(*January 5, 2020*)

Paley-Wiener theorems

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[This document is
http://www.math.umn.edu/~garrett/m/real/notes_2019-20/08g_Paley-Wiener.pdf]

1. Paley-Wiener theorem for test functions $\mathcal{D}$
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Of course, the original version [Paley-Wiener 1934] referred to L^2 functions, not distributions. The distributional aspect is from [Schwartz 1952]. Gelfand-Pettis vector-valued integral techniques are introduced. Proofs are given just for $\mathbb{R}$, where all ideas are already manifest.

As noted in the last section, as a corollary of the Paley-Wiener theorem for test functions, Fourier transforms of *arbitrary* distributions exist, and lie in the *dual* of the Paley-Wiener space PW.

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1. *Paley-Wiener theorem for test functions $\mathcal{D}$*

[1.1] Theorem: A test function f supported on a closed ball B_r of radius r at the origin in $\mathbb{R}$ has Fourier transform $\widehat{f}$ extending to an entire function on $\mathbb{C}$, with

$$|\widehat{f}(z)| \ll_N (1+|z|)^{-N}\, e^{r\cdot|y|} \qquad \text{(for } z = x+iy \in \mathbb{C}\text{, for every } N\text{)}$$

Conversely, an entire function satisfying such an estimate has Fourier transform which is a test function supported in the ball of radius r.

[1.2] Remark: Most of the proof is as expected. The interesting point is that rate-of-growth in the imaginary part determines the support of the inverse Fourier transforms.

Proof: First, the integral for $\widehat{f}(z)$ is the integral of the compactly-supported, continuous, entire-function-valued [1] function,

$$\xi \longrightarrow \Big(z \to f(\xi)\cdot e^{-i\xi z}\Big)$$

Thus, the Gelfand-Pettis integral exists, and is entire. Multiplication by z is converted to differentiation inside the integral,

$$(-iz)^N \cdot \widehat{f}(z) \;=\; \int_{B_r} \frac{\partial^N}{\partial\xi^N} e^{-iz\cdot\xi}\cdot f(\xi)\; d\xi \;=\; (-1)^N \int_{B_r} e^{-iz\cdot\xi}\cdot \frac{\partial^N}{\partial\xi^N} f(\xi)\; d\xi$$

by integration by parts. Note that differentiation does not enlarge support. Thus,

$$|\widehat{f}(z)| \ll_N (1+|z|)^{-N}\cdot \Big|\int_{B_r} e^{-iz\cdot\xi}\, f^{(N)}(\xi)\; d\xi\Big| \;\le\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}\cdot\Big|\int_{B_r} e^{-ix\cdot\xi}\, f^{(N)}(\xi)\; d\xi\Big|$$

$$\le\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}\cdot \int_{B_r} |f^{(N)}(\xi)|\; d\xi \;\ll_{f,N}\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}$$

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[1] As usual, the space of entire functions can be given the sups-on-compacts semi-norms $\sup_{z\in K}|f(z)|$. Since $\mathbb{C}$ can be covered by countably-many compacts, this topology is metrizable. Cauchy's integral formula proves *completeness*, so this space is Fréchet.

Conversely, let F be an entire function with the indicated growth and decay property, and show that

$$\varphi(\xi) \;=\; \int e^{ix\xi}\, F(x)\; dx$$

is a test function with support inside B_r. Note that the assumptions on F do *not* directly assert that F is Schwartz, so we cannot directly conclude that φ is smooth. Nevertheless, a similar obvious computation would give

$$\int (ix)^N \cdot e^{ix\xi}\, F(x)\; dx \;=\; \int \frac{\partial^N}{\partial \xi^N} e^{ix\xi}\, F(x)\; dx \;=\; \frac{\partial^N}{\partial \xi^N} \int e^{ix\xi}\, F(x)\; dx$$

Of course, moving the differentiation outside the integral is *necessary*. As expected, it is *justified* in terms of Gelfand-Pettis integrals, as follows. Since F strongly vanishes at ∞, the integrand extends continuously to the stereographic-projection one-point compactification of $\mathbb{R}$, giving a compactly-supported smooth-function-valued function on this compactification. The measure on the compactification can be adjusted to be finite, taking advantage of the rapid decay of F:

$$\varphi(\xi) \;=\; \int e^{ix\xi}\, F(x)\; dx \;=\; \int e^{ix\xi}\, F(x)\, (1+x^2)^N\; \frac{dx}{(1+x^2)^N}$$

Thus, the Gelfand-Pettis integral exists, and φ is smooth. Thus, in fact, the justification proves that such an integral of smooth functions is smooth without necessarily producing a formula for derivatives.

To see that φ is supported inside B_r, observe that, taking y of the same sign as ξ,

$$\Big| F(x+iy)\cdot e^{i\xi(x+iy)} \Big| \;\ll_N\; (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}$$

Thus,

$$|\varphi(\xi)| \;\ll_N\; \int_{\mathbb{R}} (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}\; dx \;\le\; e^{(r-|\xi|)\cdot|y|} \cdot \int_{\mathbb{R}} \frac{dx}{(1+|x|)^{-N}}$$

For $|\xi| > r$, letting $|y| \to +\infty$ shows that $\varphi(\xi) = 0$. ///

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2. *Paley-Wiener theorem for compactly-supported distributions $\mathcal{E}^*$*

[2.1] Theorem: The Fourier transform $\widehat{u}$ of a distribution u supported in B_r, of order N, is (integration against) the function $x \to u(\xi \to e^{-ix\xi})$, which is *smooth*, and extends to an *entire* function satisfying

$$|\widehat{u}(z)| \;\ll\; (1+|z|)^N \cdot e^{r\cdot|y|}$$

Conversely, an entire function meeting such a bound is the Fourier transform of a distribution of order N supported inside B_r.

Proof: Recall that the Fourier transform $\widehat{u}$ is the tempered distribution defined for Schwartz functions φ by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \;=\; u\Big(\xi \to \int_{\mathbb{R}} e^{-ix\xi}\, \varphi(x)\; dx\Big) \;=\; \int_{\mathbb{R}} u(\xi \to e^{-ix\xi})\, \varphi(x)\; dx$$

since $x \to (\xi \to e^{-ix\xi}\varphi(\xi)$ extends to a continuous smooth-function-valued function on the stereographic-projection one-point compactification of $\mathbb{R}$, and Gelfand-Pettis applies. Thus, as expected, $\widehat{u}$ is integration against $x \to u(\xi \to e^{-ix\xi})$.

The smooth-function-valued function $z \to (\xi \to e^{-iz\xi})$ is holomorphic in z. Compactly-supported distributions constitute the dual of $C^\infty(\mathbb{R})$, so application of u gives a holomorphic *scalar*-valued function $z \to u(\xi \to e^{-iz\xi})$.

Let ν_N be the N^{th}-derivative seminorm on $C^\infty(B_r)$, so

$$|u(\varphi)| \ll_\varepsilon \nu_N(\varphi)$$

Then

$$|\widehat{u}(z)| = |u(\xi \to e^{-iz\xi})| \ll_\varepsilon \nu_N(\xi \to e^{-iz\xi}) \ll \sup_{B_r} \Big|(1+|z|)^N e^{-iz\xi}\Big| \le (1+|z|)^N e^{r\cdot|y|}$$

Conversely, let F be an entire function with $|F(z)| \ll (1+|z|)^N e^{r\cdot|y|}$. Certainly F is a tempered distribution, so $F = \widehat{u}$ for a tempered distribution. We show that u is of order at most N and has support in B_r.

With η supported on B_1 with $\eta \ge 0$ and $\int \eta = 1$, make an *approximate identity* $\eta_\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon \to 0^+$. By the easy half of Paley-Wiener for test functions, $\widehat{\eta}_\varepsilon$ is entire and satisfies

$$|\widehat{\eta}_\varepsilon(z)| \ll_{\varepsilon,N} (1+|z|)^{-N} \cdot e^{\varepsilon\cdot|y|} \qquad \text{(for all } N)$$

Note that $\widehat{\eta}_\varepsilon(x) = \widehat{\eta}(\varepsilon \cdot x)$ goes to 1 as tempered distribution

By the more difficult half of Paley-Wiener for test functions, $F \cdot \widehat{\eta}_\varepsilon$ is $\widehat{\varphi}_\varepsilon$ for some test function φ_ε supported in $B_{r+\varepsilon}$. Note that $F \cdot \widehat{\eta}_\varepsilon \to F$.

For Schwartz function g with the support of $\widehat{g}$ not meeting B_r, $\widehat{g} \cdot \varphi_\varepsilon$ for sufficiently small $\varepsilon > 0$. Since $F \cdot \widehat{\eta}_\varepsilon$ is a bounded Cauchy net as tempered distributions,

$$u(\widehat{g}) = \widehat{u}(g) = \int F \cdot g = \int \lim_\varepsilon (F \cdot \widehat{\eta}_\varepsilon)\, g = \lim_\varepsilon \int (F \cdot \widehat{\eta}_\varepsilon)\, g = \lim_\varepsilon \int \widehat{\varphi}_\varepsilon\, g = \lim_\varepsilon \int \varphi_\varepsilon\, \widehat{g} = 0$$

This shows that the support of u is inside B_r. ///

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3. *Topology on Paley-Wiener spaces*

Let PW_r be the set of entire functions h such that

$$|h(z)| \ll_N (1+|z|)^{-N} e^{r\cdot|y|} \qquad \text{(for } z = x+iy \in \mathbb{C}, \text{ for every } N)$$

and

$$\mathrm{PW} = \bigcup_r \mathrm{PW}_r = \mathrm{colim}_r \mathrm{PW}_r$$

at least as a vector space. The Paley-Wiener theorem for test functions asserts that Fourier transform gives a linear bijection of the space $\mathcal{D}_r$ of test functions supported on the ball B_r to PW_r.

Topologies on PW_r completely determine the topology on the (locally convex!) colimit PW. Surely we should topologize PW_r so that Fourier transform gives a (topological) isomorphism from $\mathcal{D}_r$. Rather than topologizing PW_r indirectly by requiring this, we can examine the proof of the Paley-Wiener theorem to see that PW_r should be topologized by the family of seminorms

$$\nu_N(h) = \sup_z (1+|z|)^N e^{-r\cdot|y|} \cdot |h(z)| \qquad \text{(for } z = x+iy)$$

In particular, the usual sups-on-compacts topology on entire functions is too coarse.

[... iou ...]

4. *Fourier transforms of arbitrary distributions*

Fourier transform F gives an isomorphism $F : \mathcal{D} \to \mathrm{PW}$, which is a *restriction* of $F : \mathscr{S} \to \mathscr{S}$. By dualizing we have an isomorphism $F^* : \mathrm{PW}^* \to \mathcal{D}^*$, an *extension* of $F^* : \mathscr{S}^* \to \mathscr{S}^*$. Then $(F^*)^{-1}$ is a Fourier transform $(F^*)^{-1} : \mathcal{D}^* \to \mathrm{PW}^*$, *extending* the other direction of the isomorphism $F^* : \mathscr{S}^* \to \mathscr{S}^*$.

[... iou ...]

[Paley-Wiener 1934] R. Paley, N. Wiener, *Fourier transforms in the complex domain*, AMS Coll. Publ. XIX, NY, 1934.

[Schwartz 1950/51] L. Schwartz, *Théorie des Distributions*, I,II Hermann, Paris, 1950/51, 3rd edition, 1965.

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