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03b. Completeness of L^p spaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

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1. Examples: spaces L^p

Given a measure space X, for $1 \le p < \infty$ the usual L^p spaces are

$$L^p(X) = \{\text{measurable } f : |f|_{L^p} < \infty\} \text{ modulo } \sim$$

with the usual L^p norm

$$|f|_{L^p} \;=\; \left(\int_X |f|^p\right)^{1/p}$$

and associated metric

$$d(f,g) \;=\; |f-g|_{L^p}$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure } 0$$

[1.1] Remark: For general measure spaces this is *not* a metric until we take the quotient, since, otherwise, two different functions differing only on a set of measure 0 would be distance 0 from each other, but would not be equal.

[1.2] Remark: These L^p functions have inevitably ambiguous pointwise values, in conflict with the naive formal definition of *function*. Nevertheless, one usually does think of L^p functions as being more-or-less functions.

A simple instance of this construction, for a measure that has no sets of measure 0, so needs no quotient, is

$$\ell^p \;=\; \{\text{complex sequences } \{c_i\} \text{ with } \sum_i |c_i|^p < \infty\}$$

with norm $|(c_1, c_2, \ldots)|_{\ell^p} = (\sum_i |c_i|^p)^{1/p}$. The analogue of the following theorem for ℓ^p is more elementary.

[1.3] Theorem: The space $L^p(X)$ is a complete metric space.

[1.4] Remark: In fact, as used in the proof, a Cauchy sequence f_i in $L^p(X)$ has a subsequence converging *pointwise* off a set of measure 0 in X.

Proof: The triangle inequality here is *Minkowski's inequality.* To prove completeness, choose a subsequence f_{n_i} such that

$$|f_{n_i} - f_{n_{i+1}}|_p \;<\; 2^{-i}$$

and put

$$g_n(x) \;=\; \sum_{1 \le i \le n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) \;=\; \sum_{1 \le i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The infinite sum is not necessarily claimed to converge to a finite value for every x. The triangle inequality shows that $|g_n|_p \le 1$. *Fatou's Lemma* asserts that for $[0,\infty]$-valued measurable functions h_i

$$\int_X \Big(\liminf_i h_i\Big) \;\le\; \liminf_i \int_X h_i$$

Thus, $|g|_p \le 1$, so is finite. Thus,

$$f_{n_1}(x) + \sum_{i\ge 1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges for almost all $x \in X$. Let $f(x)$ be the sum at points x where the series converges, and on the measure-zero set where the series does not converge put $f(x) = 0$. Certainly

$$f(x) \;=\; \lim_i f_{n_i}(x) \quad \text{(for almost all } x)$$

Now prove that this almost-everywhere pointwise limit is the L^p-limit of the original sequence. For $\varepsilon > 0$ take N such that $|f_m - f_n|_p < \varepsilon$ for $m, n \ge N$. Fatou's lemma gives

$$\int |f - f_n|^p \le \liminf_i \int |f_{n_i} - f_n|^p \le \varepsilon^p$$

Thus $f - f_n$ is in L^p and hence f is in L^p. And $|f - f_n|_p \to 0$. ///

[1.5] Theorem: For a locally compact Hausdorff topological space X with positive regular Borel measure μ, the space $C_c^0(X)$ of compactly-supported continuous functions is *dense* in $L^1(X, \mu)$.

Proof: From the definition of *integral* attached to a measure, an L^1 function is approximable in the L^1-metric by a *simple* function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a *finite* linear combination of characteristic functions of measurable sets E. Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed *regularity* of the measure gives compact K and open U such that $K \subset E \subset U$ and $\mu(U - E) < \varepsilon$, for given $\varepsilon > 0$. Urysohn's lemma says that there is continuous f identically 1 on K and identically 0 off U. Thus, f approximates the characteristic function χ_E of E in L^1:

$$|f - \chi_E|_{L^1} \;=\; \int_X |f - \chi_E| \;=\; \int_{U-K} |f - \chi_E| \;\le\; \int_{U-K} 1 \;<\; \varepsilon$$

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[1.6] Corollary: For locally compact Hausdorff X with regular Borel measure μ, $L^1(X, \mu)$ is the L^1-metric completion of $C_c^o(X)$, the compactly-supported continuous functions. ///

[1.7] Remark: *Defining* $L^1(X, \mu)$ to be the L^p completion of $C_c^o(X)$ avoids discussion of ambiguous values on sets of measure zero, but also leaves ambiguity about in what sense the completion consists of *functions.*
