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Basic applications of Banach space ideas

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1. *A good trick using uniform boundedness*

The following sort of claim may seem nearly obviously true, but there is a missing key ingredient:

[1.1] Claim: Let $b = (b_1, b_2, \ldots)$ be a sequence of complex numbers such that $\sum_n b_n c_n$ is convergent for every $c = (c_1, c_2, \ldots) \in \ell^2$. Then $b \in \ell^2$.

Proof: Notably, the assumumption that the indicated sums are finite (convergent) does not directly give enough information to conclude that the map $\lambda(c) = \sum_n b_n c_n$ is a *continuous* linear functional on ℓ^2. The uniform boundedness theorem is needed to reach this conclusion.

Namely, let $\lambda_N(c) = \sum_{n \le N} b_n c_n$. These functionals *are* continuous on ℓ^2. By uniform boundedness, *either* there is a uniform bound $\beta < +\infty$ such that $\sup_N |\lambda_N(c)| \le \beta \cdot |c|$ for all $c \in \ell^2$, *or* there is a dense (hence, non-empty) G_δ such that $\sup_N |\lambda_N(c)|/|c| = +\infty$. But the assumption is that all the latter sups are finite. Thus, there must be a *uniform* bound, so $\lambda(c) = \sum_n b_n c_n$ *is* a continuous linear functional. By Riesz-Fréchet, it is given by an element of ℓ^2. ///

[1.2] Remark: If we know that the dual of L^p is L^q for σ-finite measure spaces X, then the same sort of argument applies.

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2. *Fourier series of C^o functions can diverge*

The *density* of finite Fourier series in $C^o(\mathbb{T})$ makes no claim about *which* finite Fourier series approach a given $f \in C^o(\mathbb{T})$. Indeed, the density proof given via the Féjer kernel uses finite Fourier series quite distinct from the finite partial sums of the Fourier series of f itself, namely,

$$N^{th} \text{ Féjer sum } = \frac{1}{N} \sum_{|n| \le N} (N - |n|) \cdot \widehat{f}(n) \cdot e^{2\pi i n x}$$

The Banach-Steinhaus/uniform-boundedness theorem has a decisive corollary about convergence failure of Fourier series of $C^o(\mathbb{T})$ functions:

[2.1] Corollary: There is $f \in C^o(\mathbb{T})$ whose Fourier series

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n)\, e^{inx} \qquad\qquad (\text{with } \widehat{f}(n) = \tfrac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx)$$

diverges at 0. In fact, the divergence can be arranged for a dense G_δ of continuous functions, and at any given countable set of points on $\mathbb{T}$.

Proof: To invoke Banach-Steinhaus, consider the functionals given by partial sums of the Fourier series of f, evaluated at 0:

$$\lambda_N(f) \;=\; \sum_{|n|\le N} \hat{f}(n) \;=\; \sum_{|n|\le N} \hat{f}(n)\cdot e^{2\pi i n\cdot 0}$$

There is an easy upper bound

$$|\lambda_N(f)| \;\le\; \int_0^1 \Big|\sum_{|n|\le N} e^{-2\pi i n x}\Big| \cdot \big|f(x)\big|\,dx \;\le\; |f|_{C^o}\cdot \int_0^1 \Big|\sum_{|n|\le N} e^{-2\pi i n x}\Big|\,dx \;=\; |f|_{C^o}\cdot \Big|\sum_{|n|\le N} e^{-2\pi i n x}\Big|_{L^1(\mathbb{T})}$$

We will show that equality holds, namely, that

$$|\lambda_N| \;=\; \Big|\sum_{|n|\le N} e^{-2\pi i n x}\Big|_{L^1}$$

and show that the latter L^1-norms go to ∞ as $N\to\infty$.

Summing the finite geometric series and rearranging:

$$\sum_{|n|\le N} e^{-2\pi i n x} \;=\; \frac{e^{-2\pi i N x} - e^{-2\pi i(-N-1)x}}{e^{-2\pi i x}-1} \;=\; \frac{e^{2\pi i(N+\frac{1}{2})x} - e^{-2\pi i(N+\frac{1}{2})x}}{e^{\pi i x}-e^{-\pi i x}} \;=\; \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin\frac{2\pi x}{2}}$$

The elementary inequality $|\sin t|\le|t|$ gives a lower bound

$$\int_0^1 \Big|\frac{\sin 2\pi(N+\frac{1}{2})x}{\sin\frac{2\pi x}{2}}\Big|\,dx \;\ge\; \int_0^1 \Big|\sin 2\pi(N+\tfrac{1}{2})x\Big|\cdot\frac{2}{2\pi x}\,dx \;=\; \int_0^{2\pi(N+\frac{1}{2})} |\sin x|\cdot\frac{2}{2\pi x}\,dx$$

$$\ge\; \sum_{\ell=1}^N \frac{1}{\pi\ell}\int_{2\pi(\ell-1)}^{2\pi\ell} |\sin x|\,dx \;\ge\; \sum_{\ell=1}^N \frac{1}{\pi\ell} \;\longrightarrow\; +\infty \qquad\qquad (\text{as } N\to\infty)$$

Thus, the L^1-norms do go to ∞.

We claim that the norm of the *functional* is the L^1-norm of the *kernel:* let $g(x)$ be the *sign* of the Dirichlet kernel

$$\sum_{|n|\le N} e^{-2\pi i n x} \;=\; \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin\frac{2\pi x}{2}}$$

Let g_j be a sequence of periodic continuous functions with $|g_j|\le 1$ and going to g pointwise. By *dominated convergence*

$$\lim_j \lambda_N(g_j) \;=\; \lim_j \int_0^1 g_j(x)\sum_{|n|\le N} e^{-2\pi i n x}\,dx \;=\; \int_0^1 g(x)\sum_{|n|\le N} e^{-2\pi i n x}\,dx \;=\; \int_0^1 |\sum_{|n|\le N} e^{-2\pi i n x}|\,dx$$

By Banach-Steinhaus for the Banach space $C^o(\mathbb{T})$, since (as demonstrated above) there is *no* uniform bound $|\lambda_N|\le M$ for all N, there *exists* f in the unit ball of $C^o(\mathbb{T})$ such that

$$\sup_N |\lambda_N\, v| \;=\; +\infty$$

In fact, the collection of such v is *dense* in the unit ball, and is an intersection of a *countable* collection of dense open sets (a G_δ). That is, the Fourier series of f does not converge at 0.

The result can be strengthened by using Baire's theorem again. For a dense countable set of points x_j in the interval, let $\lambda_{j,N}$ be the continuous linear functionals on $C^o(\mathbb{T})$ defined by evaluation of finite partial sums of the Fourier series at x_j's:

$$\lambda_{j,N}(f) \;=\; \sum_{|n|\le N} \hat{f}(n)\, e^{2\pi i n x_j}$$

As in the previous, the set E_j of functions f where

$$\sup_N |\lambda_{j,N} f| \;=\; +\infty$$

is a dense G_δ, so the intersection $E = \bigcap_j E_j$ is a dense G_δ, and, in particular, not empty. ///

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3. *Riemann-Lebesgue for* $f \to \widehat{f}$ *on* $L^1(\mathbb{T})$ *and* $L^1(\mathbb{R})$

The space c_o of two-sided sequences *vanishing at infinity* is

$$c_o \;=\; \{\{a_n : n\in\mathbb{Z}\} \;:\; \lim_{|n|\to\infty} a_n = 0\}$$

The space c_o is a Banach space with norm $|\{a_n\}|_{c_o} = \sup_n |a_n|$. Parametrizing the circle $\mathbb{T}$ by the interval $[0,1]$ by the exponential map $x \to e^{2\pi i x}$, the Banach space $L^1(\mathbb{T}) = L^1[0,1]$ is measurable functions f on $[0,1]$ with finite integrals $\int_0^1 |f|$ (modulo the equivalence relation of equality almost everywhere). The space $L^1[0,1]$ contains and is strictly larger than $L^2[0,1]$. On $L^2[0,1]$, Fourier transform is an isometry to $\ell^2(\mathbb{Z})$, by Parseval's theorem, and a relatively trivial form of a Riemann-Lebesgue lemma is that $\widehat{f} \in c_o$ for $f \in L^2[0,1]$. The version for L^1 is less trivial:

[3.1] Lemma: *(Riemann-Lebesgue)* $\widehat{f} \in c_o$ for $f \in L^1(\mathbb{T})$.

Proof: Finite linear combinations of exponentials are dense in $C^o(\mathbb{T})$, for example by Féjer's argument, and $C^o(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, essentially by the definition of integral and Urysohn's lemma. Thus, given $f \in L^1$ there is $g \in C^o(\mathbb{T})$ such that $|f-g|_{L^1} < \varepsilon$ and a finite linear combination h of exponentials such that $|g-h|_{C^o} < \varepsilon$. Then $|f-h|_{L^1} < 2\pi \cdot 2\varepsilon$.

Given such h, for large-enough n the Fourier coefficients are 0, by orthogonality of distinct exponentials. Thus,

$$|\widehat{f}(n)| \;=\; \frac{1}{2\pi}\Big| \int_0^{2\pi} (f(x)-h(x))\, e^{-inx}\, dx \Big| \;\le\; \frac{|f-h|_{L^1}}{2\pi} \;<\; 2\varepsilon \qquad \text{(for n large, depending on f)}$$

This proves this Riemann-Lebesgue Lemma. ///

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4. *Non-surjection of* $L^1[0,1] \to c_o$ *by* $f \to \widehat{f}$

Baire theorem and open mapping prove this.

[4.1] Corollary: *(of Baire and Open Mapping) Not* every sequence in c_o is the collection of Fourier coefficients of an $L^1(\mathbb{T})$ function.

Proof: The Fourier-coefficient map

$$Tf \;=\; \{\widehat{f}(n) : n \in \mathbb{Z}\} \;\in\; c_0$$

does map $L^1[0,1] \to c_o$, by Riemann-Lebesgue. The obvious inequality

$$|\widehat{f}(n)| \;=\; \Big|\int_0^1 f(x)\, e^{-2\pi i n x}\, dx\Big| \;\le\; \int_0^1 |f(x)|\, dx \;=\; |f|_{L^1}$$

shows $|T| \le 1$, so T is continuous. Taking $f(x) = 1$ shows $|T| = 1$.

The density of finite Fourier series in C^o and density of C^o in L^1, as in the proof of the Riemann-Lebesgue lemma, shows that T is injective. If T were also *surjective*, then the open mapping theorem would guarantee $\delta > 0$ such that for every L^1 function f

$$|\widehat{f}|_{\sup} \;\ge\; \delta \cdot |f|_{L^1}$$

However, this is impossible: with

$$f_N(x) \;=\; \sum_{|n|\le N} e^{-2\pi i n x}$$

the sup norm of $\widehat{f}_N$ is certainly 1, yet the computation about divergence of Fourier series above shows that the L^1 norm of f_N goes to ∞ like $\log N$ as $N \to +\infty$. Thus, there is no such $\delta > 0$. Thus, T cannot be surjective. ///

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5. $C^\infty(\mathbb{T})$ *is dense in* $C^o(\mathbb{T})$

Féjer's argument proves that the Cesaro-summed finite partial sums of Fourier series of a continuous function converge to that function in the $C^o(\mathbb{T})$ topology (that is, uniformly poinwise). These finite partial sums, as well as their Cesaro-summed forms, are in $C^\infty(\mathbb{T})$. Thus,

[5.1] Corollary: $C^\infty(\mathbb{T})$ is dense in $C^o(\mathbb{T})$. ///

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6. *Typical* C^o *functions are nowhere differentiable*

[6.1] Claim: In $C^o[a,b]$, there is (at least) a dense G_δ of functions which at every point fail to be differentiable.

Proof: Anticipating the application of Baire's theorem, we present everywhere-not-differentiable functions as a countable intersection of dense opens. First, for fixed large $n > 0$ and small $h \ne 0$, let

$$X_{n,h} \;=\; \{f \in C^o[a,b] : |f(x+h) - f(x)| > n \cdot |h|, \text{ for all } x \in [a,b] \text{ such that } x+h \in [a,b]\}$$

To show that $X_{n,h}$ is open, we observe that for a given $f \in X_{n,h}$, the function $|f(x+h) - f(x)| - n \cdot |h|$ is continuous in x, and is positive. Thus, since the function is continuous on the compact interval $[a,b]$, its inf is strictly positive. Thus, for g with $|g - f|_{C^o}$ sufficiently small, $|g(x+h) - g(x)| - n \cdot |h|$ is still positive. That is, $g \in X_{n,h}$.

Next, each union

$$Y_{n,h} \;=\; \bigcup_{h'\ne 0,\ |h'|<|h|} X_{n,h'}$$

$$=\; \{f \in C^o[a,b] : \text{for every } x \in [a,b], \text{ there is } 0 < h' < h \text{ such that } |f(x+h') - f(x)| > n \cdot |h'|\}$$

(where implicitly $x + h' \in [a,b]$) is a union of opens, so is open.

Density of $Y_{n,h}$ in $C^o[a,b]$ is that, for given $f \in C^o[a,b]$, there is $g \in Y_{n,h}$ near f. To prove this, first approximate f to within $\varepsilon > 0$ in sup norm by $g \in C^1[a,b]$. Among the several possible ways to do this, we choose the following. First, adjust f by subtracting a polynomial to make $f(a) = f(b)$. Extending f by periodicity, Féjer's Cesaro-summed version of the finite partial sums of its Fourier series converge to it in sup norm. These finite approximations are all C^∞, in fact, proving that we can approximate f to within $\varepsilon > 0$ in sup norm by a C^1 function g.

In particular, the derivative of g is a continuous function on $[a, b]$, so is *bounded* in absolute value, say by β.

Next, we use auxiliary piecewise-C^1 functions $\varphi_{N,\varepsilon}$ in $C^o[a, b]$ with sup norms less than a given $\varepsilon > 0$, but with absolute values of derivatives strictly greater than a given N, for any pair ε, N. For example, we can easily make piecewise-*linear* continuous functions $\varphi_{N,\varepsilon}$ with slopes $\pm(N + 1)$, changing sign so often that they stay strictly between $\pm\varepsilon$. For $N > \beta$, $g + \varphi_{2N,\varepsilon}$ is in $Y_{N,h}$ for all $h > 0$, and

$$|f - (g + \varphi_{2N,\varepsilon})|_{C^o} \;\le\; |f - g|_{C^o} + |\varphi_{2N,\varepsilon}|_{C^o} \;<\; \varepsilon + \varepsilon$$

This proves the density of every open $Y_{N,h}$ in $C^o[a, b]$.

By Baire's theorem, the countable intersection $\bigcap_{n=1,2,\ldots} Y_{n,\frac{1}{n}}$ of dense compacts is still dense. ///
