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Introduction to Levi-Sobolev spaces

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Sobolev spaces, originating with G. Levi (1906) and G. Frobenius (1907), and refined by S. Sobolev in the 1930's, provide several things. First, certain Sobolev spaces give *Hilbert spaces*, as opposed to Banach spaces, guaranteeing genuine Dirichlet/minimum principles. Second, Sobolev spaces provide a finer gradation on distributions, extending the corresponding rewritten gradation of Banach spaces C^k into a gradation of Hilbert spaces.

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1. *Prototypical Sobolev imbedding theorem*

The simplest case of a Levi-Sobolev *imbedding theorem* asserts that the +1-index Levi-Sobolev *Hilbert* space $H^1[a,b]$ described below is inside $C^o[a,b]$. The point of this is that it is possible to give Hilbert-space conditions which imply continuity, keeping in mind that there is a (true) *minimum/Dirichlet principle* in Hilbert spaces, while no such thing holds in (otherwise natural) Banach spaces such as $C^o[a,b]$

This is a corollary of a Levi-Sobolev *inequality* asserting that the $C^o[a,b]$ norm is *dominated* by the $H^1[a,b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality. The point is that there is a large *Hilbert space* $H^1[a,b]$ inside the *Banach* space $C^o[a,b]$.

We will do much more with this idea subsequently.

These are old results, and have been much elaborated in the intervening decades. However, the underlying causal mechanisms are simple and fundamental, and deserve a simple presentation.

We can think of $L^2[a,b]$ as

$$L^2[a,b] \;=\; \text{completion of } \; C^o[a,b] \text{ with respect to } |f|_{L^2} = \Big(\int_a^b |f(t)|^2\,dt\Big)^{1/2}$$

The +1-index *Levi-Sobolev* space[1] $H^1[a,b]$ is

$$H^1[a,b] \;=\; \text{completion of } \; C^1[a,b] \text{ with respect to } |f|_{H^1} = \Big(|f|^2_{L^2[a,b]} + |f'|^2_{L^2[a,b]}\Big)^{1/2}$$

[1.1] Theorem: *(Levi-Sobolev inequality)* On $C^1[a,b]$, the $H^1[a,b]$-norm *dominates* the $C^o[a,b]$-norm. That is, there is a constant C depending only on a,b such that $|f|_{C^o[a,b]} \le C\cdot |f|_{H^1[a,b]}$ for every $f\in C^1[a,b]$.

Proof: For $a\le x\le y\le b$, for $f\in C^1[a,b]$, the fundamental theorem of calculus gives

$$|f(y)-f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_x^y |f'(t)|\,dt \;\le\; \Big(\int_x^y |f'(t)|^2\,dt\Big)^{1/2}\cdot\Big(\int_x^y 1\,dt\Big)^{1/2}$$

$$\le\; |f'|_{L^2}\cdot|x-y|^{\frac12} \;\le\; |f'|_{L^2}\cdot|a-b|^{\frac12}$$

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[1] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to L^2, rather than L^p. Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906, giving a correct formulation of *Dirichlet's principle*. Sobolev's systematic development of these ideas was in the mid-1930's.

Using the continuity of $f \in C^1[a,b]$, let $y \in [a,b]$ be such that $|f(y)| = \min_x |f(x)|$. Using the previous inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x) - f(y)| \;\le\; \frac{\int_a^b |f(t)|\,dt}{|a-b|} \;+\; |f(x) - f(y)| \;\le\; \frac{\int_a^b |f| \cdot 1}{|a-b|} \;+\; |f'|_{L^2} \cdot |a-b|^{\frac{1}{2}}$$

$$\le\; \frac{|f|_{L^2}^{\frac{1}{2}} \cdot |a-b|^{\frac{1}{2}}}{|a-b|} \;+\; |f'|_{L^2} \cdot |a-b|^{\frac{1}{2}} \;=\; \frac{|f|_{L^2}^{\frac{1}{2}}}{|a-b|^{\frac{1}{2}}} \;+\; |f'|_{L^2} \cdot |a-b|^{\frac{1}{2}} \;\le\; \Big(|f|_{L^2} + |f'|_{L^2}\Big) \cdot \Big(|a-b|^{-\frac{1}{2}} + |a-b|^{\frac{1}{2}}\Big)$$

$$\le\; 2(|f|^2 + |f'|^2)^{1/2} \cdot \Big(|a-b|^{-\frac{1}{2}} + |a-b|^{\frac{1}{2}}\Big) \;=\; |f|_{H^1} \cdot 2\Big(|a-b|^{-\frac{1}{2}} + |a-b|^{\frac{1}{2}}\Big)$$

Thus, on $C^1[a,b]$ the H^1 norm dominates the C^o-norm. ///

[1.2] Corollary: *(Levi-Sobolev imbedding)* $H^1[a,b] \subset C^o[a,b]$.

Proof: Since $H^1[a,b]$ is the H^1-norm completion of $C^1[a,b]$, every $f \in H^1[a,b]$ is an H^1-limit of functions $f_n \in C^1[a,b]$. That is, $|f - f_n|_{H^1[a,b]} \to 0$. Since the H^1-norm dominates the C^o-norm, $|f - f_n|_{C^o[a,b]} \to 0$. A C^o-limit of continuous functions is continuous, so f is continuous. ///

In fact, we have a stronger conclusion than continuity, namely, a *Lipschitz condition* with exponent $\frac{1}{2}$:

[1.3] Corollary: (of proof of theorem) $|f(x) - f(y)| \le |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$ for $f \in H^1[a,b]$. ///

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2. *Sobolev theorems on* $\mathbb{T}^n$

For $0 \le k \in \mathbb{Z}$ and $f \in C^\infty(\mathbb{T}^n)$, the k^{th} Sobolev norm can be defined in terms of L^2 norms of all its derivatives up through order k:

$$|f|_{H^k}^2 \;=\; \sum_{|\alpha| \le k} |f^{(\alpha)}|_{L^2}$$

where as usual α is summed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers α_i, with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$f^{(\alpha)} \;=\; \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\, f$$

One way to define the k^{th} Sobolev space is

$$H^k(\mathbb{T}^n) \;=\; \text{completion of } C^\infty(\mathbb{T}^n) \text{ with respect to } |\cdot|_{H^k}$$

In this context, $H^{-k}(\mathbb{T}^n)$ for $-k < 0$ is defined to be the dual of $H^k(\mathbb{T}^n)$, with $H^0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism rather than $\mathbb{C}$-conjugate-linear). From the inclusion $H^{k+1} \to H^k$ for $0 \le k \in \mathbb{Z}$ dualizing gives a dual/adjoint map $H^{-k} \to H^{-k-1}$. Let

$$H^\infty(\mathbb{T}^n) \;=\; \bigcap_{k=0}^{\infty} H^k(\mathbb{T}^n) \;=\; \lim_k H^k(\mathbb{T}^n)$$

and

$$H^{-\infty}(\mathbb{T}^n) \;=\; \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{T}^n) \;=\; \text{colim}_k H^{-k}(\mathbb{T}^n)$$

The picture is

$$H^\infty(\mathbb{T}^n) \longrightarrow \ldots \longrightarrow H^1(\mathbb{T}^n) \longrightarrow H^0(\mathbb{T}^n) \longrightarrow H^{-1}(\mathbb{T}^n) \longrightarrow \ldots \longrightarrow H^{-\infty}(\mathbb{T}^n)$$

[2.1] Claim: All arrows are continuous injections with dense images.

Proof: [... iou ...] ///

A *spectral characterization* of Sobolev norms is often useful, and directly defines $H^s(\mathbb{T}^n)$ for all $s \in \mathbb{R}$: for $f \in C^\infty(\mathbb{T}^n)$, with Fourier coefficients $\widehat{f}(\xi)$,

$$|f|^2_{H^s} \;=\; \sum_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)|^2 \cdot (1+|\xi|^2)^s$$

and $H^s(\mathbb{T}^n)$ is the completion of $C^\infty(\mathbb{T}^n)$ with this norm.

[2.2] Claim: The spectral characterization gives the same topology on H^k as the characterization in terms of L^2 norms of derivatives, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

Sometimes it is convenient to give the derivative characterization slightly differently, as

$$|f|^2_{H^k} \;=\; \langle (1-\Delta)^k f,\, f\rangle_{L^2}$$

[2.3] Claim: The latter characterization gives the same topology on H^k as do the two previous characterizations, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[2.4] Theorem: *(Sobolev imbedding theorem)* $H^s(\mathbb{T}^n) \subset C^k(\mathbb{T}^n)$ for $s > \frac{n}{2}$.

Proof: [... iou ...] ///

[2.5] Corollary: $H^\infty(\mathbb{T}^n) \subset C^\infty(\mathbb{T}^n)$, and $H^{-\infty}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)^*$.

Proof: [... iou ...] ///

[2.6] Theorem: The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$,

$$\Big\langle \sum_\xi a_\xi \psi_\xi,\, \sum_\xi b_\xi \psi_\xi \Big\rangle_{H^s \times H^{-s}} \;=\; \sum_\xi a_\xi \cdot \overline{b_\xi}$$

Proof: [... iou ...] ///

That is, distributions on $\mathbb{T}^n$ admit Fourier expansions with coefficients of moderate growth, and evaluation of distributions on smooth functions can be done by a natural extension of Plancherel.

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3. *Sobolev theorems on* $\mathbb{R}^n$

The general shape of the discussion on $\mathbb{R}^n$ is similar to that on $\mathbb{T}^n$, with some unsurprising complications due to the non-compactness of $\mathbb{R}$. In particular, Fourier series are replaced by Fourier transforms and inversion.

For $0 \le k \in \mathbb{Z}$ and $f \in C_c^\infty(\mathbb{R}^n)$, the k^{th} Sobolev norm can be defined in terms of L^2 norms of all its derivatives up through order k:

$$|f|^2_{H^k} \;=\; \sum_{|\alpha| \le k} |f^{(\alpha)}|_{L^2}$$

One way to define the k^{th} Sobolev space is

$$H^k(\mathbb{R}^n) \;=\; \text{completion of } C_c^\infty(\mathbb{R}^n) \text{ with respect to } |\cdot|_{H^k}$$

In this context, $H^{-k}(\mathbb{R}^n)$ for $-k < 0$ is defined to be the dual of $H^k(\mathbb{R}^n)$, with $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism rather than $\mathbb{C}$-conjugate-linear). From the inclusion $H^{k+1} \to H^k$ for $0 \le k \in \mathbb{Z}$ dualizing gives a dual/adjoint map $H^{-k} \to H^{-k-1}$. Let

$$H^\infty(\mathbb{R}^n) \;=\; \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^n) \;=\; \lim_k H^k(\mathbb{R}^n)$$

and

$$H^{-\infty}(\mathbb{R}^n) \;=\; \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{R}^n) \;=\; \mathrm{colim}_k H^{-k}(\mathbb{R}^n)$$

The picture is the same as for $\mathbb{T}^n$:

$$H^\infty(\mathbb{R}^n) \longrightarrow \ldots \longrightarrow H^1(\mathbb{R}^n) \longrightarrow H^0(\mathbb{R}^n) \longrightarrow H^{-1}(\mathbb{R}^n) \longrightarrow \ldots \longrightarrow H^{-\infty}(\mathbb{R}^n)$$

[3.1] Claim: All arrows are continuous injections with dense images.

Proof: [... iou ...] ///

A *spectral characterization* of Sobolev norms is often useful, and directly defines $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$: for $f \in C_c^\infty(\mathbb{R}^n)$, with Fourier transform $\widehat{f}(\xi)$,

$$|f|^2_{H^s} \;=\; \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^s \; d\xi$$

and $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with this norm.

[3.2] Claim: The spectral characterization gives the same topology on H^k as the characterization in terms of L^2 norms of derivatives, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[3.3] Corollary: Distributions u in $H^{-\infty}(\mathbb{R}^n)$ have Fourier transforms that are in weighted L^2 spaces, with pointwise values almost everywhere.

Proof: [... iou ...] ///

Sometimes it is convenient to give the derivative characterization slightly differently, as

$$|f|^2_{H^k} \;=\; \langle (1 - \Delta)^k f,\, f \rangle_{L^2}$$

[3.4] Claim: The latter characterization gives the same topology on H^k as do the two previous characterizations, for $0 \le k \in \mathbb{Z}$.

Proof: [... iou ...] ///

[3.5] Theorem: *(Sobolev imbedding theorem)* $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ for $s > \frac{n}{2}$.

Proof: [... iou ...] ///

Since $\mathbb{R}^n$ is non-compact, the conclusion of the following is weaker than for $\mathbb{T}^n$, since $C^\infty(\mathbb{R}^n)$ is not equal to $\mathscr{S}(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$:

[3.6] Corollary: $H^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ and $H^{-\infty}(\mathbb{R}^n) \supset C^\infty(\mathbb{R}^n)^*$.

Proof: [... iou ...] ///

[3.7] Corollary: If we know that $\mathcal{E}(\mathbb{R}^n)^* = C^\infty(\mathbb{R}^n)^*$ is exactly compactly-supported distributions, then we can conclude that $H^{-\infty}(\mathbb{R}^n)$ contains compactly-supported distributions. ///

[3.8] Theorem: The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i\xi\cdot x}$,

$$\langle f, F\rangle_{H^s\times H^{-s}} \;=\; \int_{\mathbb{R}^n} \widehat{f}(\xi)\cdot \overline{\widehat{F}(\xi)}\; d\xi$$

Proof: [... iou ...] ///

That is, evaluation of distributions in $H^{-\infty}(\mathbb{R}^n)$ on smooth functions in $H^\infty(\mathbb{R}^n)$ can be done by a natural extension of Plancherel.
