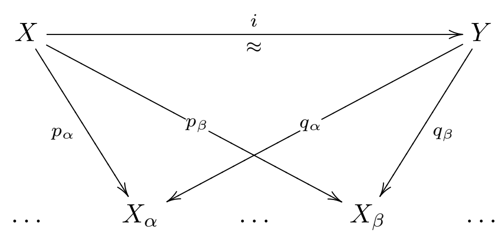


[16.2] Claim: There is at most one product of given spaces X_α, up to *unique* isomorphism.

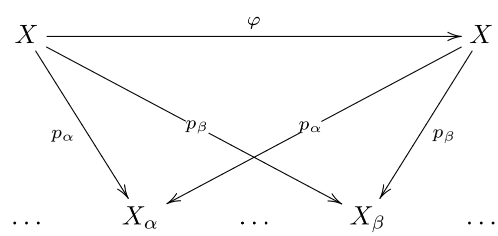
That is, given two products, X with projections p_α and Y with projections q_α, there is a unique isomorphism $i : X \to Y$ respecting the projections in the sense that

$$p_\alpha = q_\alpha \circ i \qquad \text{(for all } \alpha)$$

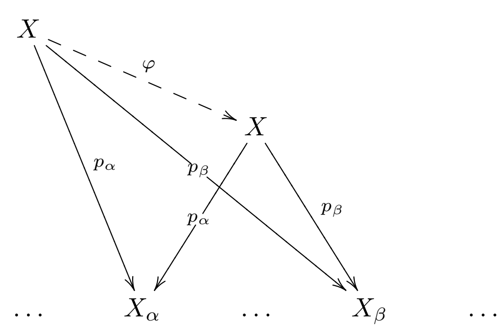
That is, there is a unique isomorphism $i : X \to Y$ such that all triangles commute in the diagram



Proof: First, we prove that products *have no (proper) endomorphisms*, meaning that the *identity map* is the only map $\varphi : X \to X$ making all triangles commute in the diagram



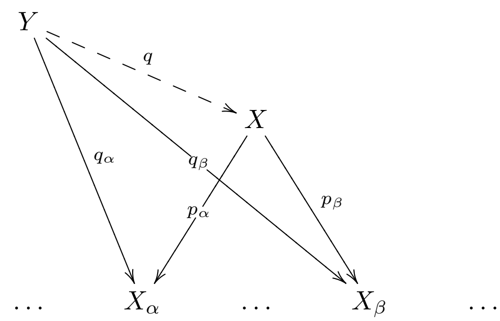
Indeed, using X and the p_α in the role of Z and f_α in the defining property of the product, we have a unique φ making all triangles commute in



[11] Often an arrow is drawn with a dotted line rather than a solid line to suggest that it is a consequence of or is derived from the other maps in the diagram.

The identity map certainly meets the requirement on φ, so, by uniqueness, the identity map on X is the *only* such map φ.

Next, let Y be another product, with projections q_α. Letting Y take the role of Z and q_α the role of f_α in the defining property of the product X (with its p_α's), we have a *unique* $q : Y \to X$ such that all triangles commute in



Reversing the roles of X and Y (and their projections), we also have a *unique* $p : X \to Y$ such that (this time in symbols rather than the picture) $p_\alpha = q_\alpha \circ p$ for all α.

Then $p \circ q : Y \to Y$ is an endomorphism of Y respecting the projections q_α, since

$$q_\alpha \circ (p \circ q) \;=\; (q_\alpha \circ p) \circ q \;=\; p_\alpha \circ q \;=\; q_\alpha$$

Thus, $p \circ q$ must be the identity on Y. Similarly, $q \circ p$ is the identity on X. Thus, p and q are mutual inverses, so both are isomorphisms. And we did have *uniqueness*, along the way. ///

[16.3] Remark: We did not use any features of topological spaces nor of continuous maps in this proof. Instead, the quantification over all Z and maps f_α indirectly described what was happening. Thus, we discover that a similar result holds for any prescribed collection of things with prescribed maps between them that allow *composition*, etc. In particular, for our present purposes we have shown that the product is unique up to *continuous* isomorphism, not just *set* isomorphism.

[16.4] Remark: Since there is at most one product, the issue of figuring out what it *is* will be simpler than if there were many possibilities. Further, we can use the mapping properties to find hints about a *construction*, thus proving existence.

[16.5] Construction of products of sets

Before addressing the *topology* on the product, we first construct it as a *set*. Of course, we secretly know that it is the usual Cartesian product, but it is interesting to see that we can *discover* this from the mapping properties, rather merely *verify* that the Cartesian product fits (which we do at the end).

Note that the uniqueness proof just given applies immediately to *sets without topologies*, as well. That is, the same diagrams but with objects just sets and maps arbitrary set maps, *define* a product X (with projections p_α) of non-empty *sets* X_α, and the same argument proves that there is at most one such thing (up to unique isomorphism). And, yes, we secretly know that the product is the usual Cartesian product, and the projections are the obvious things, but we also want to see how one might be *led to* this understanding.

To investigate properties of a product X of non-empty sets X_α (with projections p_α), we should consider various sets Z and maps $f_\alpha : Z \to X_\alpha$ to see what we learn about X as by-products. In this austere setting, there are not very many choices that suggest themselves, but, on the other hand, there are not many *wrong* choices available, either.

For example, given $x \in X$, we have the collection of all projections' values $p_\alpha(x)$, with unclear relations (or none at all) among these values. For example, to see whether things get mashed down, and as an exercise in technique, we could try to prove

[16.6] Claim: For $x \neq y$ both in X, there is at least one $\alpha \in A$ such that $p_\alpha(x) \neq p_\alpha(y)$.

Proof: Suppose that $p_\alpha(x) = p_\alpha(y)$ for all $\alpha \in A$. Let $S = \{s\}$ be a set with one element, and define $f_\alpha : S \to X_\alpha$ by

$$f_\alpha(s) = p_\alpha(x) \qquad (= p_\alpha(y), \text{ also})$$

Then, by definition, there is a unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$ for all α. But notice that f defined by $f(s) = x$ and also f defined by $f(s) = y$ have this property. By uniqueness of f, we have $x = f(s) = y$. ///

From the other side, we can wonder whether *all* possible collections of values of projections can occur. Intuitively (and secretly knowing the answer in advance) we might doubt that there are any constraints, but we are obliged to demonstrate this by exhibiting maps.

[16.7] Claim: Given choices $x_\alpha \in X_\alpha$ for all $\alpha \in A$, there is x in the product such that $p_\alpha(x) = x_\alpha$ for all α. (And by the previous claim this x is *unique*.)

Proof: Again, let $S = \{s\}$ be a set with a single element, and define $f_\alpha : S \to X_\alpha$ by $f_\alpha(s) = x_\alpha$. Then there exists a unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$. That is

$$x_\alpha = f_\alpha(s) = (p_\alpha \circ f)(s) = p_\alpha(f(s))$$

The element $x = f(s)$ is the desired one. ///

In fact, these two claims, *with their proofs*, suggest that the product X *is exactly* the collection of all choices of families $\{f_\alpha : \alpha \in A\}$ of functions $f_\alpha : \{s\} \to X_\alpha$, and α^{th} projection given by

$$p_\alpha \;:\; \{f_\alpha : \alpha \in A\} \to f_\alpha(s)$$

This is correct, and by uniqueness we know that any other construction must give an isomorphic thing, but this can be simplified, since maps from a one-element set are entirely determined by their images. Thus, finally, we have been led back to the Cartesian product

$$X = \{\{x_\beta : \beta \in A\} \;:\; x_\beta \in X_\beta\}$$

and usual projections

$$p_\alpha(\{x_\beta : \beta \in A\}) = x_\alpha$$

And, finally, we give the trivial proof of

[16.8] Claim: The Cartesian product and usual projections are a product for sets.

Proof: Given a family of maps $f_\alpha : Z \to X_\alpha$, define $f : Z \to X$ by

$$f(z) = \{f_\beta(z) : \beta \in A\}) \in X$$

This meets the defining condition, and is visibly the only map that will do so. ///

[16.9] Remark: The point here was that there is no alternative, up to (unique!) isomorphism, and that some reasonable considerations based on the mapping-property definition can lead us to a construction.

[16.10] Remark: The little fact that the collection of set maps φ from $S = \{s\}$ to a given set Y is isomorphic to Y (via $\varphi \to \varphi(s)$) is noteworthy in itself.

[16.11] Construction of product topologies

So, at last, what topology does the mapping-property definition require on the Cartesian product X of the underlying *sets* for non-empty topological spaces X_α?

One should be aware that it is possible to apparently define non-existent (impossible) objects by mapping properties, and the impossibility may not be immediately clear. Thus, one should have respect for the problem of *constructing* objects whose uniqueness (if they exist) is much easier.

First, all the projections $p_\alpha : X \to X_\alpha$ defined as expected by

$$p_\alpha(\{x_\beta : \beta \in A\}) = x_\alpha$$

must be continuous. That is, for every open set U_α in X_α, $p_\alpha^{-1}(U_\alpha)$ is open in X. Note that

$$p_\alpha^{-1}(U_\alpha) = \prod_{\beta\in A} U_\beta \qquad (\text{where } U_\beta = X_\beta \text{ for } \beta \neq \alpha)$$

Thus, the topology on X must contain *at least* these sets as opens, which does entail that the topology include finite intersections of them, which are exactly sets of the form

$$\prod_{\beta\in A} U_\beta \qquad (\text{with } U_\beta \text{ open in } X_\beta, \text{ and } U_\beta = X_\beta \text{ for all but finitely-many } \beta)$$

And arbitrary unions of these finite intersections must be included in the topology, and so on. Thus, the product topology is *at least as fine as* the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$.

From the other side, the condition concerning maps from another space Z gives a constraint on how *coarse* the topology must be, as follows. Given a family of continuous maps $f_\alpha : Z \to X_\alpha$, the corresponding $f : Z \to X$ must be continuous, which requires by definition that $f^{-1}(U)$ must be open in Z for all opens U in X. If there were *too many* opens U in X this condition could not be met. So the topology on X must be *at least as coarse* as would be allowed by $f^{-1}(U)$ being open in Z for all $f : Z \to X$. Yes, this is a less tangible constraint, since it is hard to visualize the quantification over all $f : Z \to X$.

It is reasonable to *hope* that the explicit topology on X generated by the inverse images $p_\alpha^{-1}(U)$ can be *proven* to be sufficiently coarse to meet the second condition.

To prove continuity of $f : Z \to X$, thus proving that the topology on X is sufficiently coarse, it suffices to prove that $f^{-1}(U)$ is open for all opens U *in a sub-basis*. And, happily, by $p_\alpha \circ f = f_\alpha$,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha)$$

which is open by the continuity of f_α. So f is continuous, which is to say that the topology on X is coarse enough (not too fine), so succeeds in being suitable for a product. ///

[16.12] Remark: Thus, the product topology must be *fine enough* so that inverse images $p_\alpha(U_\alpha)$ in X of projections $p_\alpha : X \to X_\alpha$ are open, and *coarse enough* so that inverse images $f^{-1}(U)$ in Z of induced maps $f : Z \to X$ are open.

[16.13] Remark: The *main* issue here is *existence* of any topology at all that will work as a product. By contrast, the *uniqueness* of products (if they exist) was proven earlier, in a standard (even clichéd) mapping-property fashion. That is, the uniqueness *up to unique isomorphism* asserts that if X and Y were two products, they must be (uniquely) homeomorphic.

[16.14] Remark: To repeat: the question of *existence* of a product topology can be viewed as being the question of whether or not the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$ might accidentally be *too fine* for all induced maps $Z \to X$ to be continuous. That is, the continuity of the projections and the continuity of the induced maps $Z \to X$ are *opposing* constraints. The for-general-reasons uniqueness tells us *a priori* that there is *at most one* simultaneous solution, so the question is whether or not there is *any*. In this case, it turns out that these conflicting constraints *do* allow a common solution. Still, it does happen in other circumstances that two opposing conditions allow *no* simultaneous satisfaction.

17. *Recap: some important function spaces and metrics*

All functions here are real-valued or complex-valued.

For a topological space X, $C^o(X)$ is the space of continuous functions on X. Some sources omit the superscript. For *compact* X, one natural norm on $C^o(X)$ is the sup-norm

$$|f|_\infty \;=\; |f|_{\sup} \;=\; |f|_{C^o} \;=\; \sup_{x\in X} |f(x)|$$

and we have seen that $C^o(X)$ is *complete* for the corresponding metric.

For not-necessarily-compact X, $C^o_{\rm bdd}(X)$ is the space of continuous *bounded* functions on X, with sup-norm.

For not-necessarily-compact X, $C^o_c(X)$ is the space of continuous functions on X *with compact support*, with sup-norm. As usual, the *support* of a function is the *closure* of the subset of X on which the function is non-zero.

For not-necessarily-compact X, $C^o_o(X)$ is the space of continuous functions on X *going to* 0 *at infinity*, with sup-norm. Going to zero at infinity means that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that, for all $x \notin K$, $|f(x)| < \varepsilon$.

For $k \in \{0, 1, 2, \ldots\}$, and for X admitting a *differentiable structure*, as do intervals $[a, b]$, spaces $\mathbb{R}^n$, and so on, $C^k(X)$ is the space of k-times continuously differentiable functions, and $C^\infty(X)$ is the space of *smooth* functions, that is, infinitely or indefinitely continuously differentiable.

For $X = [a, b]$, $C^k(X)$ has a good natural norm

$$|f|_{C^k} \;=\; |f|_{C^o} + |f'|_{C^o} + |f''|_{C^o} + \ldots + |f^{(k)}|_{C^o}$$

It takes a bit of work, but it is *complete*.

The space $C^\infty_c(\mathbb{R}^n$ of *test functions* on $\mathbb{R}^n$ is the space of compactly supported smooth functions. The topology that makes it *quasi-complete* is more complicated than most of the other function spaces mentioned here, and will be discussed later.

The L^2-metric, for example on $C^o[a, b]$, is

$$|f|_{L^2} \;=\; \sqrt{\int_a^b |f(x)|^2_{\mathbb{C}}\, dx}$$

More generally, for $1 \le p < +\infty$, the L^p-metric on $C^o[a, b]$ is

$$|f|_{L^p} \;=\; \sqrt[p]{\int_a^b |f(x)|^2_{\mathbb{C}}\, dx}$$

In a different vein,

$$\ell^2 \;=\; \{(c_1, c_2, \ldots) : \text{with } c_i \in \mathbb{C} \text{ and } \sum_n |c_n|^2 < +\infty\}$$

and the norm is $|(c_1, \ldots)|_{\ell^2} = \sqrt{|c_n|^2}$. Similarly, for $1 \le p < \infty$, we obtain the spaces ℓ^p by replacing squares and square roots by p and p^{th} roots. As a limiting case,

$$\ell^\infty \;=\; \{(c_1, c_2, \ldots) : \text{with } c_i \in \mathbb{C} \text{ and } \sup |c_n| < +\infty\}$$

with norm $|(c_1, \ldots)|_{\ell^\infty} = \sup_n |c_n|$.

[17.1] Remark: This is not an exhaustive list!.

[17.2] Remark: In some cases, such as the L^p spaces, in the present context we put the metric on *continuous* functions only because we know very well how to integrate them.
