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08. *Introduction to generalized functions (distributions)*

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1. $\mathcal{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$

On $\mathbb{R}^n$ there are many useful spaces of functions. There are at least three distinct concepts of *very nice functions*:

$$\begin{cases} \text{test functions} & = & \mathcal{D}(\mathbb{R}^n) & = & C_c^\infty(\mathbb{R}^n) \;=\; \{f \in C^\infty(\mathbb{R}^n) : f \text{ has compact support}\} \\ \text{Schwartz functions} & = & \mathscr{S}(\mathbb{R}^n) & = & \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} (1+|x|)^m |f^{(\alpha)}(x)| < \infty \text{ for all } m, \alpha\} \\ \text{smooth functions} & = & \mathcal{E}(\mathbb{R}^n) & = & C^\infty(\mathbb{R}^n) \end{cases}$$

As usual, description of operationally good topologies (*complete* in some way) on these spaces of functions is necessary, otherwise we have no idea what limits stay in the space, or fall outside. Having acknowledged that debt, we can usefully continue.

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2. *Extensions/duals*

Here, *dual* V^* of a topological vector space V means *continuous dual*, which means the collection of *continuous* linear maps from V to scalars.

The duals of the above basic spaces of nice functions have names:

$$\begin{cases} \text{distributions} & = & \mathcal{D}(\mathbb{R}^n)^* & = & C_c^\infty(\mathbb{R}^n)^* \\ \text{tempered distributions} & = & \mathscr{S}(\mathbb{R}^n)^* & & \\ \text{compactly-supported distributions} & = & \mathcal{E}(\mathbb{R}^n)^* & = & C^\infty(\mathbb{R}^n)^* \end{cases}$$

Distributions are also called *generalized functions*, since their utility mostly lies in the possibility of interpreting them as *extensions* of the notion of *function*, rather than as being in dual spaces.

In general there is no natural continuous inclusion of a topological vector space V into its dual V^*, so V^* is in no natural sense an *extension* of V. In contrast, for $\mathcal{D}$ and $\mathscr{S}$, we have inclusions $\mathcal{D} \subset \mathcal{D}^*$ and $\mathscr{S} \subset \mathscr{S}^*$ by $\varphi \to$ integrate-against-φ. That is, $\varphi \to u_\varphi$ with

$$u_\varphi(f) \;=\; \int_{\mathbb{R}^n} \varphi \cdot f$$

Via these inclusions, we *do* think of $\mathcal{D}^*$ and $\mathscr{S}^*$ as *extending* $\mathcal{D}$ and $\mathscr{S}$, respectively.

At some point, we will prove that $\mathcal{D}(\mathbb{R}^n)$ is *dense* in $\mathcal{D}(\mathbb{R}^n)^*$, and in every other space of distributions. The proof that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$ is not difficult: use smooth cut-offs with larger-and-larger compact support.

Since taking duals is inclusion-reversing, we have $\mathcal{E}^* \subset \mathscr{S}^* \subset \mathcal{D}^*$. The suggestion that $\mathcal{E}^*$ does literally consist of compactly-supported distributions, but calling it that, should be substantiated, which we do a little later.

Pictorially, with arrows being inclusions, on $\mathbb{R}^n$ we have

$$\begin{array}{ccccccccccccc}
\mathcal{D} & \longrightarrow & \mathscr{S} & \longrightarrow & \ldots & \longrightarrow & L^2 & \longrightarrow & \ldots & \longrightarrow & \mathscr{S}^* & \longrightarrow & \mathcal{D}^* \\
 & & & \searrow & & & & & & \nearrow & & & \\
 & & & & \mathcal{E} & & & & \mathcal{E}^* & & & &
\end{array}$$

Unsurprisingly, all the arrows are *continuous* maps, as we will subsequently verify. We should also eventually verify that the dual maps on the dual spaces are *injections*.

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3. *Topological details: families of seminorms*

Even though $\mathscr{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$ are complete metric spaces, their topologies are most coherently given *not* in terms of those metrics, but in terms of *countable families of seminorms*, as follows.

Fix a vectorspace V over $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* ν on V is a non-negative $\mathbb{R}$-valued function on V with properties

$$\begin{cases} \nu(c \cdot v) = |c| \cdot \nu(v) & \text{(for scalar } c \text{ and } v \in V) \\ \nu(v+w) \le \nu(v) + \nu(w) & \text{(for } v, w \in V) \end{cases}$$

Unlike a genuine norm, we do *not* require the positive-definiteness property that $\nu(v) = 0$ implies $v = 0$. To compensate, we use *separating families* of seminorms: let X be a set of seminorms on V such that, for all $0 \ne v \in V$ there is $\nu \in X$ such that $\nu(v) \ne 0$. Since we only want Hausdorff topologies on vector spaces, we will *only* consider families of seminorms, so the modifier *separating* may be dropped.

To describe the topology on V corresponding to a family F of seminorms, first we tell all the (open) neighborhoods of $0 \in V$, and to do so we give a *local sub-basis* at 0: [1] sets

$$U_{\nu,\varepsilon} \;=\; \{v \in V : \nu(v) < \varepsilon\} \qquad \text{(for } \varepsilon > 0 \text{ and } \nu \in F)$$

Then make a local sub-basis at $v \in V$ by *translating* opens from 0: a set U containing v is open if and only if the translated set

$$U - v \;=\; \{u - v : u \in U\}$$

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[1] Recall that a *local sub-basis* at a point v_o in a topological space V is a set S of opens containing v_o, such that every open containing v_o contains a *finite intersection* of opens from S.

is an open containing 0. That is, open sets containing v are all of the form $U + v$ for opens containing 0, and vice-versa.

The topology on $\mathscr{S}(\mathbb{R}^n)$ is given by the natural countable family of (semi-) norms

$$\nu_{m,n}(f) \;=\; \sup_{|\alpha|\le m} \sup_x (1+|x|)^n \, |f^{(\alpha)}(x)|$$

The topology on $\mathcal{E}(\mathbb{R}^n)$ is given by the natural countable family of seminorms

$$\nu_{N,n}(f) \;=\; \sup_{|\alpha|\le n} \sup_{|x|\le N} |f^{(\alpha)}(x)|$$

The topology on test functions is slightly more complicated, and we delay treatment of it. Of course, we cannot truly understand the spaces of (continuous!) linear functionals on any such space until we know the topology. Nevertheless, we proceed with an as-yet-unspecified topology on $\mathcal{D}(\mathbb{R}^n)$. [2]

A topology given by a *countable* (separating) family of seminorms $\{\nu_1, \nu_2, \ldots\}$ can also be given by a *metric*

$$d(v,w) \;=\; \sum_{n\ge 1} 2^{-n} \cdot \frac{\nu(v-w)}{1+\nu(v-w)}$$

This metric still is *translation-invariant*, meaning that $d(v+x, w+x) = d(v,w)$, but it does *not* have any homogeneity properties, unlike the metrics $d(v,w) = |v-w|$ made from a single norm $|\cdot|$. For that matter, the constants 2^{-n} and the shape of the expression are not canonical. Thus, it's better to say that the topology is *metrizable*, rather than *metric*.

When a topology on V is given by a countable family of seminorms, and the associated metric as above is *complete* in the usual metric-space sense, V is a *Fréchet space*.

[3.1] Claim: Both $\mathscr{S}$ and $\mathcal{E}$ are Fréchet spaces.

Proof: In both cases, if a sequence $\{f_n\}$ is Cauchy, then its restriction to a sequence of smooth functions on a closed ball B_r of radius r centered at 0 is Cauchy in the metric given by the countable collection of seminorms

$$\nu_m(f) \;=\; \sup_{|\alpha|\le m} \sup_{|x|\le r} |f^{(\alpha)}(x)|$$

Indeed, the seminorms defining the topology on $\mathscr{S}$ *dominate* these seminorms:

$$\sup_{|\alpha|\le m} \sup_{|x|\le r} |f^{(\alpha)}(x)| \;\le\; \sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \cdot |f^{(\alpha)}(x)| \qquad \text{(for any } N\ge 0)$$

Earlier we have seen that such Cauchy sequences converge (in that metric) to smooth functions on B_r. A function whose restriction to every B_r is smooth is smooth on $\mathbb{R}^n$. This completes the proof for $\mathcal{E}$.

For Schwartz functions, we need a little more than smoothness, namely, rapid decay of all derivatives. Given m and N and $\varepsilon > 0$, there is i_o such that for all $i, j \ge i_o$ we have

$$\sup_{i\ge i_o} \sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \cdot f_i^{(\alpha)}(x) \quad\le\quad 2\varepsilon$$

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[2] The topology of $\mathcal{D}(\mathbb{R}^n)$ *can* be described by a family of semi-norms, as is the case for *any* locally convex topological vector space. However, such a description does not make clear that $\mathcal{D}(\mathbb{R}^n)$ has suitable (non-metric!) *completeness*. For the latter, presenting $\mathcal{D}(\mathbb{R}^n)$ as a *strict colimit* is better, and we do this later.

Since everything is non-negative, we can exchange sups, and

$$\sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^n}\lim_{i\ge i_o}(1+|x|^2)^N\cdot f_i^{(\alpha)}(x) \;\le\; \sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^n}\sup_{i\ge i_o}(1+|x|^2)^N\cdot f_i^{(\alpha)}(x) \;\le\; 2\varepsilon$$

giving the rapid decay. ///

As in other contexts, continuity of *linear* maps $T: V\to W$ is equivalent to continuity at 0 for vector spaces V, W with *translation-invariant* topologies, meaning that a local basis $\{U_\alpha\}$ at 0 gives a local basis $\{v+U_\alpha\}$ at any other point v. Since translation invariance is equivalent to continuity of vector addition, this is the only kind of topology we will consider on vector spaces. The easy argument for the following is worth recalling:

[3.2] Claim: A linear map $T: V\to W$ is continuous if and only if it is continuous at 0.

Proof: One direction is immediate. For T is continuous at 0, for every open neighborhood N of 0 in W, there is an open neighborhood U of 0 in V such that $TU\subset N$.

With those N, U, given $v\in V$, and given a neighborhood $Tv+N$ in W, with N a neighborhood of 0 in W, $v+U$ is a neighborhood of v, and by the linearity of T

$$T(v+N) \;=\; Tv+TN \;\subset\; Tv+U$$

proving continuity of T at v. ///

[3.3] Claim: The inclusions $\mathcal{D}\to\mathscr{S}\to\mathcal{E}$ are continuous, and $\mathcal{D}$ is dense in $\mathcal{E}$.

Proof: To prove that $\mathcal{D}\to\mathscr{S}$ is continuous, it suffices to prove that $\mathcal{D}_r\to\mathscr{S}$ is continuous, for each $r>0$, where $\mathcal{D}_r$ is the Fréchet space of test functions supported on the closed ball of radius r at 0. To make $\sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N\cdot|f^{(\alpha)}(x)|$ small, for $f\in\mathcal{D}_r$, it suffices to make $(1+r^2)^N\cdot\sup_{|\alpha|\le m}\sup_{|x|\le r}|f^{(\alpha)}(x)|$ small. This is the desired continuity.

Likewise, for the continuity of $\mathscr{S}\to\mathcal{E}$, for every compact $r>0$ and for every m, we want to make $\sup_{|\alpha|\le m}\sup_{|x|\le r}|f^{(\alpha)}(x)|$ small by requiring some suitable estimate in terms of the $\mathscr{S}$ seminorms. Again,

$$\sup_{|\alpha|\le m}\sup_{|x|\le r}|f^{(\alpha)}(x)| \;\le\; \sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^0\cdot|f^{(\alpha)}(x)|$$

giving the continuity.

The density of $\mathcal{D}$ in $\mathcal{E}$ is slightly more interesting. We use a family of *smooth cut-offs*. Namely, let η be a function on $\mathbb{R}^n$ identically 1 on $\{x:|x|\le 1\}$ and identically 0 on $\{x:|x|\ge 2\}$, and for $n=1,2,\ldots$ let $\varphi_n(x)=\varphi(x/n)$. Given $f\in\mathcal{E}$, we claim that the sequence $\varphi_n\cdot f$ has $\mathcal{E}$-limit f. Indeed, given $r>0$, for $n\ge r$, $\varphi_n(x)\cdot f(x)=f(x)$ for $|x|\le r$, so $\sup_{|\alpha|\le m}\sup_{|x|\le r}|(f-\varphi_n\cdot f)^{(\alpha)}(x)|=0$. This is certainly smaller than every $\varepsilon>0$.

The most interesting issue here is the density of $\mathcal{D}$ in $\mathscr{S}$. Of course, with the same sequence of test functions φ_n, we anticipate that $\varphi_k\cdot f\to f$ in the topology on $\mathscr{S}$, for each $f\in\mathscr{S}$. Indeed, for every m, N,

$$\sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N\cdot|(f-\varphi_k\cdot f)^{(\alpha)}(x)| \;=\; \sup_{|\alpha|\le m}\sup_{|x|\ge k}(1+|x|^2)^N\cdot|(f-\varphi_k\cdot f)^{(\alpha)}(x)|$$

$$\le\; \sup_{|\alpha|\le m}\sup_{|x|\ge k}(1+|x|^2)^N\cdot|f^{(\alpha)}(x)| + \sup_{|\alpha|\le m}\sup_{|x|\ge k}(1+|x|^2)^N\cdot|(\varphi_k\cdot f)^{(\alpha)}(x)|$$

Using Leibniz' rule, and with multinomial coefficients $\binom{\alpha}{\beta}$ the second term is bounded by

$$\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\sup_{|x|\ge k}(1+|x|^2)^N\cdot|f^{\beta}(x)\cdot\varphi_k^{(\alpha-\beta)}(x)|$$

For $k = 1, 2, \ldots$,

$$\sup_{x\in\mathbb{R}^n} |\varphi_k^{(\alpha-\beta)}(x)| \;=\; \sup_{x\in\mathbb{R}^n} |k^{-|\alpha-\beta|}\cdot\varphi^{(\alpha-\beta)}(x/k)| \;\le\; \sup_{x\in\mathbb{R}^n} |\varphi^{(\alpha-\beta)}(x/k)| \;=\; \sup_{x\in\mathbb{R}^n} |\varphi^{(\alpha-\beta)}(x)|$$

there is a uniform constant (depending only on α)

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4. *Examples of generalized functions (distributions)*

Having specified topologies on $\mathscr{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$, we can illustrate continuity arguments that prove various natural functionals are in $\mathscr{S}(\mathbb{R}^n)^*$ and $\mathcal{E}(\mathbb{R}^n)$. Without having yet specified the topology on $\mathcal{D}(\mathbb{R}^n)$, granting that the inclusion $\mathcal{D}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is continuous allows proof that a given functional is in $\mathcal{D}(\mathbb{R}^n)^*$ by proving that it is in $\mathscr{S}(\mathbb{R}^n)^* \subset \mathcal{D}(\mathbb{R}^n)^*$.

[4.1] Claim: The Dirac delta functional $\delta(f) = f(0)$ is in $\mathcal{E}^*$, $\mathscr{S}^*$, and $\mathcal{D}^*$.

Proof: It suffices to use any of the seminorms $\nu_{N,0}$ on $\mathcal{E}$ as above, with $N > 0$. For example, with $\nu_{1,0}$,

$$|f(0)| \;\le\; \sup_{|x|\le 1} |f^{(0)}(x)| \;=\; \nu_{1,0}(f)$$

Thus, given $\varepsilon > 0$, for $f \in \mathcal{E}$ with $\nu_{1,0}(f) < \varepsilon$,

$$|f(0)| \;\le\; \sup_{|x|\le 1} |f^{(0)}(x)| \;=\; \nu_{1,0}(f) < \varepsilon$$

This proves the continuity. ///

[4.2] Claim: For $\varphi \in L^1(\mathbb{R}^n)$ the integrate-against functional $u_\varphi(f) = \int_{\mathbb{R}} \varphi(x)\, f(x)\, dx$ is in $\mathscr{S}^*$.

Proof: Note the elementary inequalities

$$\Big|\int_{\mathbb{R}} \varphi(x)\, f(x)\, dx\Big| \;\le\; \int_{\mathbb{R}} |\varphi(x)|\cdot|f(x)|\, dx \;\le\; \int_{\mathbb{R}} |\varphi(x)|\, dx \cdot \sup_{x\in\mathbb{R}} |f(x)| \;=\; |\varphi|_{L^1}\cdot\nu_{0,0}(f)$$

with seminorm on $\mathscr{S}$ as above. Thus, it suffices to consider φ with $|\varphi|_{L^1} \ne 0$. Given $\varepsilon > 0$, for $\nu_{0,0}(f) < \varepsilon$,

$$\Big|\int_{\mathbb{R}} \varphi(x)\, f(x)\, dx\Big| \;\le\; |\varphi|_{L^1}\cdot\nu_{0,0}(f) \;<<\; |\varphi|_{L^1}\cdot\varepsilon$$

Since $|\varphi|_{L^1}$ is just a (non-zero) constant, this gives the continuity. ///

Recall that on $\mathbb{R}^n$ the *locally integrable* functions $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are the (measurable) functions f such that $\int_K |f| < \infty$ for every compact K. Since we do not yet have a description of the topology on $\mathcal{D}(\mathbb{R}^n)$, we cannot give a precise discussion of the following, but it should be recorded as a fact, eventually provable after explication of the topology on $\mathcal{D}$:

[4.3] Claim: An $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ function f gives a distribution $u_f \in \mathcal{D}(\mathbb{R}^n)^*$ by integration-against:

$$u_f(\varphi) \;=\; \int_{\mathbb{R}^n} f(x)\,\varphi(x)\, dx \qquad\qquad (\text{for } \varphi \in \mathcal{D})$$

Proof: Later. ///

A pointwise-defined function φ is *of moderate/polynomial growth* when there is an exponent N such that

$$\sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \cdot |f(x)| \;<\; +\infty$$

Since the topology on $\mathscr{S}$ is easier to describe, we can prove:

[4.4] Claim: An L^1_{loc} function f of moderate growth gives a tempered distribution u_f by integration-against:

$$u_f(\varphi) \;=\; \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx \qquad\qquad (\text{for } \varphi\in\mathscr{S})$$

Proof: As usual, it suffices to prove continuity at 0. Let $|f(x)|\le C\cdot(1+|x|^2)^N$. Then

$$\Big|\int_{\mathbb{R}^n} f(x)\cdot\varphi(x)\,dx\Big| \;\le\; \int_{\mathbb{R}^n} |f(x)|\cdot|\varphi(x)|\,dx \;=\; \int_{\mathbb{R}^n} (1+|x|^2)^{-N}|f(x)|\cdot(1+|x|^2)^N|\varphi(x)|\,dx$$

$$\le\; C\cdot\int_{\mathbb{R}^n}(1+|x|^2)^N|\varphi(x)|\,dx \;\le\; C\cdot\int_{\mathbb{R}^n}(1+|x|^2)^{-(n+1)}\cdot(1+|x|^2)^{N+n+1}|\varphi(x)|\,dx$$

$$\le\; C\cdot\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{N+n+1}|\varphi(x)|\cdot\int_{\mathbb{R}^n}(1+|x|^2)^{-(n+1)}\,dx$$

The latter integral is finite. Thus, for the seminorm $\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{N+n+1}|\varphi(x)|$ sufficiently small, $|u_f(\varphi)|$ is as small as desired, proving the continuity of u_f at 0. ///

[4.5] Corollary: For $\mathrm{Re}(s)>-n$, the distribution given by integration-against $|x|^s$ on $\mathbb{R}^n$ is *tempered*. ///

[4.6] Remark: In particular, as we see later, the distributions $|x|^s$ have *distributional/generalized* derivatives for $\mathrm{Re}(s)>-n$, even though for $\mathrm{Re}(s)<0$ they are not classically differentiable at 0. Further, for $\mathrm{Re}(s)$ close to $-n$, their distributional derivatives are no longer in L^1_{loc}, but nevertheless give tempered distributions.

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5. *Colimit topology on* $\mathcal{D}(\mathbb{R}^n)$

Let $\mathcal{D}_r$ be test functions supported on the closed ball B_r of radius r in $\mathbb{R}^n$. Suppressing notational reference to $\mathbb{R}^n$,

$$\mathcal{D} \;=\; \bigcup_{r=1}^{\infty}\mathcal{D}_r$$

This ascending union is really a (strict) *colimit* of the Fréchet spaces $\mathcal{D}_r$. [3] That is, first, the limitands $\mathcal{D}_r$ are Fréchet spaces, with topologies given by the countable collections of seminorms

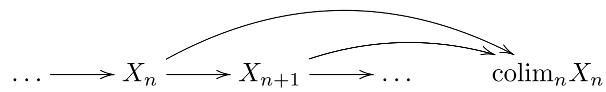
$$\nu_m(\varphi) \;=\; \sup_{|\alpha|\le m}\;\sup_{|x|\le r}|f^{(\alpha)}(x)| \qquad\qquad (\text{on } \mathcal{D}_r)$$

[3] Apparently, L. Schwartz' original description of the appropriate topology on $\mathcal{D}(\mathbb{R}^n)$ was by an explicit construction, and it was J. Dieudonné who pointed out that what Schwartz had constructed was a *colimit*. Even today, many sources still give an explicit construction, rather than the colimit characterization. This is mildly ironic, since part of the point of the colimit characterization is that we only need to understand the topologies on the limitands, and in this case the limitands $\mathcal{D}_r$ are Fréchet spaces.

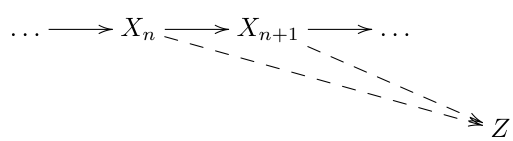
The argument that proved that smooth functions on a compact set form a Fréchet space, when given the metric induced from such a countable collection of seminorms, proves that $\mathcal{D}_r$ is complete. [4]

First, there are (partly meta-mathematical) points: first, the correct topology on the ascending union $\mathcal{D}$ should be such that a continuous linear map T on $\mathcal{D}$ is given by a family of continuous maps T_k on the limitands. That is, $Tv = T_k v$ for any sufficiently large k such that $v \in \mathcal{D}_k$. Independence of k is the compatibilities among the T_n's. Second, the topology on the ascending union should be coarse enough so that all the inclusions of limitands are continuous. It is noteworthy that these two seemingly flimsy requirements determine the topology on the ascending union, up to (unique) isomorphism.

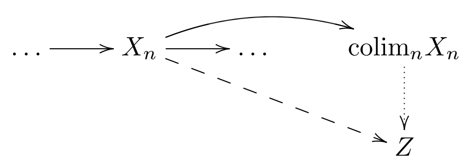
More precisely, recall the notion of *colimit* $\text{colim}_n X_n$ of a sequence of objects X_n with maps $i_n : X_n \to X_{n+1}$. To begin, the colimit $\text{colim}_n X_n$ is an object with *compatible* maps $j_n : X_n \to \text{colim}_n X_n$, meaning that all triangles



commute (in algebraic notation, $j_{n+1} \circ i_n = j_n$ for all n). Further, for every *compatible* family of maps $T_n : X_n \to Z$, meaning that all triangles



commute, there exists a *unique* $T : \text{colim}_n X_n \to Z$ such that all triangles



commute.

That is, given a compatible family of maps $X_n \to Z$, there is a unique compatible map $\text{colim}_n Z$ making all triangles commute. The converse is immediate from the fact that a map $\text{colim}_n X_n \to Z$ gives a compatible family of maps $X_m \to Z$ by composing with the maps $X_m \to \text{colim}_n X_n$.

That is, a map $\text{colim}_n X \to Z$ is exactly the same as a compatible family of maps $X_n \to Z$.

In particular, when we have the *existence* of $\text{colim}_n X_n$, then we know that the object $\text{colim}_n X_n$ and any of its attributes (such as topology) are determined uniquely, up to unique isomorphism.

Also, for purposes of discussing continuity of maps $\text{colim}_n X_n \to Z$, we need only know the topology on all the limitands X_n.

That is, to prove that a linear map $\lambda : \mathcal{D} \to \mathbb{C}$ is continuous, we need only verify the continuity of every $\lambda|_{\mathcal{D}_r}$. Notably, the limitands $\mathcal{D}_r$ are Fréchet spaces, whose topologies are relatively elementary. The more complicated topology of $\mathcal{D}$ itself need not be made explicit for such purposes, as long as we are confident it *exists*.

Existence will be proven later.

[4] Since the topology is given by seminorms, it is *locally convex*, in the sense that 0 has a local basis consisting of convex sets. The local convexity is part of the requirement of a *Fréchet space*, but for us all metrics will arise from countable families of seminorms, so the local convexity is not at issue.

6. *Distributions supported at* 0

[6.1] Theorem: A distribution u with *support* $\{0\}$ is a (finite) linear combination of Dirac's δ and its derivatives.

The *support* of a distribution u is the *complement* of the *union* of all open sets $U \in \mathbb{R}^n$ such that

$$u(f) \;=\; 0 \qquad\qquad (\text{for } f \in \mathcal{D}_K \text{ with compact } K \subset U)$$

Proof: We keep in mind that the space $\mathcal{D}$ of test functions on $\mathbb{R}^n$ is $\mathcal{D} = \bigcup_K \mathcal{D}_K$, really a *colimit* where $\mathcal{D}_K$ is test functions supported on compact K. The latter is a Fréchet space, with *norms*

$$\nu_{k,K}(f) \;=\; \sup_{i\le k,\; x\in K} |f^{(i)}(x)|$$

Thus, it suffices to classify u in $\mathcal{D}_K^*$ with support $\{0\}$.

A continuous linear functional from a *limit* of Banach spaces (such as $\mathcal{D}_K$) to $\mathbb{C}$ factors through a limitand. Thus, there is an *order* $k \ge 0$ such that u factors through

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(\alpha)} \text{ vanishes on } \partial K \text{ for all } \alpha \text{ with } |\alpha| \le k\}$$

We need an auxiliary gadget. Fix a smooth compactly-supported function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1}x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in \mathcal{D}(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for some constant C (depending on k and K, but not on f)

$$|\psi_\varepsilon f|_k \;=\; \sup_{x\in K}\,\sup_{|\alpha|\le k} |(\psi_\varepsilon f)^{(\alpha)}(x)| \;\le\; C\cdot \sup_{|i|\le k}\,\sup_{x}\,\sup_{0\le j\le i} \varepsilon^{-|j|}\left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right|$$

For f vanishing to order k at 0, that is, $f^{(\alpha)}(0) = 0$ for all multi-indices α with $|\alpha| \le k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, for some constant C

$$|f(x)| \;\le\; C\cdot |x|^{k+1}$$

and, generally, for α^{th} derivatives with $|\alpha| \le k$,

$$|f^{(\alpha)}(x)| \;\le\; C\cdot |x|^{k+1-|\alpha|}$$

For some constant C

$$|\psi_\varepsilon f|_k \;\le\; C\cdot \sup_{|i|\le k}\,\sup_{0\le j\le i} \varepsilon^{-|j|}\cdot \varepsilon^{k+1-|i|+|j|} \;\le\; C\cdot \varepsilon^{k+1-|i|} \;\le\; C\cdot\varepsilon^{k+1-k} \;=\; C\cdot\varepsilon$$

Thus, for all $\varepsilon > 0$, for smooth f vanishing to order k at 0,

$$|u(f)| \;=\; |u(\psi_\varepsilon f)| \;\le\; C\cdot\varepsilon$$

Thus, $u(f) = 0$ for such f.

That is, u is 0 on the intersection of the kernels of δ and its derivatives $\delta^{(\alpha)}$ for $|\alpha| \le k$. Generally,

[6.2] Proposition: A continuous linear function $\lambda \in V^*$ vanishing on the intersection of the kernels of a finite collection $\lambda_1, \ldots, \lambda_n$ of continuous linear functionals on V is a linear combination of the λ_i.

Proof: The linear map

$$q : V \longrightarrow \mathbb{C}^n \quad \text{by} \quad v \longrightarrow (\lambda_1 v, \ldots, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) \;=\; c_1 z_1 + \ldots + c_n z_n \qquad \text{(for some constants } c_i\text{)}$$

Thus,

$$\lambda(v) \;=\; (L \circ q)(v) \;=\; L(\lambda_1 v, \ldots, \lambda_n v) \;=\; c_1 \lambda_1(v) + \ldots + c_n \lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

The following resolves a potential ambiguity.

[6.3] Lemma: For compact K inside the *complement* of the support of a distribution u,

$$u(f) \;=\; 0 \qquad \text{(for } f \in \mathcal{D}_K\text{)}$$

Proof: This is plausible, but not utterly trivial. Let $\{U_i : i \in I\}$ be open sets such that for compact K' inside any single U_i and $f \in \mathcal{D}_{K'}$ we have $u(f) = 0$. Let $\{\psi_i : i \in I\}$ be a smooth locally finite *partition of unity*[5] subordinate to $\{U_i : i \in I\}$. Take $f \in \mathcal{D}_{K'}$ for K' compact inside $U = \bigcup_i U_i$. Then

$$f \;=\; f \cdot 1 \;=\; \sum_i f \cdot \psi_i$$

and the sum is *finite*. Then

$$u(f) \;=\; u(\sum_i f \cdot \psi_i) \;=\; \sum_i u(f \cdot \psi_i) \;=\; \sum_i 0 = 0$$

(The fact that the sum is finite allows interchange of summation and evaluation.) ///

7. *Weak dual topologies (weak $*$-topologies)*

Among other possibilities, we can give dual spaces V^* their *weak dual topologies* (also called *weak *-topologies*): for a topological vector space V with (continuous linear) dual V^*, the weak dual topology on V^* is given by the family of seminorms

$$\nu_v(\lambda) \;=\; |\lambda(v)| \qquad \text{(for } v \in V,\, \lambda \in V^*\text{)}$$

Thus, in the weak dual topology, $\lambda_n \to \lambda$ if and only if $|\lambda_n(v) - \lambda(v)| \to 0$ for all $v \in V$.

[5] That is, the functions ψ_i are smooth, take values between 0 and 1, sum to 1 at all points, and on any compact there are only finitely-many which are non-zero. The existence of such partitions of unity is not completely trivial to prove.

The following very important density assertion gives an explanation of the sense in which distributions are generalized *functions*: they are obtainable as suitable *limits* of very nice functions:

[7.1] Claim: Test functions $\mathcal{D}(\mathbb{R}^n)$ are sequentially *dense* in $\mathcal{E}(\mathbb{R}^n)^*$, in $\mathscr{S}(\mathbb{R}^n)^*$, and in $\mathcal{D}(\mathbb{R}^n)^*$, with their respective weak dual topologies. That is, the distributions u_φ given by integration against test functions φ are dense.

Proof: [... iou ...] ///

There are several easier-to-prove useful results:

[7.2] Claim: For a continuous linear map $T : V \to W$ of topological vector spaces, the dual map $T^* : W^* \to V^*$ given by $(T^*\mu)(v) = \mu(Tv)$ is continuous, when V^*, W^* have the weak dual-topologies.

Proof: Let ν_v be the seminorm on V^* given by $\nu_v(\lambda) = |\lambda(v)|$, and η_w the analogous seminorm on W^* for $w \in W$. Given $\varepsilon > 0$, it would suffice to find $w \in W$ and $\delta > 0$ such that $\eta_w(\mu) < \delta$ implies $\nu_v(T^*\mu) < \varepsilon$, for all $\mu \in W^*$. Indeed,

$$\nu_v(T^*\mu) \;=\; |(T^*\mu)(v)| \;=\; |\mu(Tv)| \;=\; \eta_{Tv}(\mu)$$

Thus, taking $\delta = \varepsilon$ and $w = Tv$, $\eta_{Tv}(\mu) < \varepsilon$ implies that $\nu_v(T^*\mu)$, giving the continuity. ///

As a corollary of density assumptions or conclusions:

[7.3] Claim: For $T : V \to W$ with TV *dense* in W, the dual map T^* is *injective*.

Proof: If $(T^*\mu)(v) = 0$ for a given $\mu \in W^*$ and for all $v \in V$, then $\mu(Tv) = 0$ for all $v \in V$. Since μ is continuous, this implies that $\mu(w) = 0$ for all w in the closure of TV. By the density assumption, the closure of TV is W, so $\mu = 0$. ///

[7.4] Corollary: The dual maps $\mathcal{E}^* \to \mathscr{S}^* \to \mathcal{D}^*$ are continuous injections.

Proof: Earlier, we saw that $\mathcal{D} \to \mathscr{S} \to \mathcal{E}$ are continuous inclusions, with dense images. Thus, with weak dual topologies, $\mathcal{E}^* \to \mathscr{S}^* \to \mathcal{D}^*$ make sense, are continuous, and the immediately previous claim gives the injectivity. ///

---

8. *Differentiation of generalized functions (distributions)*

A large part of the purpose of generalized functions is to be able to differentiate not-classically-differentiable functions in a way that nevertheless fits coherently with classical differentiation. That is, as it turns out, it is possible and useful to differentiate functions even when the (numerical) limits of difference quotients $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ do not exist.

Granting that $\mathcal{E}(\mathbb{R}^n)^* \subset \mathscr{S}(\mathbb{R}^n)^* \subset \mathcal{D}(\mathbb{R}^n)^*$, it suffices to define differentiation on $\mathcal{D}(\mathbb{R}^n)^*$:

By duality: For $u \in \mathcal{D}(\mathbb{R}^n)^*$, define

$$\Big(\frac{\partial}{\partial x_j}u\Big)(f) \;=\; -u\Big(\frac{\partial}{\partial x_j}f\Big) \qquad\qquad (\text{for all } f \in \mathcal{D}(\mathbb{R}^n))$$

The sign flip is for compatibility with integration by parts, when u is integration-against a test function or Schwartz function. That is, the definition by duality is adjusted for consistency with the definition *by extension-by-continuity*:

By extension-by-continuity: Granting from above that test functions are (sequentially) dense in $\mathscr{S}^*$, $\mathcal{E}^*$, and $\mathcal{D}^*$, for a sequence $\{u_n\}$ of test functions that is Cauchy in the corresponding weak dual topology, define

$$\frac{\partial}{\partial x_j}(\lim_n u_n) \;=\; \lim_n \Big(\frac{\partial}{\partial x_j} u_n\Big)$$

where the left-hand side is defined by the right, where both limits are in the respective weak dual topologies, and the differentiations on the right-hand side are the classical limits of difference quotients.

In particular, letting u_f be the integration-against-f functional,

$$\frac{\partial}{\partial x_j} u_f \;=\; u_{\partial f/\partial x_j}$$

where the left side is distributional derivative, and $\partial f/\partial x_j$ is the classical limit-of-difference-quotients. That is, the distributional differentiation extended-by-continuity genuinely is an extension corresponding to the inclusions $\mathcal{D} \subset \mathcal{D}^*$, $\mathcal{D} \subset \mathscr{S} \subset \mathscr{S}^*$, and $\mathcal{D} \subset \mathcal{E}^*$.

[8.1] Claim: These two definitions give the same maps $\frac{\partial}{\partial x_j}$ of $\mathcal{D}^*$, $\mathscr{S}^*$, and $\mathcal{E}^*$ to themselves. These maps are continuous in the weak dual topologies.

Proof: [... iou ...] ///

The iconic example: $H' = \delta$**:** Let H be the Heaviside step function, $H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$. (There is no point in trying to define $H(0)$.) We claim that $H'(\varphi) = \varphi(0)$ for test functions φ. That is, with Dirac's delta-function $\delta(\varphi) = \varphi(0)$, $H' = \delta$. Indeed, by the duality characterization of derivative,

$$H'(\varphi) \;=\; -H(f') \;=\; -\int_0^\infty f'(x)\,dx \;=\; -\big(\lim_{x\to+\infty} f(x) \;-\; f(0)\big) \;=\; f(0)$$

[8.2] Claim: If $u \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the tempered distribution given by (integration against) u has distributional derivative equal to (integration against) the classical derivative of u.

Proof: [... iou ...] ///

The derivative δ' **of the Dirac** δ **on** $\mathbb{R}$**:** On one hand, we can directly define

$$\delta'(f) \;=\; -f'(0) \qquad\qquad \text{(with sign flip as suggested above)}$$

and verify that this is a compactly supported distribution, since

$$|\delta'(f)| \;=\; |-f'(0)| \;\le\; \sup_{|x|\le 1} |f'(x)|$$

and the right-hand side is one of the seminorms defining the topology on $\mathcal{E}(\mathbb{R})$. On the other hand, the previous claim systematically bundles-up such arguments, so we know in advance that δ' is a compactly-supported distribution.

The derivative of $u(x) = 1/\sqrt{|x|}$ **on** $\mathbb{R}$**:** Certainly u is locally integrable, so gives a distribution. Away from 0, it is smooth, and away from 0 its derivative is $1/2|x|^{3/2}$. The potential confusion is that this derivative is no longer locally integrable at 0, so we cannot say that the corresponding distribution is literally integrate-against this function. (The same issue arises with the Cauchy principal-value integrals against $1/x$.) Yes, we *can* correctly say that $u'(\varphi) = -u(\varphi')$. If we try to integrate by parts, we will obtain terms that blow up. Hadamard had the insight to discard such terms (!), keeping only the *finite part*. A few years later, F. Riesz showed that this amounted to meromorphic continuation of the *family* of distributions $1/|x|^s$.

Non-moderate growth tempered distributions: Although the idea that (integrate-against) moderate-growth functions gives tempered distributions is accurate, there are definitely more tempered distributions than functions with moderate pointwise growth. For example, $u(x) = e^{ie^x}$ is continuous and bounded, so locally L^1. Thus, it gives a tempered distribution. Unsurprisingly, its distributional derivative is correctly computed by taking classical/pointwise derivatives:

$$u' \;=\; \text{(integration against)} \quad ie^x\, e^{ie^x}$$

That function is not of moderate growth, but since u' is the derivative of a tempered distribution, it is a tempered distribution. Evidently, the extreme oscillation causes a great deal of cancellation.

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9. *Multiplication of generalized functions by smooth functions*

Another part of the purpose of generalized functions is to be able to multiply them by nicer functions in a way that cannot literally be *pointwise*, but nevertheless fits coherently with classical pointwise multiplication.

By duality: For $u \in \mathcal{D}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathcal{D}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication. Observe that the product $\varphi \cdot f$ is still in $\mathcal{D}(\mathbb{R}^n)$. Somewhat similarly, for $u \in \mathscr{S}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$ so that it and all its derivatives are *of moderate growth*, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathscr{S}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication, and $\varphi \cdot f \in \mathscr{S}(\mathbb{R}^n)$ by the hypothesis on φ. Last, for $u \in \mathcal{E}(\mathbb{R}^n)^*$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$, define

$$(\varphi \cdot u)(f) \;=\; u(\varphi \cdot f) \qquad\qquad (\text{for all } f \in \mathcal{E}(\mathbb{R}^n))$$

where $\varphi \cdot f$ is literal pointwise multiplication, and $\varphi \cdot f \in \mathcal{E}(\mathbb{R}^n)^*$.

By extension-by-continuity: Granting that test functions are (sequentially) dense in $\mathscr{S}^*$, $\mathcal{E}^*$, and $\mathcal{D}^*$, for suitable $\varphi \in \mathcal{E}(\mathbb{R}^n)^*$ and a sequence $\{u_n\}$ of test functions Cauchy in the corresponding weak dual topology, define

$$\varphi \cdot (\lim_n u_n) \;=\; \lim_n (\varphi \cdot u_n)$$

where the left-hand side is defined by the right, where both limits are in the respective weak dual topologies, and the multiplications on the right-hand side are the pointwise multiplications.

In particular, letting u_f be the integration-against-f functional for nice functions f,

$$\varphi \cdot u_f \;=\; u_{\varphi \cdot f}$$

where the left-hand side is distributional multiplication, and on the right-hand side is classical pointwise multiplication. That is, the distributional multiplication extended-by-continuity is indeed an extension of classical pointwise multiplication of nice functions.

[9.1] Claim: These two definitions give the same maps of $\mathcal{D}^*$, $\mathscr{S}^*$, and $\mathcal{E}^*$ to themselves. These maps are continuous in the weak dual topologies.

Proof: [... iou ...] ///

Example: $x \cdot \delta = 0$ For f, by the duality definition,

$$(x \cdot \delta)(f) \;=\; \delta(x \cdot f) \;=\; (xf)(0) \;=\; 0 \cdot f(0) \;=\; 0$$

[9.2] Theorem: For a distribution u and smooth function f, if $f \cdot u = 0$, then the support of u is contained in the 0-set $Z = \{x \in \mathbb{R}^n : f(x) = 0\}$ of f.

Proof: Let φ be a test function with support E not meeting the 0-set of f. Certainly $f|_E$ is a smooth, non-vanishing function, so the *idea* is that

$$u(\varphi) \;=\; u(f \cdot \frac{1}{f|_E} \cdot \varphi) \;=\; (f \cdot u)(\frac{1}{f|_E} \cdot \varphi) \;=\; 0(\frac{1}{f|_E} \cdot \varphi) \;=\; 0$$

To make this argument precise, use smooth partitions of unity. Namely, using the compactness of the support E of φ, let U, V be disjoint open subsets of $\mathbb{R}^n$ with $U \supset E$ and $V \supset Z$, and $W = \mathbb{R}^n \cap E^c \cap Z^c$, where the superscript c denotes complement. Let $\{U_\alpha\}$ be a locally finite refinement of $\{U, V, W\}$, and $\{\eta_\alpha\}$ a smooth partition of unity subordinate to $\{U_\alpha\}$. Let $\varphi_1 \in \mathcal{D}$ be the (finite) sum of all η_α whose supports meet U, and φ_2 the (locally finite) sum of all other η_α's. Then φ_1 is identically 1 on U, and identically 0 on V. The point is that φ_1/f is a smooth function. Then

$$\varphi \;=\; 1 \cdot \varphi \;=\; \varphi_1 \cdot \varphi \;=\; f \cdot \frac{\varphi_1}{f} \cdot \varphi$$

Thus,

$$u(\varphi) \;=\; u(f \cdot (\frac{\varphi_1}{f} \cdot \varphi) \;=\; (f \cdot u)(\frac{\varphi_1}{f} \cdot \varphi) \;=\; 0(\frac{\varphi_1}{f} \cdot \varphi) \;=\; 0$$

Thus, $\operatorname{spt} u \subset Z$. ///

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10. *Fourier transforms of tempered distributions*

The Plancherel extension of the literal Fourier transform integral from $L^1(\mathbb{R}^n)$ or $\mathscr{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n$ is a precursor of the extension of Fourier transform to tempered distributions.

By duality: For $u \in \mathscr{S}(\mathbb{R}^n)^*$, define

$$\widehat{u}(f) \;=\; u(\widehat{f}) \qquad\qquad (\text{for all } f \in \mathscr{S}(\mathbb{R}^n))$$

where $\widehat{f}\mathscr{S}(\mathbb{R}^n)$ is the literal Fourier transform integral.

By extension-by-continuity: Granting from above that Schwartz functions are (sequentially) dense in tempered distributions, for a sequence $\{u_n\}$ of Schwartz functions Cauchy in the weak dual topology on $\mathscr{S}(\mathbb{R}^n)^*$, define

$$(\lim_n u_n)\widehat{\ } \;=\; \lim_n(\widehat{u}_n)$$

with the left-hand side is defined by the right, where both limits are in the weak dual topology on $\mathscr{S}(\mathbb{R}^n)^*$, and the Fourier transforms on the right-hand side are the literal integrals.

In particular, letting u_f be the integration-against-f functional for nice functions f,

$$\widehat{u_f} \;=\; u_{\widehat{f}}$$

where the left-hand side is distributional Fourier transform, and on the right-hand side is the literal Fourier transform integral. That is, the distributional multiplication extended-by-continuity is indeed an extension of classical pointwise multiplication of nice functions.

[10.1] Claim: These two definitions give the same map of $\mathscr{S}^*$ to itself, and are continuous in the weak dual topology.

Proof: [... iou ...] ///

The following amounts to the assertion that the seemingly natural way to compute Fourier transforms of $u \in \mathcal{E}^* \subset \mathscr{S}^*$ is correct:

[10.2] Claim: For $u \in \mathcal{E}(\mathbb{R}^n)^*$, $\widehat{u}$ is a pointwise-valued function, and $\widehat{u}(\xi) = u(e^{-2\pi i\xi\cdot x})$, noting that the exponential functions are in $\mathcal{E}$.

Proof: [... iou ...] ///

[10.3] Example:

$$\widehat{\delta}(\varphi) \;=\; \delta(\widehat{\varphi}) \;=\; \widehat{\varphi}(0) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i 0\cdot x}\,\varphi(x)\,dx \;=\; \int_{\mathbb{R}^n} 1\cdot\varphi(x)\,dx \;=\; 1(\varphi)$$

That is, $\widehat{\delta} = 1$. By Fourier inversion, since both are *even*, $\widehat{1} = \delta$.

[10.4] Example: More generally, Fourier transforms of polynomials are finite linear combinations of corresponding derivatives of Dirac δ. That is, in one dimension, invoking the previous claim,

$$\widehat{\delta^{(k)}} \;=\; \delta^{(k)}(e^{-2\pi i\xi x}) \;=\; (-1)^k\cdot\delta(\frac{\partial^k}{\partial x^k}e^{-2\pi i\xi x}) \;=\; (-1)^k\cdot(-2\pi i\xi)^k\cdot\delta(e^{-2\pi i\xi x}) \;=\; (2\pi i\xi)^k \;=\; (2\pi i)^k\cdot\xi^k$$

The obvious analogue holds in higher dimensions, describable in multi-index notation.

[10.5] Example: For distributions u, let $T_x u$ be the *translation* by $x \in \mathbb{R}^n$ extending the notion of translation of a classical function, so $(T_x u)(\varphi) = u(T_{-x}\varphi)$, because of the identify for test functions φ, ψ

$$\int T_x\varphi\cdot\psi \;=\; \int \varphi(y+x)\,\psi(y)\,dy \;=\; \int \varphi(y)\,\psi(y-x)\,dy$$

by changing variables. We claim that for *tempered* u,

$$\widehat{(T_x u)} \;=\; e^{2\pi i\xi\cdot x}\cdot\widehat{u}$$

That is, Fourier transform *intertwines* translation and multiplication by the corresponding exponential. Indeed, using the duality characterization of both Fourier transform and translation, for $\varphi\mathscr{S}$,

$$(T_x u)\widehat{\;}(\varphi) \;=\; (T_x u)(\widehat{\varphi}) \;=\; u(T_{-x}\widehat{\varphi})$$

Compute

$$T_{-x}\widehat{\varphi}(\xi) \;=\; T_{-x}\int e^{-2\pi i\xi\cdot y}\,\varphi(y)\,dy \;=\; \int e^{-2\pi i(\xi-x)\cdot y}\,\varphi(y)\,dy$$

$$=\; \int e^{-2\pi i\xi\cdot y}\Big(e^{2\pi i x\cdot y}\,\varphi(y)\Big)\,dy \;=\; \big(e^{2\pi i x\cdot y}\cdot\varphi(y)\big)\widehat{\;}(\xi)$$

Thus,

$$(T_x u)\widehat{\;}(\varphi) \;=\; u\Big(\big(e^{2\pi i x\cdot y}\cdot\varphi(y)\big)\widehat{\;}\Big) \;=\; \widehat{u}\Big(e^{2\pi i x\cdot y}\cdot\varphi(y)\Big) \;=\; \big(e^{2\pi i x\cdot y}\widehat{u}\big)(\varphi)$$

as claimed. ///

[10.6] Example: We claim that the Fourier transform of the principal-value integral η against $1/x$ is a constant multiple of the sign function. First, $x\cdot\eta = 1$, by direct computation. Taking Fourier transforms,

$$\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\widehat{\eta} \;=\; \delta$$

Since $\frac{\partial}{\partial \xi}\text{sgn}(x) = 2\delta$, we have

$$\frac{\partial}{\partial \xi}\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial \xi}\widehat{\eta} - \tfrac{1}{2}\text{sgn}(\xi)\Big) \;=\; 0$$

The following extension of the Mean Value Theorem should be no surprise:

[10.7] Claim: For a distribution u, if $\frac{\partial u}{\partial x} = 0$, then u is (integration against) a constant.

Proof: For such u, for every test function φ, $0 = u'(\varphi) = u(\varphi')$. Thus, on $\mathbb{R}$ at least, we want to know which test functions ψ are derivatives of other test functions φ. In the context of this question, we have to hope that $\psi = \varphi'$ if and only if $\int_{\mathbb{R}} \psi = 0$. The latter vanishing condition is that $c(\psi) = 0$ for (integration against) any constant c.

Indeed, on one hand, if $\psi = \varphi'$, then $\int_{-\infty}^{x} \psi(y)\, dy = \varphi(x) \to 0$ as $x \to +\infty$. On the other hand, if $\int_{\mathbb{R}} \psi = 0$, then the function $\varphi(x) = \int_{-\infty}^{x} \psi(y)\, dy$ is smooth, and since ψ is compactly supported, is 0 for x sufficiently negative, and also for x sufficiently positive, since the compact support of ψ implies that φ is eventually constant, and the vanishing of the integral implies that that constant is 0. ///

Returning to the example,

$$\frac{\partial}{\partial \xi}\Big(\frac{1}{-2\pi i}\widehat{\eta} - \tfrac{1}{2}\text{sgn}(\xi)\Big) \;=\; 0$$

implies that $\frac{1}{-2\pi i}\widehat{\eta} - \frac{1}{2}\text{sgn}(\xi)$ is (integration against) a constant C. Since both $\widehat{\eta}$ and sgn are *odd*, $C = 0$.

That is, the Fourier transform of the Cauchy principal value functional η attached to $1/x$ is

$$\widehat{\eta}(\xi) \;=\; -\pi i\,\text{sgn}(\xi)$$

///

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11. $\mathcal{E}^*$ is exactly compactly-supported distributions

The space of $\mathcal{E} = \mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ is a Fréchet space, namely, topology given by a countable collection of seminorms

$$\nu_{m,n}(f) \;=\; \sup_{|\alpha|\le m}\, \sup_{|x|\le n} |f^{(\alpha)}(x)|$$

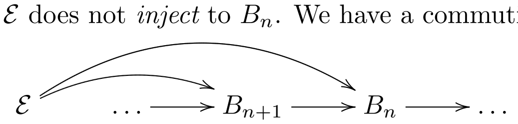
For that matter, it is convenient to consider a *diagonal* subset

$$\nu_{n}(f) \;=\; \sup_{|\alpha|\le n}\, \sup_{|x|\le n} |f^{(\alpha)}(x)|$$

of seminorms. Since $\nu_n(f) \ge \nu_{m,n}(f)$ for all $f \in \mathcal{E}$ and for all $n \ge m$, this diagonal set of seminorms gives the same topology as does the larger set of seminorms.

[11.1] Claim: The dual space $\mathcal{E}^*$ is exactly the set of distributions with compact support.

Proof: Let B_n be the Banach space obtained by completing $\mathcal{E}$ with respect to ν_n. Yes, this allows considerable collapsing, so, no, $\mathcal{E}$ does not *inject* to B_n. We have a commuting diagram



We claim that any continuous linear map $T : \mathcal{E} \to V$ to a *normed* vector space V must factor through (be continuous for) the topology of some limitand B_n. [6] The same holds for arbitrary limits of Banach spaces.

Given $\varepsilon > 0$, by continuity of T there is a neighborhood U of 0 in $\mathcal{E}$ such that TU is contained in the ε-ball $N_\varepsilon \subset V$. By one construction of the topology on the limit, there is a local basis at 0 in $\mathcal{E}$ consisting of opens

$$U_{n,\delta} \;=\; \{f \in \mathcal{E} : \nu_n(f) < \delta\} \qquad\qquad (\text{for } n = 1, 2, \ldots \text{ and } \delta > 0)$$

Thus, for some n and δ, necessarily $TU_{n,\delta} \subset N_\varepsilon$.

For *arbitrary* $\varepsilon' > 0$, by linearity of T,

$$TU_{n,\varepsilon'/\varepsilon} \;=\; T(\frac{\varepsilon'}{\varepsilon} \cdot U_{n,\delta}) \;=\; \frac{\varepsilon'}{\varepsilon} \cdot TU_{n,\delta} \;\subset\; \frac{\varepsilon'}{\varepsilon} \cdot N_\varepsilon \;=\; N_{\varepsilon'}$$

That is, for every $\varepsilon' > 0$, there is a sufficiently small open U in the B_n topology (on $\mathcal{E}$) such that that $TU \subset N_{\varepsilon'}$.

Thus, any $u \in \mathcal{E}^*$ factors through the B_n topology on $\mathcal{E}$, for some sufficiently large n. That is, $u(\varphi) = 0$ for a test function φ supported *off* the closed n-ball $\{x : |x| \leq n\}$. That is, the support of u is contained in the n-ball. ///

[11.2] Remark: Further, the proof shows that estimates on u only depend on derivatives up to some finite order. Such distributions are said to be *of finite order*. That is,

[11.3] Corollary: Compactly-supported distributions are of finite order. ///

There are $u \in \mathcal{D}^*$ which are *not* of finite order such as

$$\sum_{k=0}^{\infty} \delta_k^{(k)}$$

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12. *Orders of distributions*

The rough idea of *order* k of a distribution $u \in \mathcal{D}^*$ is that u does not depend on any derivatives above the k^{th}, with k a non-negative integer. In the previous section, we already saw that compactly-supported distributions are of finite order, in a precise sense. An analogous result holds for tempered distributions:

[12.1] Claim: Given a tempered distribution u, there is some m, and N, such that, given $\varepsilon > 0$, there is $\delta > 0$ such that $\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N \cdot |f^{(\alpha)}(x)| < \delta$ implies $|u(f)| < \varepsilon$, for all $f \in \mathscr{S}$.

Proof: This is another example of the general fact that a continuous linear map u from a limit of Banach spaces to a normed space (here just $\mathbb{C}$) is in fact continuous in the topology of one of the limitands. ///

The smallest m such that there exists an N making this inequality hold for all $\varphi \in \mathscr{S}$ is the *order* of u.

In contrast, for general distributions u, the order is a little more complicated to describe. From the characterization that $\mathcal{D} = \bigcup_r \mathcal{D}_r = \text{colim}_r \mathcal{D}_r$, an element u of the dual is exactly a compatible family

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[6] This does not quite imply that T factors through B_n, since not every continuous linear map on a subspace of a Banach space extends to the whole. On the other hand, for target space $V = \mathbb{C}$, Hahn-Banach assures existence of such an extension. Also, if we happen to know that the limit $\mathcal{E}$ is dense in every limitand B_n (which is true in the present example), we can *extend by continuity* to obtain a continuous linear $B_n \to V$.

u_r of elements of the duals of the limitands. For each r, the topology on $\mathcal{D}_r$ is given by seminorms $\sup_{|\alpha|\le m} \sup_{|x|\le r} |\varphi^{(\alpha)}(x)|$, with m possibly depending on r. Letting m_r be the smallest m that works for given r, the *order* of u is $\sup_r m_r$. This may be $+\infty$.

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13. *Appendix: smooth partitions of unity*

We prove existence of *smooth partitions of unity subordinate to locally finite refinements of an open cover*, as explained below.

We only consider topological manifolds, for example $\mathbb{R}^n$, that are countably-based (sometimes called *second-countable*), to avoid some pathologies irrelevant to our present interests.

A *refinement* of a cover $\{U_\alpha : \alpha \in A\}$ of a topological manifold M is a cover $\{V_\beta : \alpha \in B\}$ such that every V_β lies inside some U_α. [7] A cover $\{U_\alpha : \alpha \in A\}$ of M is *locally finite* when each $x \in M$ lies inside only finitely-many U_α. Recall the following standard fact:

[13.1] Claim: An open cover of a smooth manifold admits a locally finite refinement. Specifically, given an open cover $\{U_\alpha : \alpha \in A\}$, there is a countable, locally finite refinement $\{W_i : i = 1, 2, \ldots\}$ and diffeomorphism $\varphi_i : W_i \approx B_3$ to the open ball of radius 3 in $\mathbb{R}^n$ so that $V_i = \varphi_i^{-1} B_1$ is still an open cover of M.

Proof: The case of compact M is easy, so we only consider non-compact M.

Let $E_1, E_2, \ldots$ be a countable basis for M consisting of opens with compact closures. Let $K_1 = \overline{E}_1$, and $K_{i+1} = \overline{E}_1 \cup \ldots \cup \overline{E}_r$ where $r > 1$ is the smallest with $K_i \subset E_1 \cup \ldots \cup E_r$.

The smooth manifold M is the union of the compacts $K_i - K_{i-1}^o$, where K^o is the interior of a set K. Thus, any x in any U_α is in $K_{i+2} - K_{i-1}^o$ for some index. Let $W_{x,\alpha,i} \subset U_\alpha \cap (K_{i+2} - K_{i-1}^o)$ be an open neighborhood of x, with smoth map $\varphi_{x,\alpha,i} : W_{x,\alpha,i} \approx B_3$. Let $V_{x,\alpha,i} = \varphi_{x,\alpha,i}^{-1}(B_1)$.

For fixed i, as x, α vary, the neighborhoods $V_{x,\alpha,i}$ cover the compact $K_{i+1} - K_i^o$ and are all inside $K_{i+2} - K_{i-1}^o$. For each i, take a finite subcover relabelled as $V_{i,1}, \ldots, V_{i,k_i}$. Upon relabelling, the opens $W_{i,1}, \ldots, W_{i,k_i}$ (with attached diffeomorphisms) constitute the desired countable, locally finite refinement. ///

[13.2] Remark: The previous claim has an obvious analogue for not-necessarily-smooth, topological manifolds.

[13.3] Claim: Given a cover $\{W_i\}$ with diffeomorphisms $\{\varphi_i\}$ as in the previous claim, there exists a smooth partition of unity $\{\varphi_i\}$ subordinate to $\{W_i\}$, that is, such that $\mathrm{spt}\, \varphi_i \subset W_i$.

Proof: For any smooth function f with real values $0 \le f(x) \le 1$, with $f(x) = 1$ for $|x| \le 1$, and $f(x) = 0$ for $|x| \ge 2$, let $f_i = f \circ \varphi_i$. Thus, $F = \sum_i f_i$ is a locally finite sum, and is positive everywhere. Then $g_i = f_i/F$ is the desired partition of unity subordinate to $\{W_i\}$. ///

Now we have what might be viewed as a smooth analogue of *Urysohn's lemma* from the purely topological setting:

[13.4] Corollary: For E a closed subset of a smooth manifold, and U an open containing E, there exists a smooth, real-valued function f taking values $0 \le f(x) \le 1$ such that f is 1 on E and f is 0 off U.

Proof: The cover of the manifold M consisting of two opens, U and $M - E$, has a locally finite subcover $\{W_i\}$ as in the claim, with homeomorphisms $\{\varphi_i\}$, with subordinate partition of unity $\{f_i\}$. ///

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[7] It is necessary to allow the possibility that each U_α contains more than one V_β, etc. That is, it is not quite that each U_α shrinks to a smaller V_α.

[13.5] Remark: The earlier-mentioned subtlety in the notion of *refinement* of a cover is manifest in the proof of the latter corollary.
