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08e. Fourier transform of sech

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[This document is
http://www.math.umn.edu/~garrett/m/real/notes_2019-20/08e_Fourier_transform_sech.pdf]

The following computations deserve to be in one's bestiary of known Fourier transforms:

[0.1] Claim:

$$\int_{\mathbb{R}} \frac{e^{ix\xi}\,dx}{\cosh x} = \frac{\pi}{\cosh \frac{\pi}{2}\xi} \qquad \text{(with } \cosh x = \tfrac{e^x+e^{-x}}{2} \text{ as usual)}$$

[0.2] Remark: After seeing how constants work out, we renormalize to obtain a symmetrical assertion.

Proof: Since $\cosh x$ is *even*, we may as well take $\xi > 0$. Then $e^{ix\xi}$ is nicely decreasing for x in the upper half-plane, and we can evaluate the integral by residues.

The poles of $1/\cosh x$ are at the zeros of $\cos(ix)$, namely, $\frac{\pi}{2}i$, $\frac{3\pi}{2}i$, $\frac{5\pi}{2}i$, ... The residues alternate in sign, with the residue at $\frac{\pi}{2}i$ determined by

$$\frac{1}{\cosh(x+\frac{\pi}{2}i)} = \frac{2}{e^{(x+\frac{\pi}{2}i)}+e^{-(x+\frac{\pi}{2}i)}} = \frac{2}{ie^x+ie^{-x}} = \frac{-2i}{e^x-e^{-x}}$$

$$= \frac{-2i}{(1+x+\ldots)-(1-x+\ldots)} = \frac{-i}{x}+\ldots$$

Thus, the residue at $\frac{\pi}{2}i$ is $-i$, the residue at $\frac{3\pi}{2}i$ is $+i$, and so on.

Thus, by residues, the integral is

$$\int_{\mathbb{R}} \frac{e^{ix\xi}\,dx}{\cosh x} = 2\pi i\Big(-i\cdot e^{i\cdot\frac{\pi}{2}i\xi}+i\cdot e^{i\cdot\frac{3\pi}{2}i\xi}-i\cdot e^{i\cdot\frac{5\pi}{2}i\xi}+\ldots\Big) = \frac{2\pi e^{-\frac{\pi}{2}\xi}}{1+e^{-\pi\xi}} = \frac{2\pi}{e^{\frac{\pi}{2}\xi}+e^{-\frac{\pi}{2}\xi}} = \frac{\pi}{\cosh\frac{\pi}{2}\xi}$$

as claimed. ///

[0.3] Corollary:

$$\int_{\mathbb{R}} \frac{e^{-2\pi ix\xi}\,dx}{\cosh \pi x} = \frac{1}{\cosh \pi\xi}$$

Proof: To symmetrize the previous result, replace ξ by 2ξ, and x by $-\pi x$:

$$\pi\int_{\mathbb{R}} \frac{e^{-2\pi ix\xi}\,dx}{\cosh \pi x} = \frac{\pi}{\cosh \pi\xi}$$

Cancelling the factors π gives the assertion. ///