

## 08c. Extension by continuity

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1. Recapitulation: the metric case
2. Quasicompleteness

(The proofs here are not surprising, once the proper context is understood.)

It is natural to imagine extending a linear map  $T : X \rightarrow Y$  to some  $\tilde{T} : \tilde{X} \rightarrow Y$  for some sort of completion  $\tilde{X}$  of  $X$ , with suitably complete  $Y$ , by the obvious

$$\tilde{T}(X\text{-}\lim x_\alpha) = Y\text{-}\lim Tx_\alpha$$

With uniformly continuous maps  $T$  on metric spaces, checking the viability of this notion of extension was done earlier. For maps among general topological vector spaces, there is a bit more overhead, especially to attend to the (arguably correct) notion of *quasi-completeness* (also called *local completeness*) of non-metric topological vector spaces.

We recall a metric version first, and a version for quasi-complete spaces afterward. By *topological vector space* we mean a (real or complex) vector space with a (of course locally convex<sup>[1]</sup>) topology in which scalar multiplication and vector addition are continuous.

The more general case of *quasi-complete* (locally convex) topological vector spaces is essential to accommodate spaces of test functions, and weak dual spaces, among other function spaces occurring in practice.

Notably, while a *metric* space has a *completion* that is itself complete, it seems that outside of the metric situation the analogous *quasicompletion* of a not-quasicomplete topological vector space may fail to be quasicomplete. Fortunately, many naturally occurring spaces, while non-metric, are already quasi-complete. Thus, difficulties with an alleged process of quasi-completion are not an obstacle.

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### 1. Recapitulation: the metric case

For the following, pre-Fréchet means that a topological vector space has a translation-invariant, locally convex topology, but is not necessarily complete.

[1.1] **Theorem:** Let  $T : X \rightarrow Y$  be a continuous linear map from a pre-Fréchet space  $X$  to a Fréchet space  $Y$ . Then  $T$  extends uniquely to a continuous linear map  $\tilde{T} : \tilde{X} \rightarrow Y$  on the completion  $\tilde{X}$  of  $X$ , by

$$\tilde{T}(X\text{-}\lim_n x_n) = Y\text{-}\lim_n Tx_n$$

for Cauchy sequences  $\{x_n\}$ .

*Proof:* First, we prove that  $\{x_n\}$  Cauchy implies  $\{Tx_n\}$  Cauchy. Given  $\varepsilon > 0$ , let  $\delta > 0$  be such that  $d_X(x, 0) < \delta$  implies  $d_Y(Tx, 0) < \varepsilon$ . By translation-invariance of the metrics,  $d_X(x, x') < \delta$  implies  $d_X(x - x', 0) < \delta$ , so

$$d_Y(Tx, Tx') = d_Y(T(x - x'), 0) < \varepsilon$$

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[1] A topology on a vector space is *locally convex* when every point has a local basis consisting of convex sets. As usual, a set  $E$  in a vector space is convex when, for every  $x, y \in E$  and  $0 \leq t \leq 1$ ,  $tx + (1 - t)y \in E$ . Further, for *translation-invariant* topologies, that is, where a local basis  $\{U_\alpha\}$  at 0 gives a local basis at  $x$  by  $\{x + U_\alpha\}$ , local convexity at 0 is sufficient for local convexity at every point. It is non-trivial to construct not-locally-convex metric vector spaces, but they do exist.

Since  $\{x_n\}$  is Cauchy, there exists  $n_o$  such that  $m, n \geq n_o$  implies  $d(x_m, x_n) < \delta$ , and then  $d(Tx_m, Tx_n) < \varepsilon$ . Thus,  $\{Tx_n\}$  is Cauchy.

To prove that  $\tilde{T}$  is well-defined, let  $\{x_n\}$  and  $\{x'_n\}$  be two Cauchy sequences in  $X$  with the same limit  $z$  in  $\tilde{X}$ . That is,  $\lim_n d_X(x_n, x'_n) = 0$ . Given  $\delta > 0$ , let  $n_o$  be large enough so that, for all  $n \geq n_o$ ,  $d_X(x_n, z) < \delta/2$  and  $d_X(x'_n, z) < \delta/2$ . Then  $d_X(x_n, x'_n) < \delta$ , and  $d_Y(Tx_n, Tx'_n) < \varepsilon$ . This holds for all  $\varepsilon > 0$ , so

$$Y\text{-}\lim_n Tx_n = Y\text{-}\lim_n Tx'_n$$

and  $\tilde{T}$  is well-defined.

For the continuity of  $\tilde{T}$ , using the translation-invariance of the metrics, it suffices to prove continuity at 0. Recall that  $d_{\tilde{X}}(z, 0) = \lim_n d_X(x_n, 0)$  for a sequence  $\{x_n\}$  in  $X$  approaching  $z \in \tilde{X}$ . Given  $\varepsilon > 0$ , let  $\delta > 0$  be such that  $d_X(x, 0) < \delta$  with  $x \in X$  implies  $d_Y(Tx, 0) < \varepsilon$ . Then, for  $z \in \tilde{X}$  with  $d_{\tilde{X}}(z, 0) < \delta/2$ , for a sequence  $\{x_n\}$  in  $X$  approaching  $z$ , without loss of generality  $d_X(x_n, 0) < \delta$  for all  $n$ . Thus, using the continuity of  $T$ ,

$$d_Y(\tilde{T}z, 0) = d_Y(\lim_n Tx_n, 0) = \lim_n d_Y(Tx_n, 0) \leq \lim_n \varepsilon = \varepsilon$$

Last, we prove that  $\tilde{T}$  is linear. Given  $z = \lim_n x_n$  and  $z' = \lim_n x'_n$  with  $z, z' \in \tilde{X}$  and  $x_n, x'_n \in X$ , using the continuity of  $\tilde{T}$ ,

$$\tilde{T}(z + z') = \tilde{T}(\lim_n x_n + \lim_n x'_n) = \tilde{T}(\lim_n x_n) + \tilde{T}(\lim_n x'_n) = \tilde{T}(z) + \tilde{T}(z')$$

For scalar multiplication: for  $c \in \mathbb{C}$ ,

$$\tilde{T}(c \cdot z) = \tilde{T}(c \cdot \lim_n x_n) = \tilde{T}(\lim_n c \cdot x_n) = \lim_n T(c \cdot x_n) = \lim_n c \cdot Tx_n = c \cdot \lim_n Tx_n = c \cdot \tilde{T}(z)$$

This finishes the proof for the metric case. ///

## 2. Quasi-completeness

Now we discuss extension-by-continuity of continuous linear maps on not-necessarily-metric topological vector spaces.

*Sequential* completeness is insufficient in scenarios where a topological vector space does not have a countable local basis.

One formalization of the notion of *sequence* in a set  $X$  is as a function  $f : \{1, 2, 3, \dots\} \rightarrow X$ , where  $\{1, 2, 3, \dots\}$  has the usual ordering. In that context, a useful and necessary generalization is that a *net* in a set  $X$  is a function  $f : A \rightarrow X$ , where  $A$  is a *directed set*, meaning that  $A$  has an order relation  $<$  and for all  $a, b \in A$  there exists  $c \in A$  such that  $c > a$  and  $c > b$ .

Extending the notion of *Cauchy sequence*, a net  $f : A \rightarrow X$  in a topological vector space  $X$  is *Cauchy* when, for every neighborhood  $U$  of  $0 \in X$ , there is  $a_o \in A$  such that  $f(a) - f(b) \in U$  for all  $a \geq a_o$  and  $b \geq a_o$ . That net *converges to*  $z \in X$  when, for every neighborhood  $U$  of  $0 \in X$ , there is  $a_o \in A$  such that, for all  $a \geq a_o$ ,  $f(a) - z \in U$ .

The strongest notion of *completeness* for topological vector spaces is that every Cauchy net converges. For metric spaces, this is provably equivalent to sequential completeness. However, for more general topological vector spaces, such as weak dual spaces, full completeness definitely fails. This failure occurs already with  $\ell^2$  with the weak topology, and with the weak dual topologies on various spaces of distributions (generalized functions).

A notion of completeness weaker than convergence of *all* Cauchy nets is requiring convergence of *bounded* Cauchy nets, where a subset  $E$  of a topological vector space  $X$  is bounded if, for every neighborhood  $U$  of  $0 \in X$ , there is  $0 < t_o \in \mathbb{R}$  such that, for every  $\alpha \in \mathbb{C}$  with  $|\alpha| \geq t_o$ ,  $E \subset \alpha \cdot U$ . The property that *bounded* Cauchy nets converge is *quasi-completeness* (also called *local completeness*).

Complete metric topological vector spaces are certainly quasi-complete. Spaces of test functions are *not* complete metric, but *are* quasi-complete. Weak duals of Fréchet spaces are quasi-complete. Weak duals of spaces of test functions are quasi-complete. We give proofs of these facts elsewhere.

Quasi-completeness is a sufficient notion of completeness for Gelfand-Pettis vector-valued integrals, and for Cauchy-Goursat-Grothendieck theory for vector-valued holomorphic functions.

Given a locally convex topological vector space  $X$ , a *quasi-completion*  $\tilde{X}$  of  $X$  is a quasi-complete topological vector space with an injection  $X \rightarrow \tilde{X}$ , so that every point in  $\tilde{X}$  is a limit of a bounded Cauchy net in  $X$ . [2]

[2.1] **Theorem:** A continuous linear map  $T : X \rightarrow Y$  from a locally convex topological vector space  $X$  to a quasi-complete locally convex topological vector space  $Y$  extends uniquely to a continuous linear map  $\tilde{T} : \tilde{X} \rightarrow Y$  from a quasi-completion  $\tilde{X}$  of  $X$  to  $Y$ .

*Proof:* For a bounded Cauchy net  $f : A \rightarrow X$ , attempt to define the extension  $\tilde{T}$  by

$$\tilde{T}(\tilde{X}\text{-}\lim_{a \in A} f(a)) = Y\text{-}\lim_{a \in A} T(f(a))$$

To see that this is well-defined, let  $f : A \rightarrow X$  and  $g : B \rightarrow X$  be two bounded Cauchy nets with the same limit  $z \in \tilde{X}$ . Given a neighborhood  $N$  of  $0 \in Y$ , let  $U$  be a neighborhood of  $0 \in X$  such that  $TU \subset N$ . Replace  $U$  by  $U \cap (-U)$ , so that  $U = -U$ . Let  $a_o \in A$  and  $b_o \in B$  such that  $f(a) - z \in U/2$  for  $a \geq a_o$ , and  $g(b) - z \in U/2$  for  $b \geq b_o$ . Then  $f(a) - g(b) \in U/2 - U/2 = U$ . Thus, for such  $a, b$ ,

$$Tf(a) - Tg(b) = T(f(a) - g(b)) \in TU \subset N$$

This holds for every neighborhood  $N$  of  $0$ , so  $\tilde{T}z$  is well-defined.

Continuity and linearity of  $\tilde{T}$  still need to be checked... ///

[2] It seems that there is no general method to prove existence of such  $\tilde{X}$  for non-metric  $X$ . Certainly we can mimic the construction of (sequential) completions for the metric case: the idea would be to let  $\tilde{X}$  be the set of bounded Cauchy nets modulo bounded Cauchy nets going to  $0$ . To define vector addition and scalar multiplication in the natural way, by

$$c \cdot \lim f(a) = \lim c \cdot f(a) \qquad \lim_{a \in A} f(a) + \lim_{b \in B} g(b) = ???$$

we need to compare directed sets  $A$  and  $B$ . Fortunately, in topological vector spaces there is no necessity to use arbitrary directed sets. Rather, we can use a single one, the directed set  $A$  of neighborhoods of  $0$  in  $X$ . Then define

$$\lim_{a \in A} f(a) + \lim_{a \in A} g(a) = \lim_a f(a) + g(a)$$

One should check that this is well-defined, that is, independent of the bounded Cauchy nets with given limits. To tell the topology of  $\tilde{X}$ , it suffices to give a local basis at  $0$ . For each open  $N$  of  $X$  containing  $0$ , let  $\tilde{N}$  be the set of points in  $\tilde{X}$  obtainable as limits of bounded Cauchy nets *inside*  $N$ , and *not* obtainable as limits of bounded Cauchy nets *outside*  $N$ . (This extends the analogue for metric spaces and their completions.) Manifestly, the original topology on  $X$  is the same as the subspace topology from  $\tilde{X}$ . One should check continuity of vector operations on  $\tilde{X}$ , and the local convexity of this topology. One should also check the quasi-completeness of  $\tilde{X}$  with the indicated topology, which seems to fail, in general.