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03. Measure and integral

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1. *Borel-measurable functions and pointwise limits*

Pointwise limits of continuous functions on $\mathbb{R}$ or on intervals $[a, b]$ need not be continuous. We want a class of functions closed under taking pointwise limits of sequences. The following is the simplest form of a general discussion.

The collection of *Borel subsets* of $\mathbb{R}$ is the smallest collection of subsets of $\mathbb{R}$ closed under taking *countable unions*, under *countable intersections*, under *complements*, and containing all open and closed subsets of $\mathbb{R}$. This is also called the Borel *σ-algebra* in $\mathbb{R}$. We must check that this description makes sense, in the claim below.

More generally, in *any* set X, a σ-algebra is a set A of subsets of X, including ϕ and X, and so that countable unions and countable intersections of elements of A are again in A. Note that X need not have a topology or metric or any other structure for this notion of σ-algebra to make sense.

The following is the analogue for σ-algebras of the analogous assertion for groups and subgroups, and many other situations.

[1.1] Claim: Let X be an arbitrary non-empty set. Intersections of σ-algebras of subsets of X are σ-algebras. Thus, the *smallest* σ-algebra containing a given set of sets is the intersection of all σ-algebras containing it.

Proof: Let S be a set of subsets of a set X, and $\{A_i : i \in I\}$ a collection of σ-algebras containing S. Let A be the intersection $\bigcap_i A_i$. Given a countable collection $E_1, E_2, \ldots$ of sets in A, for every $i \in I$ the set E_j are in A_i, so their intersection and union are in A_i. Since this holds for every $i \in I$, that intersection and union are in A. The argument for complements is even simpler. ///

There is traditional terminology for certain simple types of Borel sets. For example a *countable intersection of open sets* is a G_δ set, while a *countable union of closed sets* is an F_σ. The notation can be iterated: a $G_{\delta\sigma}$ is a countable union of countable intersections of opens, and so on. We will not need this.

A simple useful choice of larger class of functions than continuous is: a real-valued or complex-valued function f on $\mathbb{R}$ is *Borel-measurable* when the inverse image $f^{-1}(U)$ is a Borel set for every open set U in the target space.

First, we verify some immediate desirable properties:

[1.2] Claim: The sum and product of two Borel-measurable functions are Borel-measurable. For non-vanishing Borel-measurable f, $1/f$ is Borel-measurable.

Proof: As a warm-up to this argument, it is useful to rewrite the $\varepsilon-\delta$ proof, that the sum of two continuous functions is continuous, in terms of the condition that inverse images of opens are open.

For Borel-measurable f,g on $\mathbb{R}$, let $f\oplus g$ be the $\mathbb{R}\times\mathbb{R}$-valued function on $\mathbb{R}\times\mathbb{R}$ defined by $(f\oplus g)(x,y) = (f(x),g(y))$. Let $s:\mathbb{R}\times\mathbb{R}$ be the sum map, $s(x,y)=x+y$. Let $\Delta:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ be the diagonal map $\Delta(x)=(x,x)$. Both s and Δ are continuous, and

$$(f+g)^{-1} \;=\; \Delta^{-1}\circ(f\oplus g)^{-1}\circ s^{-1}$$

Since s is continuous, for open $U\subset\mathbb{R}$, $s^{-1}(U)$ is open in $\mathbb{R}\times\mathbb{R}$, and is a countable union of open rectangles $(a_i,b_i)\times(c_i,d_i)$. Then

$$(f\oplus g)^{-1}(s^{-1}(U)) \;=\; \bigcup_i (f\oplus g)^{-1}((a_i,b_i)\times(c_i,d_i)) \;=\; \bigcup_i f^{-1}(a_i,b_i)\times g^{-1}(c_i,d_i)$$

and every inverse image $f^{-1}(a_i,b_i)$ and $g^{-1}(c_i,d_i)$ is *Borel measurable.* Then

$$\Delta^{-1}\Big(f^{-1}(a_i,b_i)\times g^{-1}(c_i,d_i)\Big) \;=\; f^{-1}(a_i,b_i)\cap g^{-1}(c_i,d_i)\Big) \;=\; \text{(Borel measurable)}$$

The countable union indexed by i is still Borel-measurable, so $(f+g)^{-1}(U)$ is measurable. The arguments for product and inverse are nearly identical, since product and inverse (away from 0) are continuous. ///

It is sometimes useful to allow the target space for functions to be the *two-point compactification* $Y = \{-\infty\}\cup\mathbb{R}\cup+\infty$ of the real line, with neighborhood basis $-\infty\cup(-\infty,a)$ at $-\infty$ and $(a,+\infty)\cup\{+\infty\}$ at $+\infty$ when we need to allow functions to blow up in some fashion. But $\pm\infty$ are not numbers, and do not admit consistent manipulation as though they were.

A more serious positive indicator of the reasonable-ness of Borel-measurable functions as a larger class containing continuous functions:

[1.3] Theorem: Every pointwise limit of Borel-measurable functions is Borel-measurable. More generally, every countable *inf* and countable *sup* of Borel-measurable functions is Borel-measurable, as is every countable *liminf* and *limsup.*

Proof: We prove that a countable $f(x)=\inf_n f_n(x)$ is measurable. Observe that $f(x)<b$ if and only if there is some n such that $f_n(x)<b$. Thus,

$$f^{-1}(-\infty,b) \;=\; \bigcup_n f_n^{-1}(-\infty,b) \;=\; \text{(countable union of measurables)} \;=\; \text{(measurable)}$$

Further,

$$f^{-1}(-\infty,a] \;=\; \bigcap_n f^{-1}(-\infty,a+\frac{1}{n}) \;=\; \text{(countable intersection of measurables)} \;=\; \text{(measurable)}$$

and then

$$f^{-1}(a,b) \;=\; f^{-1}(-\infty,b)-f^{-1}(-\infty,a] \;=\; f^{-1}(-\infty,b)\cap(\mathbb{R}-f^{-1}(-\infty,a])$$

$$=\; \text{(intersection of measurable with complement of measurable)} \;=\; \text{(measurable)}$$

A nearly identical argument proves measurability of countable *sup*s of measurable functions.

A slight enhancement of this argument treats *liminfs* and *limsups*: $\limsup_n f_n(x) < b$ if and only if, for all n_o, there is $n \geq n_o$ such that $f_n(x) < b$:

$$\{x : \liminf_n f_n(x) < b\} \;=\; \bigcap_{n_o \geq 1} \Big(\bigcup_{n \geq n_o} f_n^{-1}(-\infty, b) \Big)$$

$$= \text{(countable intersection of countable unions of measurables)} \;=\; \text{(measurable)}$$

The rest of the argument for measurability of pointwise *liminfs* is identical to that for *infs*, and also for *limsups*. When pointwise $\lim_n f_n(x)$ exists, it is $\liminf_n f_n(x)$, showing that countable limits of measurable are measurable. ///

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2. *Lebesgue-measurable functions and almost-everywhere pointwise limits*

A sequence $\{f_n\}$ of Borel-measurable functions on $\mathbb{R}$ converges (pointwise) *almost everywhere* when there is a Borel set $N \subset \mathbb{R}$ of measure 0 such that $\{f_n\}$ converges pointwise on $\mathbb{R} - N$. One of Lebesgue's discoveries was that ignoring what may happen on sets of measure zero was an essential simplifying point in many situations.

However, there are sets of Lebesgue measure 0 that are not Borel sets. Thus, *almost-everywhere* pointwise limits of Borel-measurable functions may fall into a larger class. That is, there is a larger σ-algebra than that of Borel sets. Indeed, the description of the Lebesgue (outer) measure suggests that *any subset F of a Borel set E of measure zero should itself be measurable, with measure zero.*

The smallest σ-algebra containing all Borel sets in $\mathbb{R}$ and containing all subsets of Lebesgue-measure-zero Borel sets is the σ-algebra of *Lebesgue-measurable* sets in $\mathbb{R}$.

[2.1] Claim: Finite sums, finite products, and inverses (of non-zero) Lebesgue-measurable functions are Lebesgue-measurable.

Proof: The proofs in the previous section did not use any specifics of the σ-algebra of Borel sets, so the same proofs succeed. ///

[2.2] Theorem: Every pointwise-almost-everywhere limit of Lebesgue-measurable functions f_n is Lebesgue-measurable.

Proof: Again, the proofs in the previous section did not use any specifics of the σ-algebra of Borel sets. ///

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3. *Borel measures*

A *Borel measure* μ is an assignment of (often *non-negative*) real numbers $\mu(E)$ (measures) to Borel sets E, in a fashion that is *countably additive* for disjoint unions:

$$\mu(E_1 \cup E_2 \cup E_3 \cup \ldots) \;=\; \mu(E_1) + \mu(E_2) + \mu(E_3) + \ldots \qquad \text{(for } \textit{disjoint} \text{ Borel sets } E_1, E_2, E_3, \ldots)$$

The most important prototype of a Borel measure is *Lebesgue (outer) measure* of a Borel set $E \subset \mathbb{R}$, described by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

That is, it is the *inf* of the sums of lengths of the intervals in a countable cover of E by open intervals. For example, any countable set has (Lebesgue) measure 0.

That is, there is a σ-algebra A including Borel sets (equivalently, including open sets), and μ is a (often non-negative real-valued) function on A with the countable additivity above.

[... iou ...]

[3.1] Remark: Assuming the Axiom of Choice, one can prove that there is no Borel measure μ with σ-algebra containing *all* subsets of $\mathbb{R}$. So our ambitions for assigning measures should be more modest.

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4. *Lebesgue integrals*

With such notion of *measure*, there is a corresponding *integrability* and *integral*, due to Lebesgue. It amounts to replacing the literal rectangles used in Riemann integration by more general rectangles, with bases not just intervals, but measurable sets, as follows.

The *characteristic function* or *indicator function* ch_E or χ_E of a measurable subset $E \subset \mathbb{R}$ is 1 on E and 0 off. A *simple function* is a finite, positive-coefficiented, linear combination of characteristic functions of bounded measurable sets, that is, is of the form

$$\text{(simple function) } s \;=\; \sum_{i=1}^{n} c_i \cdot \mathrm{ch}_{E_i} \qquad \text{(with } c_i \ge 0)$$

The *integral* of s is what one would expect:

$$\int s\, d\mu \;=\; \int \Big(\sum_{i=1}^{n} c_i \cdot \mathrm{ch}_{E_i}\Big)\, d\mu \;=\; \sum_i c_i \cdot \mu(E_i)$$

Next, the integral of a *non-negative* function f is the *sup* of the integrals of all simple functions between f and 0:

$$\int f\, d\mu \;=\; \sup_{0 \le s \le f} \int s\, d\mu \qquad \text{(sup over simple } s \text{ with } 0 \le s(x) \le f(x) \text{ for all } x)$$

After proving that the positive and negative parts f_+ and f_- of Borel measurable real-valued f are again Borel measurable,

$$\int f\, d\mu \;=\; \int f_+\, d\mu - \int (-f_-)\, d\mu$$

Similarly, for complex-valued f, break f into real and imaginary parts.

There are details to be checked:

[4.1] Theorem: Borel-measurable functions f, g taking values in $[0, +\infty]$ are *integrable*, in the sense that the previous prescription yields an assignment $f \to \int_{\mathbb{R}} f \in [0, +\infty]$ such that for positive constants a, b

$$\int_{\mathbb{R}} (af + bg) \;=\; a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g \qquad \text{(for all } a, b \ge 0)$$

For complex-valued Borel-measurable f, g, the absolute values $|f|$ and $|g|$ are Borel-measurable. Assuming $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |g| < \infty$, for any complex a, b

$$\int_{\mathbb{R}} (af + bg) \;=\; a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$$

Proof: [... iou ...] ///

For a Borel-measurable function f on $\mathbb{R}$ and Borel-measurable set $E \subset \mathbb{R}$, the *integral of f over E* is

$$\int_E f \;=\; \int_{\mathbb{R}} \mathrm{ch}_E \cdot f$$

where ch_E is the characteristic function of E.

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5. *Abstract integration, abstract measure spaces*

An elementary but fundamental result is

[5.1] Proposition: Let f be a $[0, +\infty]$-valued measurable function on X. Then there are simple functions $s_1, s_2, s_3, \ldots$ *with non-negative real coefficients* so that for all $x \in X$, $s_1(x) \le s_2(x) \le s_3(x) \le \ldots \le f(x)$, and for all $x \in X$, $\lim_n s_n(x) = f(x)$.

Note: Some authors distinguish between *positive* measures and *complex* measures, where the distinction is meant to be that the former are $[0, \infty]$-valued, while the latter are constrained to assume only *'finite'* complex values.

The *integral of a characteristic function* χ_E is taken to be simply

$$\int_X \chi_E \, d\mu = \mu(E)$$

Then the *integral of a simple function*

$$s(x) = \sum_{1 \le i \le n} c_i \chi_{E_i}$$

(with $c_i \ge 0$) is defined to be

$$\int_X \sum_{1 \le i \le n} c_i \chi_{E_i} = \sum_{1 \le i \le n} c_i \int_X \chi_{E_i} \, d\mu = \sum_{1 \le i \le n} c_i \int_X \mu E_i$$

For a $[0, +\infty]$-valued function f, we write

$$0 \le s \le f$$

for a *simple* function s if s has *non-negative real* coefficients, and if for all $x \in X$

$$0 \le s(x) \le f(x)$$

Then the *Lebesgue integral* of f is defined to be

$$\int_X f \, d\mu = \sup_{s : 0 \le s \le f} \int_X s \, d\mu$$

Note that at this point we can only integrate *non-negative real-valued* functions.

The standard space

$$L^1(X, \mu) = \{\text{complex-valued measurable } f \text{ so that } \textstyle\int_X |f| \, d\mu < \infty\}$$

Since $|f|$ is non-negative real-valued, we can indeed make sense of this. This is the collection of *integrable* functions f. Then write

$$f(x) = u(x) + iv(x)$$

where both u, v are real-valued, and write

$$u = u_+ - u_- \qquad v = v_+ - v_-$$

where u_+, v_+ are the 'positive parts' and where u_-, v_- are the 'negative parts' of these functions. Define the *Lebesgue integral*

$$\int_X f \, d\mu = \int_X u_+ \, d\mu - \int_X u_- \, d\mu + i \int_X v_+ \, d\mu - i \int_X v_- \, d\mu$$

Then we have to check that this definition, in terms of integrals of non-negative functions, really has the presumed properties. It is in proving such that we need the *integrability*.

For brevity, when there is no chance of confusion we will often simply write

$$\int_X f$$

rather than either of

$$\int_X f \, d\mu, \qquad \int_X f(x) \, d\mu(x)$$

for the integral of f on the measure space X with respect to the measure μ.

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6. *Convergence theorems: Fatou, Lebesgue monotone, Lebesgue dominated*

Easy, natural examples show that *pointwise* limits $f = \lim_n f_n$ of measurable functions f_n, while still measurable, need *not* satisfy $\int f = \lim \int f_n$. That is, this failure is not a pathology, but, rather, is completely reasonable. Hence additional conditions are essential to know that the integral of a pointwise limit is the limit of the integrals.

First, a relatively simple initial step:

[6.1] Theorem: *(Fatou's lemma)* For Borel-measurable f_n with values in $[0, +\infty]$, the pointwise $f(x) = \liminf_n f_n(x)$ is Borel-measurable, and

$$\int \liminf_n f_n(x) \, dx \;\le\; \liminf_n \int f_n$$

Proof: [... iou ...] ///

[6.2] Theorem: *(Lebesgue: monotone convergence)* Let $f_1, f_2, \ldots$ be a sequence of non-negative real-valued Lebesgue-measurable functions on $[a, b]$, with $f_1(x) \le f_2(x) \le \ldots$ for all x. Then $\int_a^b \lim_n f_n(x) \, dx = \lim_n \int_a^b f_n(x) \, dx$. This includes the possibility that some of the limits of the pointwise values are $+\infty$, and that the integral of the limit is $+\infty$.

Proof: [... iou ...] ///

[6.3] Theorem: *(Lebesgue: dominated convergence)* Let $f_1, f_2, \ldots$ be a sequence of complex-valued Lebesgue-measurable functions on $[a, b]$, with $|f_n(x)| \le g(x)$ for all x, for some measurable g with $\int_a^b g(x) \, dx < +\infty$. Then $\int_a^b \lim_n f_n(x) \, dx = \lim_n \int_a^b f_n(x) \, dx$.

Proof: [... iou ...] ///

7. *Completions of measures*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*, in the sense that they are countable unions of finite-measure sets.

First, the *product* σ-algebra is the σ-algebra in $X \times Y$ generated by all products $E \times F$ with $E \in A$ and $F \in B$.

For *iterated integrals* to make sense, we need to check a few things. For $E \in A \times B$, for $x \in X$ and $y \in Y$, let

$$E_x \;=\; \{y \in Y : (x, y) \in E\} \qquad \text{and} \qquad E^y \;=\; \{x \in X : (x, y) \in E\}$$

As a consistency check, we have

[7.1] Theorem: For $E \in A \times B$, for $x \in X$ and $y \in Y$, $E_x \in A$ and $E^y \in B$. The function $x \to \nu(E_x)$ is μ-measurable, $y \to \mu(E^y)$ is ν-measurable, and

$$\int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

Proof: [... iou ...] ///

Then the *product measure* $\mu \times \nu$ can be defined in the expected fashion: for $E \in A \times B$,

$$(\mu \times \nu)(E) \;=\; \int_X \nu(E_x)\, d\mu(x) \;=\; \int_Y \mu(E^y)\, d\nu(y)$$

8. *Fubini-Tonelli theorem(s)*

Let X, μ and Y, ν be measure spaces with corresponding σ-algebras A, B. Assume X and Y are σ-*finite*.

[8.1] Theorem: *(Fubini-Tonelli)* For complex-valued measurable f, g, if any one of

$$\int_X \int_Y |f(x,y)|\, d\mu(x)\, d\nu(y) \qquad \int_Y \int_X |f(x,y)|\, d\nu(y)\, d\mu(x) \qquad \int_{X \times Y} |f(x,y)|\, d\mu{\times}\nu$$

is finite, then they *all* are finite, and are equal. For $[0, +\infty]$-valued functions f,

$$\int_X \int_Y f(x,y)\, d\mu(x)\, d\nu(y) \;=\; \int_Y \int_X f(x,y)\, d\nu(y)\, d\mu(x) \;=\; \int_{X \times Y} f(x,y)\, d\mu{\times}\nu$$

although the values may be $+\infty$.

Proof: [... iou ...] ///

For proof of the theorem, we need the notion of *monotone class*. A monotone class in a set X is a set $\mathcal{M}$ of subsets of X closed under countable ascending unions and under countable descending intersections. That is, if

$$M_1 \subset M_2 \subset M_3 \subset \ldots$$

$$N_1 \supset N_2 \supset N_3 \supset \ldots$$

are collections of sets in $\mathcal{M}$, then

$$\bigcup_i M_i \qquad \bigcap_i N_i$$

both lie in $\mathcal{M}$, as well. Another characterization of $\mathcal{A}\times\mathcal{B}$ is that it is the smallest monotone class containing all products $E \times F$ with $E \in \mathcal{A}$ and $F \in \mathcal{B}$.

Let f be a $\mathcal{A}\times\mathcal{B}$-measurable function on $X \times Y$. (Note that this does not depend upon having a 'product measure', but only upon the sigma-algebra!) Then all the functions

$$x \to f(x,y) \qquad \text{(for fixed } y \in Y$$

$$y \to f(x,y) \qquad \text{(for fixed } x \in X$$

are measurable (in appropriate senses). In particular, we could apply this to the *characteristic function* of a set $G \in \mathcal{A}\times\mathcal{B}$.

Now we come to the point where the sigma-finiteness of X and Y is necessary. For $G \in \mathcal{A}\times\mathcal{B}$, let

$$f(x) = \nu(G_x) \qquad g(y) = \mu(G_y)$$

where G_x, G_y are as above. We have already noted that f, g are *measurable*. Further,

$$\int_X f(x)\; d\mu(x) = \int_Y g(y)\; d\nu(y)$$

This is proven by showing that the collection of G for which the conclusion is true is a *monotone class* containing all products $E \times F$.

In light of the latter equality, we can define the *product measure* $\mu \times \nu$ on $G \in \mathcal{A}\times\mathcal{B}$ by

$$(\mu\times\nu)(G) = \int_X f(x)\; d\mu(x) = \int_Y g(y)\; d\nu(y)$$

with notation as just above. The *countable additivity* follows from a preliminary version of Fubini's theorem, namely that if f_i are countably-many $[0, +\infty]$-valued functions on a measure space Ω, then

$$\int_\Omega \sum_i f_i = \sum_i \int_\Omega f_i$$

which itself is a little corollary of the monotone convergence theorem.

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9. *Completions of measures*

For example, a reasonable measure on $\mathbb{R}^m \times \mathbb{R}^n$ should include many sets not expressible as countable unions of products $E \times F$ where $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$. For example, diagonal subsets of the form $D = \{(x,x) : 0 \le x \le 1\} \subset \mathbb{R}^2$ are not countable unions of products, but should surely be measurable.

One way to accomplish this is by *completion* of the product measure.

Then the *completion* of $\mu\times\nu$ further assigns measure 0 to *any* subset S of $T \in A \times B$ with $(\mu\times\nu)(T) = 0$, and adjoins all such sets to the σ-algebra $A \times B$.

[9.1] Claim: Lebesgue measure on $\mathbb{R}^m \times \mathbb{R}^n$ is the completion of the product of Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$.

Proof: [... *iou* ...] ///

Completing a product measure is usually what we want, but it slightly complicates the statement of the corresponding Fubini-Tonelli theorem:

[9.2] Theorem: Let X, A, μ and Y, B, ν be *complete* measure spaces, with X, Y σ-finite. Let f be a function on $X \times Y$ measurable with respect to the *completion* of the product measure. Then $x \to f(x,y)$ and $y \to f(x,y)$ are μ-measurable and ν-measurable (only) *almost everywhere*.

Proof: [... iou ...] ///

[9.3] Remark: To be precise, *completeness* is a property of σ-algebras, not of measures.

10. *Comparison to continuous functions: Lusin's theorem*

One aspect of the following theorem is that we have not inadvertently needlessly included functions wildly unrelated to continuous functions:

[10.1] Theorem: *(Lusin)* Continuous functions approximate Borel-measurable functions well: given Borel-measurable real-valued or complex-valued f on $\mathbb{R}$, for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a relative closed $E \subset \Omega$ such that $\mu(\Omega - E) < \varepsilon$, and $f|_E$ is *continuous*.

Proof: [... iou ...] ///

Not much better can be done than Lusin's theorem says: for example, continuous approximations to the Heaviside step function

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

have to go from 0 to 1 *somewhere*, by the Intermediate Value Theorem, so will be in $(\frac{1}{4}, \frac{3}{4})$ on an open set of strictly positive measure.

[10.2] Remark: It turns out that the everyday use of measure theory, measurable functions, and so on, does *not* proceed by way of Lusin's theorem or similar direct connections with continuous functions, but, rather, by direct interaction with the more general ideas.

11. *Comparison to uniform pointwise convergence: Severini-Egoroff*

[11.1] Theorem: *(Severini, Egoroff)* Pointwise convergence of sequences of Borel-measurable functions is approximately *uniform* convergence: given a almost-everywhere pointwise-convergent sequence $\{f_n\}$ of Borel-measurable functions on $\mathbb{R}$, for every $\varepsilon > 0$ and for every Borel subset $\Omega \subset \mathbb{R}$ of finite Lebesgue measure, there is a Borel subset $E \subset \Omega$ such that $\{f_n\}$ converges *uniformly* pointwise on E.

Proof: [... iou ...] ///

[11.2] Remark: Despite the connection that the Severini-Egoroff theorem makes between pointwise and *uniform* pointwise convergence, this idea turns out *not* to be the way to understand convergence of measurable functions. Instead, the game becomes ascertaining additional conditions that guarantee convergence of integrals, as earlier.

12. *Lebesgue-Radon-Nikodym theorem*

Let μ, ν be two positive measures on a common sigma algebra $\mathcal{A}$ on a set X. Say that ν is *absolutely continuous* with respect to μ if $\mu(E) = 0$ implies $\nu(E) = 0$ for all measurable sets E. This is often written $\nu < \mu$. The measure μ is *supported on* or *concentrated on* a subset X_o of X if, for all measurable E,

$$\mu(E) = \mu(E \cap X_o)$$

The two measures μ, ν are *mutually singular* if μ is supported on X_1 and ν is supported on X_2 and $X_1 \cap X_2 = \emptyset$. This is often written $\mu \perp \nu$.

[12.1] Theorem: Theorem. Let μ, ν be positive measures on a common sigma-algebra $\mathcal{A}$ on a set X. There is a unique pair of positive measures ν_a and ν_s so that

$$\nu_a < \mu \qquad \nu_s \perp \mu$$

Further, there is $\varphi \in L^1(X, \mu)$ so that for any measurable set E

$$\nu_a(E) = \int_X \varphi \, d\mu$$

The function φ is the *Radon-Nikodym derivative* of ν_a with respect to μ, and is often written as

$$\varphi = \frac{d\nu_a}{d\mu}$$

The pair (ν_a, ν_s) is the *Lebesgue decomposition* of ν with respect to μ.