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Fourier transform examples, Sobolev space examples

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[This document is
http://www.math.umn.edu/˜garrett/m/real/notes_2019-20/08ab_Sobolev-Fourier_examples.pdf]

We want to show that there are functions in $H^{s+\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$, but not in $H^s(\mathbb{R})$, and give further practical examples of Fourier transforms.

First, a useful family of Fourier transforms. Let

$$f_\alpha(x) \;=\; \begin{cases} x^\alpha e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

Of course, as usual, Euler's integral for the Γ-function is

$$\Gamma(s) \;=\; \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}\,(s) > 0)$$

and extended by meromorphic continuation outside the region of absolute convergence.

[1.1] Claim: $\widehat{f}_\alpha(\xi) = \Gamma(\alpha+1)\cdot \dfrac{1}{(1+i\xi)^{\alpha+1}}$ with $\mathrm{Re}\,(\alpha) > -1$ for L^1-ness so that the Fourier transform is given by the literal integral.

Proof: The normalization of the Fourier transform that superficially suppresses the most constants for immediate purposes here is

$$\widehat{f}_\alpha(\xi) \;=\; \int_{\mathbb{R}} e^{-i\xi x} f_\alpha(x)\, dx$$

This is

$$\int_0^\infty e^{-i\xi x}\, x^\alpha\, e^-x\; dx \;=\; \int_0^\infty e^{-x(1+i\xi)}\, x^{\alpha+1}\, \frac{dx}{x}$$

By changing variables, a similar integral is readily evaluated: for $0 < y \in \mathbb{R}$,

$$\int_0^\infty e^{-xy}\, x^{\alpha+1}\, \frac{dx}{x} \;=\; \frac{1}{y^{\alpha+1}} \cdot \int_0^\infty e^{-x}\, x^{\alpha+1}\, \frac{dx}{x} \;=\; \frac{1}{y^{\alpha+1}} \cdot \Gamma(\alpha+1)$$

The identity principle (also known as *perpetuation of analytic relations*) gives

$$\int_0^\infty e^{-x(1+i\xi)}\, x^{\alpha+1}\, \frac{dx}{x} \;=\; \frac{1}{(1+i\xi)^{\alpha+1}} \cdot \Gamma(\alpha+1)$$

as claimed. ///

As a corollary,

[1.2] Corollary: The function f_α is in $H^{\mathrm{Re}\,(\alpha)+\frac{1}{2}-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$, and is definitely *not* in $H^{\mathrm{Re}\,(\alpha)+\frac{1}{2}}(\mathbb{R})$.

Proof: The s^{th} spectral Sobolev norm-squared is

$$|f_\alpha|^2_{H^s} \;=\; |\Gamma(\alpha+1)|^2 \int_{\mathbb{R}} \Big|\frac{1}{(1+i\xi)^{\alpha+1}}\Big|^2 \cdot (1+\xi^2)^s\, d\xi \;=\; |\Gamma(\alpha+1)|^2 \int_{\mathbb{R}} \frac{1}{(1+\xi^2)^{\mathrm{Re}\,(\alpha)+1-s}}\, d\xi$$

This is finite *exactly* when $2\cdot(\mathrm{Re}\,(\alpha)+1-s) > 1$, which is $\mathrm{Re}\,(\alpha) - s > -\frac{1}{2}$, or $s < \mathrm{Re}\,(\alpha) + \frac{1}{2}$. ///
