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08f. Poisson summation by distribution theory

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[This document is
http://www.math.umn.edu/~garrett/m/real/notes_2019-20/08f_distributional_Poisson_summation.pdf]

Let $\psi(y) = e^{2\pi i y}$ and $\psi_x(y) = \psi(xy)$. The additive group $\mathbb{R}$ acts on $\mathscr{S}$ and on $\mathcal{D}$ continuously by the *regular representation* $R_g f(x) = f(x+g)$. The natural duality gives the (continuous) *dual* or *contragredient* representation on $\mathscr{S}^*$ and $\mathcal{D}^*$ by

$$(R_g^* u)(f) \;=\; u(R_{g^{-1}} f)$$

There are two fundamental identities regarding this regular representation and Fourier transforms (for $f \in \mathscr{S}$):

$$(R_x f)\widehat{\ } \;=\; \psi_x \, \widehat{f} \qquad\qquad (\psi_x \cdot f) \;=\; R_{-x}\widehat{f}$$

From these and from the definition of Fourier transform for tempered distributions, the same identities hold for tempered distributions, as well.

For a collection Φ of smooth functions on $\mathbb{R}$ with common zero set Z, and a distribution u such that $\varphi u = 0$ for all $\varphi \in \Phi$, $\mathrm{spt} u \subset Z$. In particular, for Φ a subset of $C_o^\infty(\mathbb{R})$ having a single point $\{0\}$ as common zero set. Let $\mathcal{O}_0$ be the ring of germs of smooth functions at 0, and let $\mathbf{m}$ be its unique maximal ideal, consisting of smooth functions vanishing at 0. Suppose that the image in $\mathcal{O}_0$ of the ideal generated by Φ in $C_o^\infty(\mathbf{R})$ is exactly $\mathbf{m}^n$. That is, we suppose that all functions in Φ vanish at 0 to order at least n, and every germ of a smooth function at 0 vanishing to order at least n is a linear combination over $\mathcal{O}_0$ of elements of Φ. Let u be a distribution so that $\varphi u = 0$ for all $\varphi \in \Phi$. *Then u is a complex linear combination of* $\delta_0, \ldots, \delta_0^{(n-1)}$.

Consider the tempered distributions

$$u(f) \;=\; \sum_{n \in \mathbf{Z}} f(n) \qquad\qquad v(f) \;=\; \sum_{n \in \mathbf{Z}} \hat{f}(n)$$

The *Poisson summation formula* asserts that $u = v$. We will identify properties possessed by both u and v, and prove that there is a unique tempered distribution with these properties.

Certainly $u(\psi_n f) = u(f)$ for $n \in \mathbb{Z}$, and $u(R_n f) = u(f)$ for $n \in \mathbb{Z}$. Thus,

$$\psi_n u \;=\; u \qquad R_n u = u \qquad\qquad (\text{for all } n \in \mathbb{Z})$$

The two identities above which *intertwine* Fourier transform and the regular representation imply that v has the same properties. Further, letting $\gamma(x) := e^{-\pi x^2}$, we have $\hat{\gamma} = \gamma$, and so $u(\gamma) \;=\; v(\gamma)$.

Now we prove that the space of tempered distributions w such that

$$\psi_n w \;=\; w \qquad\qquad R_n w \;=\; w \qquad\qquad (\text{for all } n \in \mathbb{Z})$$

is *one-dimensional* over $\mathbb{C}$. This, together with the evaluation of both u and v on γ, will prove the Poisson summation formula.

The common zero set of the collection $\Phi = \{\psi_n - 1 : n \in \mathbb{Z}\}$ is $\mathbb{Z}$, so any distribution w annihilated by multiplication by all $\psi_n - 1$ must be supported on $\mathbb{Z}$. Let $\varphi \in C_o^\infty(\mathbb{R})$ be such that $\mathrm{spt}\,\varphi \cap \mathbb{Z} = \{0\}$ and $\varphi =\equiv 1$ on some neighborhood of 0. Then $\mathrm{spt}(\varphi w) = \{0\}$. Further, since the $\psi_n - 1$ generate the whole maximal ideal in the ring of germs of smooth functions at 0, we conclude that φw is a constant multiple of δ. By use of a partition of unity to localize the issues, we find that

$$w \;=\; \sum \, c_n \, \delta_n$$

for some constants c_n. The translation invariance of w implies that all the constants c_n are the same. Thus, there is a constant c so that

$$w = c \sum \delta_n$$

This is the desired uniqueness (one-dimensionality) assertion. ///
