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08i. Peano and Borel on arbitrary Taylor-Maclaurin coefficients

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1. Arbitrary Taylor coefficients
2. Taylor-Maclaurin expansions with remainder

Here we'll prove Giuseppe Peano's theorem from 1884, rediscovered by Emile Borel in 1895, that for any sequence $\{c_n\}$ of complex numbers there is an infinitely differentiable function f near 0 (or on $\mathbb{R}$, if one wants) so that $c_n = f^{(n)}(0)/n!$. The point is that there is no growth condition whatsoever imposed upon the sequence $\{c_n\}$. In particular, if the coefficients grow very quickly the radius of convergence of the infinite Taylor series will be 0, of course.

The proof we give is not strictly in the style of either Peano or Borel. In particular, it is anachronistic in that it uses test functions. Peano's proof [Genocchi-Peano 1884], resurrected in [Besenyei 2014], is also given in [Caicedo 2013], from which I learned of the existence of Peano's argument. Borel's argument (from his thesis) is linked-to in that same reference.

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1. Arbitrary Taylor-Maclaurin coefficients

Fix a test function ψ which is supported on $[-1,+1]$, and is identically equal to 1 on some neighborhood of 0. Given a sequence $\{c_n\}$ of complex numbers, define

$$f(x) = \sum_{0\le n<\infty} c_n\, x^n\, \psi(\max(n,|c_n|)\cdot x)$$

We claim that this defines an infinitely differentiable function so that

$$f^{(n)}(0) = c_n$$

To prove this, we need to consider a slightly more general expression. Let

$$F(x) = \sum_{0\le n<\infty} c_n\, x^n\, \varphi(\max(n,|c_n|)\cdot x)$$

where φ is a test function with support inside $[-1,+1]$. We claim that $F(x)$ is continuously differentiable for any real x, and has derivative given by the obvious formula

$$F'(x) = \sum_{0\le n<\infty} c_n\, \big(x^n\, \max(n,|c_n|)\varphi'(\max(n,|c_n|)\cdot x) + nx^{n-1}\, \varphi(\max(n,|c_n|)\cdot x)\big)$$

In fact, such expressions are readily seen to be infinitely differentiable for $x \ne 0$, since for x in a compact subset of $\mathbb{R}$ not containing 0 the sum is (uniformly) *finite*. To see this, note that $\varphi(\max(n,|c_n|)\, x) = 0$ unless

$$\max(n,|c_n|)\,|x| < 1$$

which certainly requires

$$n < \frac{1}{|x|}$$

There are only finitely-many positive integers n which meet this condition, and uniformly so for x in compacta away from 0, which proves that the sum is uniformly finite. Further, this uniform finiteness proves that for

$x \not= 0$ the derivative is the expression given above, since it is certainly legitimate to differentiate termwise a finite sum.

It remains to prove the differentiability at 0, and that the derivative at 0 is the value $c_1\varphi(0)$ at 0 of the obvious expression. Let

$$g(x) = F(x) - (c_0 + c_1\, x\, \varphi(\max(1, |c_1|)\, x)))$$

Then for $|x| < 1$ we have

$$|g(x)| \le \sum_{2\le n<\infty} |c_n|\; |x|^n\; \varphi(\max(n, |c_n|)\cdot x)$$

Since the support of φ is $[-1, 1]$, $\varphi(\max(n, |c_n|)\cdot x) = 0$ unless

$$|c_n| < \frac{1}{|x|}$$

Thus,

$$|g(x)| \le \sum_{2\le n<\infty} \frac{1}{|x|}\; |x|^n \le \frac{|x|^2}{1-|x|}$$

Thus, g is certainly continuous and differentiable at 0, and has derivative 0 at 0:

$$|\frac{g(h) - g(0)}{h}| \le \frac{|h|^2/(1-|h|)}{|h|} \to 0$$

as $h \to 0$. Thus,

$$F(x) = g(x) + c_0 + c_1\, x\, \varphi(\max(1, |c_1|)\, x))$$

is continuous and differentiable at 0, and has derivative at 0 given by

$$F'(0) = g'(0) + 0 + c_1\,[1\cdot\varphi(\max(1, |c_1|)\cdot 0)) + 0\cdot \max(1, |c_1|)\cdot\varphi(\max(1, |c_1|)\cdot 0)]$$

$$= 0 + 0 + c_1\,[\varphi(0) + 0] = c_1\varphi(0)$$

Thus, by induction, any such expression is in fact infinitely differentiable, with derivative given in the obvious way by differentiation term-by-term.

In particular, we can inductively compute the values of the derivatives of F at 0, and (by induction, by what we've shown) they are given by the obvious formula, differentiating term-by-term. Using the fact that $\psi(0) = 1$, we find that the n^{th} derivative of $f(x)$ at 0 is $n!\cdot c_n$, as desired.

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2. *Taylor-Maclaurin expansions with remainder*

For the reader's convenience, we recall a proof of on form of Taylor-Maclaurin expansion with remainder. We do this in just one variable, for notational simplicity:

[2.1] **Theorem:** For f k-times continuously differentiable on some interval $I = [x_1, x_2]$ containing a point a, for $x \in I$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(c)}{k!}(x-a)^{k+1}$$

for some c between a and x.

Proof: For $t \in I$, let

$$P_t(x) = f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \ldots + \frac{f^{(k)}(t)}{k!}(x-t)^k$$

be the order-k Taylor polynomial of f expanded at t. Let

$$F(t) = (f(x) - P_t(x)) - \left(\frac{f(x)-P_a(x)}{(x-a)^{k+1}}(x-t)^{k+1}\right)$$

Certainly $F(x) = 0$, and also

$$F(0) = (f(x) - P_a(x)) - \left(\frac{f(x)-P_a(x)}{x^{k+1}}x^{k+1}\right) = (f(x)-P_a(x)) - (f(x)-P_a(x)) = 0$$

Therefore, by the so-called Rolle's Theorem, there is c between a and x so that

$$F'(c) = 0$$

That is,

$$0 = -f'(c) + (-f''(c)(x-c) + f'(c)) + \left(-\frac{f'''(c)}{2!}(x-c)^2 + \frac{f''(c)}{2!}2(x-c)\right)$$

$$\ldots + \left(-\frac{f^{(k+1)}(c)}{k!}(x-c)^k + \frac{f^{(k)}(c)}{k!}k(x-c)^{k-1}\right) + \frac{f(x)-P_a(x)}{(x-a)^{k+1}}(k+1)(x-c)^k$$

This telescopes to

$$0 = -\frac{f^{(k+1)}(c)}{k!}(x-c)^k + \frac{f(x)-P_a(x)}{(x-a)^{k+1}}(k+1)(x-c)^k$$

Therefore, multiplying out by $(x-a)^{k+1}$, cancelling the common factor of $(x-c)^k$, and dividing by $k+1$, we have

$$f(x) = P_a(x) + \frac{f^{k+1}(c)}{(k+1)!}(x-a)^{k+1}$$

As a corollary, we also conclude that $f(x) - P_a(x)$ is a differentiable function which is bounded (for x bounded) by

$$|f(x) - P_a(x)| < \text{constant} \times |x-a|^{k+1}$$

for $|x-a| \le \max(|x_1 - a|, |x_2 - a|)$ with the constant being

$$\text{constant} = \sup_{c \in I} |f^{k+1}(c)|$$

Bibliography

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