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06c. Introduction to convolutions

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/06c_intro_convolutions.pdf]

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1. Convolution on $L^1(\mathbb{R}^n)$

The formulaic definition of *convolution* of $f, g \in L^1(\mathbb{R}^n)$ is as a pointwise (a.e.) function

$$(f*g)(x) \;=\; \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

For each fixed x, it is not *a priori* clear that the integral converges. If $f, g \in L^2$, then Cauchy-Schwarz-Bunyakowsky could be invoked to show that the integral converges absolutely, but on $\mathbb{R}^n$ there are L^1 functions that are not L^2. So we need

[1.1] Claim: For $f, g \in L^1(\mathbb{R}^n)$, $f*g \in L^1(\mathbb{R}^n)$, and

$$|f*g|_{L^1} \;\le\; |f|_{L^1} \cdot |g|_{L^1} \qquad\qquad \text{(almost everywhere)}$$

Proof: Attempting to compute the L^1 norm of $f*g$,

$$\int_{\mathbb{R}^n} |(f*g)(x)|\, dx \;=\; \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \Big|\, dx \;\le\; \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy\, dx$$

By Fubini-Tonelli, if we allow the possible value $+\infty$ for the latter iterated integral, we can change the order of integration, giving

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dx\, dy$$

Then this allows a change of variables, replacing x by $x+y$, obtaining

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)|\, |g(y)|\, dx\, dy \;=\; \int_{\mathbb{R}^n} |f(x)\, dx \cdot \int_{\mathbb{R}^n} |g(y)|\, dy \;=\; |f|_{L^1} \cdot |g|_{L^1}$$

as claimed. ///

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2. $\widehat{f*g} = \widehat{f}*\widehat{g}$

The idea is that Fourier transform converts convolution to pointwise multiplication.

[2.1] Claim: For $f, g \in L^1(\mathbb{R}^n)$,

$$\widehat{f*g} \;=\; \widehat{f}*\widehat{g} \qquad\qquad \text{(pointwise everywhere)}$$

Indeed, from above, $f * g \in L^1$, so the Fourier transform integral converges absolutely. Also, $\widehat{f}$ and $\widehat{g}$ are in fact *continuous*, by Riemann-Lebesgue, so there is no ambiguity in talking about multiplication of point-wise values.

Proof: Computing directly,

$$\widehat{f * g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}(f*g)(x)\;dx \;=\; \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\int_{\mathbb{R}^n} f(x-y)\,g(y)\;dy\;dx$$

Since $f * g \in L^1$, by Fubini-Tonelli we can change the order of integration, and then replace x by $x+y$, to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i(x+y)\cdot\xi} f(x)\,g(y)\;dx\;dy \;=\; \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\;dx\cdot\int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi} f(y)\;dy \;=\; \widehat{f}(x)\cdot\widehat{g}(y)$$

as claimed. ///

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3. *Convolution of Schwartz functions*

Letting $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ for brevity,

[3.1] Claim: $f * g \in \mathscr{S}$ for $f, g \in \mathscr{S}$.

Proof: [... iou ...] ///

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4. *δ as unit in convolution algebras*

Without developing distribution theory, we can anticipate that in several useful circumstances the Dirac δ acts as an identity for convolution. (This will be discussed later, as well, in a more-developed context.)

With suitable hypotheses on f, for example that $f \in \mathscr{S}$,

[4.1] Claim: $\delta * f = f * \delta = f$ and $\delta' * f = f * \delta' = f'$.

Proof: [... iou ...] ///

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5. *Cautionary example*

Possibly disturbingly, *associativity* does not hold for arbitrary triples of *distributions*:

Let H be the Heaviside step function, 0 left of 0, and 1 right of 0. Since

$$(1*\delta')*H \;=\; 1'*H \;=\; 0*H \;=\; 0 \;\neq\; 1 \;=\; 1*\delta \;=\; 1*(\delta'*H)$$

associativity fails in this example. This is not a pathology, and is *not* a serious failing in notions of *generalized functions* (distributions).

Yes, in general we expect to extend classical operations to generalized functions. However, too-naive conceptions of classical operations can lead to difficulties, as this example illustrates.

Here, the underlying problem is that the bare formulaic notion of convolution does not explain its *function.* In reality, when (generalized) functions φ, ψ on $\mathbb{R}^n$ (for example) *act* (continuously) on a (topological) vector space V (possibly consisting of functions on $\mathbb{R}^n$ or elsewhere), denoted $\varphi \times v \to \varphi \cdot v$, *convolution* is the (uniquely determined, if it exists at all) operation on the acting functions such that

$$(\varphi * \psi) \cdot v \;=\; \varphi \cdot (\psi \cdot v)$$

In such a context, associativity of convolution is immediate.

Thus, whenever a more elementary, but formulaic, associativity for convolution (of various sorts of generalized functions) fails, we might suspect that there simply is no *action* of those generalized functions on reasonable topological vector spaces. This will be clarified later.

In contrast, one traditional attitude is to simply prove that when at least two out of three distributions α, β, γ is compactly supported, then

$$(\alpha * \beta) * \gamma \;=\; \alpha * (\beta * \gamma)$$

with a formulaic version of *convolution*. Yes, this can be proven, but does not explain why two-out-of-three, nor what the true obstacle might be.
