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04. Riesz-Markov-Kakutani theorem

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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1. *Regularity of measures*

A *Borel* measure μ on (a σ-algebra A in) a topological space X is *inner regular* when, for every $E \in A$

$$\mu(E) \;=\; \sup_{\text{compact } K\subset E} \mu(K)$$

The *Borel* measure μ is *outer regular* when, for every $E \in A$

$$\mu(E) \;=\; \inf_{\text{open } U\supset E} \mu(U)$$

The measure μ is *regular* when it is *both* inner and outer regular.

Although exploration of potential pathologies obviously has some interest, for applications we are almost entirely interested in regular Borel measures.

Again, a measure μ, or, really, also the σ-algebra A on which it's defined, is *complete* when $\mu(E) = 0$ implies that every subset of E is A, μ-measurable, with measure 0. In fact, it seems that our intuitive notion of *measure* implicitly includes completeness. However, it does not do so formally.

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2. *Riesz-Markov-Kakutani theorem*

Let X be a locally compact, Hausdorff, topological space. Further, suppose X is σ-compact, in the sense that it is a countable union of compact subsets.

A map $f \to \lambda(f)$ of continuous, compactly supported functions $C_c^o(X)$ to scalars is *positive* when $\lambda(f) \ge 0$ for $f \in C_c^o(X)$ taking values in $[0, +\infty)$.

[2.1] Theorem: *(Riesz, Markov, Kakutani, independently)* Given a positive functional λ on $C_c^o(X)$, there is a σ-algebra A containing all Borel sets, and a unique positive, complete, regular Borel measure μ on A, such that

$$\lambda(f) \;=\; \int_X f(x)\, d\mu(x) \qquad\qquad (\text{for all } f \in C_c^o(X))$$

Proof: (Standard... *[... iou ...]*) ///

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3. *Lebesgue measure reprise*

As a corollary of the Riesz-Markov-Kakutani theorem we have a different description of the Lebesgue measure and integral, as an extension of the Riemann integral, with the very useful side effect of proving inner and outer regularity.

In the Riesz-Markov-Kakutani theorem, take $X = \mathbb{R}^n$, and $\lambda(f)$ to be the usual Riemann integral for $f \in C_c^o(\mathbb{R}^n)$, and let Lebesgue measure be the associated *positive, regular, Borel* measure. With this description of Lebesgue measure, as opposed to the more tangible (but also more awkward) Lebesgue outer measure, we must verify that all the expected properties do hold.

[3.1] Corollary: Let μ be Lebesgue measure, induced by the Riesz-Markov-Kakutani theorem from the Riemann integral on $C_c^o(\mathbb{R}^n)$.
• μ is *translation-invariant* in the sense that $\mu(E + x) = \mu(E)$ for all $x \in \mathbb{R}^n$.
• The Lebesgue measure of a cube $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is the product $\prod_i |b_i - a_i|$, and similarly for closed and half-open intervals and their products.

Proof: (Standard... *[... iou ...]*) ///
