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Distributions: examples

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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1. *Distributions* $|x|^s$ *and* $\mathrm{sgn}(x)\cdot|x|^s$ *on* $\mathbb{R}$

On $\mathbb{R}$, the functions $|x|^s$ and $\mathrm{sgn}(x)\cdot|x|^s$ are locally integrable for $\mathrm{Re}(s) > -1$, so for s in that range give distributions, by integration-against:

$$u_s(\varphi) \;=\; \int_{\mathbb{R}} |x|^s \cdot \varphi(x)\, dx \qquad\qquad v_s(\varphi) \;=\; \int_{\mathbb{R}} \mathrm{sgn}(x)\cdot|x|^s \cdot \varphi(x)\, dx \qquad\qquad (\text{for } \varphi \in \mathcal{D})$$

where $\mathcal{D} = C_c^\infty(\mathbb{R})$ is the space of test functions. Let $\mathscr{S}$ be Schwartz functions, and $\mathscr{S}^*$ tempered distributions, with $\mathcal{D}^*$ the collection of *all* distributions.

[1.1] The differential equations For $\mathrm{Re}(s) > 1$, these distributions are integration-against classically differentiable functions, so their derivatives can be computed as classical limits-of-difference-quotients:

$$\frac{d}{dx}|x|^s \;=\; s\cdot\mathrm{sgn}(x)\cdot|x|^{s-1} \qquad\qquad \frac{d}{dx}\mathrm{sgn}(x)\cdot|x|^s \;=\; s\cdot|x|^{s-1}$$

Thus,

$$x\frac{d}{dx}|x|^s \;=\; s\cdot|x|^s \qquad\qquad x\frac{d}{dx}\mathrm{sgn}(x)\cdot|x|^s \;=\; s\cdot\mathrm{sgn}(x)\cdot|x|^s$$

[1.2] Holomorphy in $\mathrm{Re}(s) > 0$ From the appearance of the integral formulas describing them, it is surely *plausible* that for $\mathrm{Re}(s) > 0$ the distribution-valued functions $s \to u_s$ and $s \to v_s$ are *holomorphic* $\mathcal{D}^*$-valued functions. Further, considering that their pointwise values grow polynomially at infinity, it seems plausible that they are holomorphic $\mathscr{S}^*$-valued functions.

We make stronger claims: for $\mathrm{Re}(s) \gg 1$, $s \to u_s$ is a holomorphic C^o-valued function, hence a holomorphic $\mathcal{D}^*$-valued function. In fact, in that region, it is a holomorphic function C^o_{mod}-valued function (continuous functions of moderate growth), hence is a holomorphic $\mathscr{S}^*$-valued function. The second assertion is stronger than the first, but it is useful to prove the weaker, simpler statement first.

For $x \in \mathbb{R}$ bounded away from 0, then the issue becomes simpler, because $(x, s) \to x^s$ is locally holomorphic in both variables. But sets including 0 require a bit of finesse.

We show this for u_s by direct computation by checking complex differentiability in the natural sups-on-compacts Fréchet-space structure of C^o. By even-ness of u_s, it suffices to treat the compacts $K = \{x : 0 \leq x \leq 1\}$ and $\{x : 1 \leq x \leq N\}$ for $N = 1, 2, \ldots$, since these cover $\mathbb{R}$. The compact K, including 0, requires the most attention.

Take $\mathrm{Re}(s) \geq 3$. Given fixed $h \in \mathbb{C}$ with $0 < |h| < 1$, break K into two parts:

$$K_0 \;=\; \{x : 0 \leq x \leq \sqrt{|h|}\} \qquad\qquad K_1 \;=\; \{x : \sqrt{|h|} \leq x \leq 1\}$$

For $x \in K_0$,

$$\Big|\frac{x^{s+h} - x^s}{h} - x^s \cdot \log x\Big| \;\leq\; \frac{2\sqrt{|h|}^3}{|h|} + \sqrt{|h|}^3 \cdot |\log x| \;\ll\; \sqrt{|h|}$$

with constant uniform in $|h| \leq 1$. For $x \in K_1$, expanding in power series,

$$\frac{x^{s+h} - x^s}{h} - x^s \cdot \log x \;=\; x^s \cdot \Big(\frac{h(\log x)^2}{2!} + \frac{h^2(\log x)^3}{3!} + \ldots\Big)$$

For $1 \geq x \geq \sqrt{|h|}$, this gives

$$\Big|\frac{x^{s+h} - x^s}{h} - x^s \cdot \log x\Big| \;\leq\; |x^s| \cdot \Big(\frac{|h| \cdot |\frac{1}{2}\log|h||^2}{2!} + \frac{|h|^2 \cdot |\frac{1}{2}\log|h||^3}{3!} + \ldots\Big)$$

$$\leq\; |x^s| \cdot \sqrt{|h|} \cdot \Big(\frac{\sqrt{|h|} \cdot |\log\sqrt{|h|}|^2}{2!} + \frac{|h|^{3/2} \cdot |\log\sqrt{|h|}|^3}{3!} + \ldots\Big) \;\ll\; |x^s| \cdot \sqrt{|h|}$$

since the infinite series has a finite sum, due to $\sup_{0<t\leq 1} t \cdot |\log t| = 1/e < \infty$. Thus, we have a suitable estimate for complex differentiability in s for $0 \leq x \leq 1$.

For $1 \leq x \leq N$, similarly, with $|h| < 1$,

$$\Big|\frac{x^{s+h} - x^s}{h} - x^s \cdot \log x\Big| \;\leq\; |x^s| \cdot \Big(\frac{|h| \cdot (\log x)^2}{2!} + \frac{|h|^2 \cdot (\log x)^3}{3!} + \ldots\Big)$$

$$\leq\; |x^s| \cdot |h|\Big(\frac{(\log x)^2}{2!} + \frac{|h| \cdot (\log x)^3}{3!} + \ldots\Big) \;\leq\; |x^s| \cdot |h|\Big(\frac{(\log N)^2}{2!} + \frac{|h| \cdot (\log N)^3}{3!} + \ldots\Big) \;\ll_N\; |x^s| \cdot |h|$$

This completes the proof of complex differentiability of the C^o-function-valued function $s \to u_s$ in $\mathrm{Re}(s) \geq 3$.

The proof of complex differentiability for $s \to u_s$ as a function taking values in the space C^o_{mod} of C^o functions *of moderate growth* is a variant of the previous. That space C^o_{mod} is an LF-space, described as follows. For $f \in C^o_c$ and $N \geq 0$, define a seminorm by

$$|f|_{C^o_N} \;=\; \sup_{x \in \mathbb{R}} |f(x)| \cdot (1 + x^2)^{-N}$$

and [1]

$$C^o_N \;=\; \text{completion of } C^o_c \text{ with respect to } |\cdot|_{C^o_N}$$

and

$$C^o_{\mathrm{mod}} \;=\; \mathrm{colim}_N\, C^o_N \qquad\qquad \text{(in the category of locally convex spaces)}$$

For example, *bounded* continuous functions are not necessarily in C^o_0, but are in C^o_ε for every $\varepsilon > 0$.

[1] In many other contexts, understanding of the limitands as being as portrayed here, is an analogue of the technical problems with considering the space of bounded continuous functions on $\mathbb{R}$ with sup norm. It is certainly a Banach space, but the translation action of $\mathbb{R}$ on it is *not continuous*. A correct substitute to avoid that problem is to consider the continuous functions going to 0 at infinity, instead. This is the sup-norm completion of C^o_c, so the action of $\mathbb{R}$ is continuous. For controlled growth rates, an analogous refinement is necessary to have continuous action. Luckily, these distinctions disappear in the colimit, so the situation is one of those where naivete is fortunately not immediately disastrous.

Estimates on $(x^{s+h} - x^s)/h - x^s \log x$ for $0 \le x \le 1$ proceed as in the previous discussion.

For estimates for $x \in [1, +\infty)$, fix a compact subset of $\mathrm{Re}(s) > 0$, inside a strip $0 < 3 \le \mathrm{Re}(s) \le b$, and take $N \gg b$.

$$(1+x^2)^N \cdot \Big|\frac{x^{s+h} - x^s}{h} - x^s \cdot \log x\Big| \;=\; (1+x^2)^N \cdot |x^s| \cdot \Big(\frac{h \cdot (\log x)^2}{2!} + \frac{h^2 \cdot (\log x)^3}{3!} + \ldots\Big) \;\ll_{b,N}\; |h|$$

This gives complex differentiability as a C^o_{mod}-valued function of s. ///

[1.3] Meromorphic extension to $s \in \mathbb{C}$

From the equations

$$\frac{d}{dx}|x|^s \;=\; s \cdot \mathrm{sgn}(x) \cdot |x|^{s-1} \qquad\qquad \frac{d}{dx}\mathrm{sgn}(x) \cdot |x|^s \;=\; s \cdot |x|^{s-1}$$

by the identity principle from complex analysis for distribution-valued functions, [2] the same identities hold whenever both sides make sense. Conveniently, the notation/convention distinction between degree of homogeneity for a function versus the distribution of integration-against that function disappears when looking at the corresponding differential equation, because $\frac{d}{dx}u(tx) = t\frac{du}{dx}(tx)$. Rearranging and iterating,

$$\mathrm{sgn}(x) \cdot |x|^{s-1} \;=\; \frac{\frac{d}{dx}|x|^s}{s} \;=\; \frac{\frac{d^2}{dx^2}\mathrm{sgn}(x) \cdot |x|^{s+1}}{s(s+1)} \qquad\qquad |x|^{s-1} \;=\; \frac{\frac{d}{dx}\mathrm{sgn}(x) \cdot |x|^s}{s} \;=\; \frac{\frac{d^2}{dx^2}|x|^{s+1}}{s(s+1)}$$

gives strip-wise meromorphic continuations as tempered distributions, since derivatives of tempered distributions are again tempered distributions. From the explicit step-wise meromorphic continuations, the poles are at most *simple*. For the moment, we have only a weak result that the poles of u_s and v_s are contained in $\{-1, -2, -3, -4, \ldots\}$.

[1.4] Residues of $s \to u_s$ and $s \to v_s$ are distributions

The residues and other Laurent and power series coefficients of $s \to u_s$ and $s \to v_s$ are also tempered distributions, seen as follows.

The Grothendieck-Schwartz extension of Cauchy-Goursat theory gives Laurent or power series expansions

$$f(s) \;=\; \sum_{n \ge -N} c_n (s - s_o)^n$$

for $\mathscr{S}^*$-valued meromorphic functions $f(s)$, with contour integral expressions

$$c_n \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w - s_o)^{n+1}}\, dw$$

for the coefficients. The integrands are compactly supported, continuous, $\mathscr{S}^*$-valued, so as Gelfand-Pettis integrals exist in the same space $\mathscr{S}^*$.

[1.5] Residues and leading Laurent terms inherit homogeneity

Residues or leading Laurent terms, if non-zero, inherit the homogeneity, parity, and still satisfy the differential equations noted earlier. For

[2] Beyond the classical Cauchy-Goursat complex function theory, this uses the Schwartz-Grothendieck extension to holomorphic functions taking values in a locally convex, quasi-complete topological vector space, which itself uses Gelfand-Pettis integrals, and uses the quasi-completeness of spaces of distributions or tempered distributions, as duals of LF-spaces or of Fréchet spaces.

example, for a simple pole of u_s at $s=s_o$ with residue w, applying $x\frac{d}{dx}$ to the Laurent expansion of u_s near $s=s_o$,[3]

$$x\frac{d}{dx}u_s \;=\; x\frac{d}{dx}\Big(\frac{w}{s-s_o}+c_o+c_1(s-s_o)+\ldots\Big) \;=\; \frac{x\frac{d}{dx}w}{s-s_o}+x\frac{d}{dx}c_o+x\frac{d}{dx}c_1(s-s_o)+\ldots$$

and this is equal, as $\mathcal{D}^*$-valued Laurent expansion, to

$$s\cdot\Big(\frac{w}{s-s_o}+c_o+c_1(s-s_o)+\ldots\Big) \;=\; (s-s_o)\Big(\frac{w}{s-s_o}+c_o+c_1(s-s_o)+\ldots\Big)+s_o\Big(\frac{w}{s-s_o}+c_o+c_1(s-s_o)+\ldots\Big)$$

$$=\; \Big(w+c_o(s-s_o)+c_1(s-s_o)^2+\ldots\Big)+\Big(\frac{s_ow}{s-s_o}+s_oc_o+s_oc_1(s-s_o)+\ldots\Big)$$

Equating residues at $s=s_o$ gives

$$x\frac{d}{dx}w \;=\; s_o\cdot w$$

as claimed. An identical argument would apply to the leading Laurent term at higher-order poles.

[1.6] Further restriction on poles For $\varphi\in\mathscr{S}$ with support away from 0, $u_s(\varphi)$ is entire, since the integrands do not blow up at 0. Thus, for any s_o, $(s-s_o)u_s(\varphi)|_{s=s_o}=0$. Thus, any possible residues have support $\{0\}$. The classification of distributions supported on $\{0\}$ is that they are exactly finite linear combinations of Dirac δ and its derivatives. The even-order derivatives are *even*, and the odd-order derivatives are *odd*.

Thus, poles of u_s can occur at most at s_o such that some even-order derivative $\delta^{(2k)}$ satisfies $x\frac{d}{dx}u=s_o\cdot u$, and poles of v_s can occur at most at s_o such that some odd-order derivative $\delta^{(2k+1)}$ satisfies $x\frac{d}{dx}u=s_o\cdot u$.

We claim that $x\frac{d}{dx}\delta^{(k)}=(-k-1)\delta^{(k)}$. In fact, first, an induction will show that $x\delta^{(k)}=-k\cdot\delta^{(k-1)}$: the base case is $x\cdot\delta=0$, and then

$$(x\cdot\delta^{(k)})(\varphi) \;=\; \delta^{(k)}(x\cdot\varphi) \;=\; -\delta^{(k-1)}(\frac{d}{dx}x\varphi) \;=\; -\delta^{(k-1)}(\varphi+x\varphi') \;=\; -\delta^{(k-1)}(\varphi)+(x\delta^{(k-1)})(\varphi')$$

$$=\; -\delta^{(k-1)}(\varphi)+(k-1)\cdot\delta^{(k-2)}(\varphi') \;=\; -\delta^{(k-1)}(\varphi)-(k-1)\cdot\delta^{(k-1)}(\varphi) \;=\; -k\cdot\delta^{(k-1)}(\varphi)$$

Then

$$x\frac{d}{dx}\delta^{(k)} \;=\; \frac{d}{dx}x\delta^{(k)}-\delta^{(k)} \;=\; \frac{d}{dx}(-k)\delta^{(k-1)}-\delta^{(k)} \;=\; (-k)\delta^{(k)}-\delta^{(k)} \;=\; (-k-1)\cdot\delta^{(k)}$$

Thus, we have proven

[1.7] Theorem: u_s has poles *at most* at $-1,-3,-5,\ldots$, and v_s has poles *at most* at $-2,-4,-6,\ldots$, and the residues, if any, are necessarily constant multiples of corresponding derivatives of δ.

[1.8] Poles actually occur To see that u_s really does have poles at $-1,-3,-5,\ldots$ and v_s at $-2,-4,-6,\ldots$, it suffices to see that $u_s(\varphi)$ has these poles for some choice of $\varphi\in\mathscr{S}$, and similarly for v_s. For example, taking $\varphi(x)=e^{-\pi x^2}$ since this is its own Fourier transform and we can re-use the computation just below,

$$u_s(e^{-x^2}) \;=\; \int_{\mathbb{R}^\times}|x|^s\cdot e^{-\pi x^2}\,dx \;=\; 2\int_0^\infty |x|^s\cdot e^{-\pi x^2}\,dx \;=\; 2\int_0^\infty |x|^{s+1}\cdot e^{-\pi x^2}\,\frac{dx}{x}$$

[3] Again, the legitimacy of applying d/dx termwise in the Laurent expansion is assured by the Grothendieck-Schwartz extension of Cauchy-Goursat.

$$= \int_0^\infty |x|^{\frac{s+1}{2}} \cdot e^{-\pi x} \, \frac{dx}{x} \;=\; \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})$$

This does have poles at $s = -1, -3, -5, \ldots$. Similarly, for v_s,

$$v_s(xe^{-\pi x^2}) \;=\; \int_{\mathbb{R}^\times} \operatorname{sgn}(x) \cdot |x|^s \cdot xe^{-\pi x^2} \, dx \;=\; \int_{\mathbb{R}^\times} \operatorname{sgn}(x) \cdot |x|^s \cdot xe^{-\pi x^2} \, dx$$

$$= \int_{\mathbb{R}^\times} |x|^{s+1} \cdot e^{-\pi x^2} \, dx \;=\; \pi^{-\frac{s+2}{2}} \Gamma(\frac{s+2}{2})$$

so v_s does have poles at $-2, -4, -6, \ldots$.

Thus, the residues are non-zero constant multiples of the corresponding derivatives of δ. To determine the constants,

[1.9] Determination of constants

Granting that the residues of u_s at $s = -1, -3, -5, \ldots$ and of v_s at $s = -2, -4, -6, \ldots$ are constant multiples c_k of corresponding derivatives of δ. We can determine the constants as follows.

On one hand, the previous section computes the residue applied to Gaussian $e^{-\pi x^2}$ in the even case, and applied to $xe^{-\pi x^2}$ in the odd case:

$$c_{2k+1} \cdot \delta^{(2k)}(e^{-\pi x^2}) \;=\; \operatorname{Res}_{s=-(2k+1)} \, \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}) \qquad \text{(for } k = 1, 3, 5, \ldots)$$

and

$$c_{2k+2} \cdot \delta^{(2k+1)}(e^{-\pi x^2}) \;=\; \operatorname{Res}_{s=-(2k+2)} \, \pi^{-\frac{s+2}{2}} \Gamma(\frac{s+2}{2}) \qquad \text{(for } k = 2, 4, 6, \ldots)$$

On the other hand, other hand, via Plancherel-Parseval,

$$\delta^{(2k)}(e^{-\pi x^2}) \;=\; \widehat{\delta^{(2k)}}(\widehat{e^{-\pi x^2}}) \;=\; (-2\pi i)^{2k} \int_{\mathbb{R}} x^{2k} \, e^{-\pi x^2} \, dx \;=\; 2(-2\pi i)^{2k} \int_0^\infty x^{2k+1} \, e^{-\pi x^2} \, \frac{dx}{x}$$

$$= (-2\pi i)^{2k} \int_0^\infty x^{\frac{2k+1}{2}} \, e^{-\pi x} \, \frac{dx}{x} \;=\; (-2\pi i)^{2k} \pi^{-\frac{2k+1}{2}} \Gamma(\frac{2k+1}{2})$$

and

$$\delta^{(2k+1)}(xe^{-\pi x^2}) \;=\; \widehat{\delta^{(2k+1)}}(\widehat{xe^{-\pi x^2}}) \;=\; -i(-2\pi i)^{2k+1} \int_{\mathbb{R}} x^{2k+1} \, xe^{-\pi x^2} \, dx$$

$$= -2i(-2\pi i)^{2k+1} \int_0^\infty x^{2k+3} \, e^{-\pi x^2} \, \frac{dx}{x} \;=\; -i(-2\pi i)^{2k+1} \int_0^\infty x^{\frac{2k+3}{2}} \, e^{-\pi x} \, \frac{dx}{x}$$

$$= -i(-2\pi i)^{2k+1} \pi^{-\frac{2k+3}{2}} \Gamma(\frac{2k+3}{2})$$

Thus,

$$c_{2k+1} \;=\; \frac{\operatorname{Res}_{s=-(2k+1)} \, \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})}{(-2\pi i)^{2k} \pi^{-\frac{2k+1}{2}} \Gamma(\frac{2k+1}{2})} \qquad \text{(for } k = 1, 3, 5, \ldots)$$

and similarly for c_{2k+2}.

[1.10] Group-theoretic homogeneity

For classical point-wise functions f, write $(f \circ t)(x) = f(tx)$ for $t \in \mathbb{R}^\times$ and $x \in \mathbb{R}$. For distributions u, by duality for $\varphi \in \mathcal{D}$ we can define $(u \circ t)(\varphi) = u(\varphi \circ t^{-1})$, or $u \circ t$ can be defined by extension by continuity, using the density of test functions in distributions. In $\operatorname{Re}(s) > -1$, the distributions are given by the literal integrals, and changing variables in the integrals gives homogeneity [4]

$$(u_s \circ t)(\varphi) \;=\; u_s(\varphi \circ t^{-1}) \;=\; |t|^{s+1} \cdot u_s(\varphi) \qquad\qquad (v_s \circ t)(\varphi) \;=\; v_s(\varphi \circ t^{-1}) \;=\; \mathrm{sgn}(t) \cdot |t|^{s+1} \cdot v_s(\varphi)$$

Again, the homogeneity with respect to the multiplication action of positive reals imples satisfaction of corresponding differential equations:

$$x\frac{d}{dx}u_s \;=\; s \cdot u_s \qquad\qquad x\frac{d}{dx}u_s \;=\; s \cdot u_s$$

[1.11] Theorem: *Uniqueness:* Up to scalar multiples, for $s \neq -1, -3, -5, \ldots$, u_s is the unique even distribution u satisfying $x\frac{d}{dx}u = s \cdot u$, and at those points/eigenvalues the unique distributions of that parity and satisfying the differential equation are $\delta, \delta^{(2)}, \delta^{(4)}, \ldots$. Up to scalar multiples, for $s \neq -2, -4, -6, \ldots$, v_s is the unique odd distribution u satisfying $x\frac{d}{x}u = s \cdot u$, and at those points the unique distributions of that parity and satisfying the differential equation are $\delta', \delta^{(3)}, \delta^{(5)}, \ldots$ [5] We treat just u_s, as v_s is almost identical.

[1.12] Remark: This existence and uniqueness proof does *not* use anything about meromorphic continuation, nor about any other means of *regularizing* non-convergent integrals, such as the principal-value integral. That is, the proof shows that the functionals given by *restricting* u_s and v_s to test functions vanishing to infinite order at 0 (which are defined by integrals converging for all $s \in \mathbb{C}$), *have unique extensions* to the full space of test functions.

[1.13] Remark: To prove uniqueness for just for tempered distributions would allow simpler arguments, using Fourier transforms, but the stronger uniqueness conclusion is useful. We have already shown that u_s and v_s (meromorphically continued) are tempered.

Proof: First, we grant that a distribution on $\mathbb{R}^n$ annihilated by all differential operators $\partial/\partial x_j$ is (integration against) a constant. [6] On $\mathbb{R}$, for distribution u satisfying $\frac{d}{dx}u = \lambda \cdot u$, multiplying by the smooth function $e^{-\lambda x}$ gives $\frac{d}{dx}(e^{-\lambda x} \cdot u) = 0$. Thus, u is a constant multiple of $e^{\lambda x}$. Similarly, on $\mathbb{R}^\times$, the invariant differential operator is $x\frac{d}{dx}$, and a similar argument proceeds on the connected component $(0, +\infty)$ of the identity in $\mathbb{R}^\times$: distributions on $(0, +\infty)$ satisfying $x\frac{d}{dx}u = s \cdot u$ are constant multiples of $|x|^s$.

The space $\mathcal{D}_0$ of even distributions vanishing to infinite order at 0 is *closed* in the space $\mathcal{D}_+$ of all even distributions. Since the literal integral converges very well, $w_s = u|_{\mathcal{D}_0}$ is given by that integral for all $s \in \mathbb{C}$. By the uniqueness result of the previous paragraph, on the space $\mathcal{D}(\mathbb{R}^\times)_+$ of even distributions supported away from 0, w_s is the unique element in $\mathcal{D}(\mathbb{R}^\times)_+^*$ satisfying the differential equation. This functional extends by continuity to the closure $\mathcal{D}_0$ of $\mathcal{D}(\mathbb{R}^\times)_+$ in $\mathcal{D}_+$.

[4] The normalization for homogeneity of distributions has a different normalization than that for homogeneity of functions, due to change-of-measure under dilations.

[5] I first saw this use of the Snake Lemma applied to an exact sequence of very small complexes, for issues of existence and uniqueness of extensions, in W. Casselman's discussion of extended automorphic forms.

[6] More generally, on a connected Lie group G, annihilation of a distribution by all elements of the Lie algebra implies that the distribution is (integration against) a constant. On $G = \mathbb{R}^n$, this annihilation is just annihilation by all partial derivative operators $\partial/\partial x_j$.

We prove existence[7] and uniqueness of an extension of $u|_{\mathcal{D}_0}$ to an element of $\mathcal{D}_+^*$.

Dualizing the (very short) exact sequence

$$0 \longrightarrow \mathcal{D}_0 \longrightarrow \mathcal{D}_+$$

and invoking Hahn-Banach, gives

$$0 \longrightarrow X \longrightarrow \mathcal{D}_+^* \longrightarrow \mathcal{D}_0^* \longrightarrow 0$$

where X is the space of even distributions supported at 0, namely, the space $\bigoplus_k \mathbb{C} \cdot \delta^{(k)}$ of finite linear combinations of Dirac δ and its even-order derivatives. Thus, we have

$$0 \longrightarrow \bigoplus_k \mathbb{C} \cdot \delta^{(k)} \longrightarrow \mathcal{D}_+^* \longrightarrow \mathcal{D}_0^* \longrightarrow 0$$

The operator $T = T_s = x\frac{d}{dx} - s$ stabilizes all the spaces involved, so we have a complex

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_k \mathbb{C} \cdot \delta^{(k)} & \longrightarrow & \mathcal{D}_+^* & \longrightarrow & \mathcal{D}_0^* & \longrightarrow & 0 \\ & & \downarrow T_s & & \downarrow T_s & & \downarrow T_s & & \\ 0 & \longrightarrow & \bigoplus_k \mathbb{C} \cdot \delta^{(k)} & \longrightarrow & \mathcal{D}_+^* & \longrightarrow & \mathcal{D}_0^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

From a complex of the form

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow T & & \downarrow T & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

the Snake Lemma gives a long exact sequence

$$0 \longrightarrow \ker T|_A \longrightarrow \ker T|_B \longrightarrow \ker T|_C \xrightarrow{\eta} \frac{A}{TA} \longrightarrow \frac{B}{TB} \longrightarrow \frac{C}{TC} \longrightarrow 0$$

with *connecting homomorphism* η, whose details we do not need. Here, we have an element of $\ker T|_C$, and we ask whether it is an image of an element from $\ker T|_B$, and, if so, whether it is uniquely so.

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[7] The meromorphically continued u_s agrees with w_s on $\mathcal{D}_0$, so gives a tangible extension of w_s, away from poles of u_s. But the technique of the uniqueness argument gives a stronger existence assertion, and the argument does not presume a meromorphic continuation.

Existence is equivalent to surjectivity of $\ker T|_B \to \ker T|_C$, which is implied by $A/TA = \{0\}$, without any information about the connecting homomorphism η. We know the distributions supported on $\{0\}$, in particular the even ones. By direct calculation, T_s is surjective on $A = \bigoplus_k \mathbb{C} \cdot \delta^{(2k)}$ except for $s = -1, -3, -5, \ldots$:

$$T_s\Big(\sum_{0\le k\in\mathbb{Z}} c_k \cdot \delta^{(2k)}\Big) \;=\; \sum_{0\le k\in\mathbb{Z}} c_k \cdot (-(2k+1) - s) \cdot \delta^{(2k)}$$

The sums are finite, and by hypothesis $-(2k+1) - s \neq 0$, so the map is invertible.

Uniqueness would follow from $\ker T|_A = \{0\}$. In the present circumstance, triviality of this kernel is that there is no *even* distribution supported at 0 and annihilated by $T = T_s$. As it happens, this is the same condition as that for existence, namely, $s \neq -1, -3, -5, \ldots$.

At $s = -1, -3, -5, \ldots$, the even distributions $\delta, \delta^{(2)}, \delta^{(4)}, \ldots$ satisfy $x\frac{d}{dx}u = s \cdot u$. To finish the uniqueness, since up to constants w_s is the unique functional on $\mathcal{D}_0$ satisfying the differential equation, it must be shown that w_s on $\mathcal{D}_0$ does *not* extend to a functional on $\mathcal{D}_+$ for these values of s. That is, we must show that $\ker T|_B \to \ker T|_C$ is not surjective. Since $\ker T|_C$ is always one-dimensional, it suffices to show that the connecting homomorphism is not the zero map. Thus, it suffices to show that $A/TA \to B/TB$ is not injective. That is, there is $u \in A \cap TB$ that is not in TA.

It suffices to find a *tempered* distribution u (of suitable parity) such that $(x\frac{d}{dx} + 2k)u = \delta^{(2k)}$. Thus, we can use Fourier transforms: under Fourier transform, the equation becomes

$$(-\frac{d}{dx} \circ x + 2k)\widehat{u} \;=\; (-2\pi i x)^{2k}$$

or

$$(-(x\frac{d}{dx} + 1) + 2k)\widehat{u} \;=\; (-2\pi i x)^{2k}$$

which is

$$(x\frac{d}{dx} - (2k-1))\widehat{u} \;=\; -(-2\pi i)^{2k} \cdot x^{2k}$$

Thus, $\widehat{u}$ satisfies an *inhomogeneous* version of the eigenfunction differential equation, where the inhomogeneity x^{2k} is itself a solution of the same differential equation.

Since $|x|^s$ itself satisfies $(x\frac{d}{dx} - s)|x|^s = 0$, the standard device [8] is to differentiate [9] this expression with respect to the spectral parameter s, obtaining

$$(x\frac{d}{dx} - s)\Big(|x|^s \cdot \log|x|\Big) \;=\; |x|^s$$

At $s = -2k$ this gives

$$(x\frac{d}{dx} + 2k)\Big(|x|^{2k} \cdot \log|x|\Big) \;=\; |x|^{2k}$$

Thus, up to irrelevant constants, the desired u is the Fourier transform of $|x|^{2k} \cdot \log|x|$. Since $|x|^{2k} \cdot \log|x|$ is not a polynomial, its Fourier transform is not a derivative of δ. By the classification of distributions supported at $\{0\}$, it cannot have support just $\{0\}$. Thus, $\delta^{(2k)}$ is in TB, but not in TA, since all distributions in TA have support $\{0\}$. ///

[8] This includes a method sometimes called *variation of parameters*. For example, for a differential operator L and a family of eigenfunctions u_λ meromorphically (or real-differentiably) parametrized by a complex variable λ with $Lu_\lambda = \lambda \cdot u_\lambda$, differentiate the latter expression in λ to obtain $L\frac{\partial u_\lambda}{\partial\lambda} = u_\lambda + \lambda\frac{\partial u_\lambda}{\partial\lambda}$. That is, $(L - \lambda)\frac{\partial u_\lambda}{\partial\lambda} = u_\lambda$.

[9] This differentiation is justified, once again, by the Schwartz-Grothendieck extension of Cauchy-Goursat.

[1.14] Distributions u with $x^k \cdot u = 0$ and $x\frac{d}{dx}u = \lambda \cdot u$ Suppose $x^k \cdot u = 0$ but $x^{k-1} \cdot u \neq 0$, and $x\frac{d}{dx}u = \lambda \cdot u$. Applying d/dx to $x^k \cdot u = 0$ gives

$$0 \;=\; kx^{k-1} \cdot u + x^k \cdot \frac{du}{dx} \;=\; kx^{k-1} \cdot u + x^{k-1} \cdot x\frac{du}{dx} \;=\; kx^{k-1} \cdot u + x^{k-1} \cdot \lambda \cdot u \;=\; (k+\lambda)x^{k-1} \cdot u$$

Since $x^{k-1} \cdot u \neq 0$, necessarily $\lambda = -k$.

This would give another proof that there are no distributions supported at $\{0\}$ satisfying the eigenfunction differential equation except for special eigenvalues, without necessarily classifying distributions supported at $\{0\}$.

There are also more physical/concrete proofs. For example,

[1.15] Claim: (Integration against) $1/|x|$ on $\mathbb{R}$ does not extend to a homogeneous, even distribution.

Proof: Suppose it did extend to a distribution u. Of course, by Hahn-Banach, if we disregard the homogeneity condition and the parity, there is an extension, but this is not the question. Let f be a test function identically 1 on $[-1,1]$, identically 0 outside $[-2,2]$, and monotonically decreasing (and real-valued) on $[1,2]$, and oppositely on $[-2,-1]$. Let $f_t(x) = f(tx)$. The homogeneity condition gives $u(f) = u(f_t)$ for all $t > 0$. With $g_t(x) = f(x) - f_t(x)$, this is $u(g_t) = 0$ for all $0 < t \in \mathbb{R}$. On the other hand, g_t vanishes identically on a neighborhood of 0, so $u(g)$ must be a constant multiple, say by constant c, of integration against $1/|x|$ (by uniqueness of invariant functionals). For $t > 1$, g_t is strictly positive, so the integral against $1/|x|$ is non-zero. Thus,

$$0 \;=\; u(f - f_t) \;=\; u(g_t) \;=\; c \cdot \int_{\mathbb{R}} \frac{g_t(x)}{|x|} dx$$

implies that $c = 0$. ///

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2. *Fourier transforms, principal value integrals, Frullani integrals*

The strong form of the uniqueness assertion is useful in qualitative computations, as in the following.

[2.1] Corollary: v_{-1} is the *principal-value* functional

$$h(\varphi) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx \qquad\qquad \text{(for } \varphi \text{ Schwartz)}$$

Proof: Visibly, h is *odd*. To ascertain its homogeneity, for $t > 0$,

$$(h \circ t)(\varphi) \;=\; h(\varphi \circ t^{-1}) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(t^{-1}x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|tx| \geq \varepsilon} \frac{\varphi(x)}{tx}\, d(tx)$$

$$=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon/t} \frac{\varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx \;=\; h(\varphi)$$

This is the same homogeneity as v_{-1}. By uniqueness, h is a constant multiple of v_{-1}. Applying both to $\varphi(x) = xe^{-\pi x^2}$ shows that the constant is 1. ///

[2.2] Fourier transform of principal value integral To determine the Fourier transform of the principal value integral $h(f) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)}{x}\, dx$, recall that this distribution is homogeneous of degree -1, and is *odd*. By the above, its Fourier transform is of degree $-(1-1) = 0$, and is odd. By uniqueness, $\widehat{\varphi}$ must be a $c \cdot \mathrm{sgn}(x)$ of *sign*. To determine the constant, use the odd Schwartz function $\varphi(x) = xe^{-\pi x^2}$, whose Fourier transform is $-ixe^{-\pi x^2}$. On one hand,

$$\widehat{h}(\varphi) \;=\; h(\widehat{\varphi}) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{-ixe^{-\pi x^2}}{x}\,dx \;=\; -i\cdot \lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} e^{-\pi x^2}\,dx \;=\; -i\int_{\mathbb{R}} e^{-\pi x^2}\,dx \;=\; -i$$

On the other hand,

$$\int_{\mathbb{R}} \mathrm{sgn}(x)\cdot xe^{-\pi x^2}\,dx \;=\; 2\int_0^\infty xe^{-\pi x^2}\,dx \;=\; -\frac{2}{-2\pi}\int_0^\infty e^{-\pi x^2}\,dx \;=\; \frac{1}{\pi}\int_{\mathbb{R}} e^{-\pi x^2}\,dx \;=\; \frac{1}{\pi}$$

Thus,

$$-i \;=\; \widehat{h}(\varphi) \;=\; c\cdot(\mathrm{sgn} x)(\varphi) \;=\; c\cdot\frac{1}{\pi}$$

and $c = -i\pi$. That is, $\widehat{h} = -i\pi\mathrm{sgn}(x)$.

[2.3] Fourier transform of $\frac{1}{|x|^s}$ Most of the work to determine the Fourier transforms of u_s and v_s was done earlier, in the course of proving uniqueness of distributions satisfying parity and homogeneity conditions. Since Fourier transform interacts well with parity and homogeneity, the Fourier transforms will have the same parity and homogeneity determined as follows. The homogeneity can be invoked either at the group level by the way Fourier transform interacts with dilations, or at the algebra level by the way Fourier transform interacts with differentiation and multiplication by x. For the latter, letting $\partial = d/dx$, Fourier transform converts the equation

$$s\cdot u_s \;=\; (x\circ\partial)u_s$$

to

$$s\cdot \widehat{u}_s \;=\; -(\partial\circ x)\widehat{u}_s \;=\; -(x\circ\partial+1)\widehat{u}_s$$

so

$$(x\circ\partial)\widehat{u}_s \;=\; -(s+1)\cdot\widehat{u}_s$$

By the strong uniqueness, $\widehat{u}_s$ is a constant multiple of u_{-1-s}. Similarly, $\widehat{v}_s$ is a constant multiple of v_{-1-s}. To determine the constant for u_s, evaluate on $\varphi(x) = e^{-\pi x^2}$, the latter Gaussian being its own Fourier transform:

$$\widehat{u}_s(\varphi) \;=\; u_s(\widehat{\varphi}) \;=\; u_s(\varphi) \;=\; \int_{\mathbb{R}} |x|^s\, e^{-\pi x^2}\,dx \;=\; \int_0^\infty x^{\frac{s+1}{2}}\, e^{-\pi x}\,\frac{dx}{x} \;=\; \pi^{-\frac{s+1}{2}}\,\Gamma(\frac{s+1}{2})$$

Also,

$$u_{-1-s}(\varphi) \;=\; \pi^{-\frac{(-s-1)+1}{2}}\,\Gamma(\frac{(-1-s)+1}{2}) \;=\; \pi^{\frac{s}{2}}\,\Gamma(\frac{-s}{2})$$

Thus,

$$\widehat{u}_s \;=\; \frac{\pi^{-\frac{s+1}{2}}\,\Gamma(\frac{s+1}{2})}{\pi^{\frac{s}{2}}\,\Gamma(\frac{-s}{2})}\cdot u_{-1-s}$$

Replacing s by $-s$ gives a more memorable behavior of the exponent s:

$$\widehat{\frac{1}{|x|^s}} \;=\; \frac{\pi^{\frac{s-1}{2}}\,\Gamma(\frac{1-s}{2})}{\pi^{\frac{-s}{2}}\,\Gamma(\frac{s}{2})}\cdot\frac{1}{|x|^{1-s}}$$

[2.4] Fourier transform of $\mathrm{sgn}(x)\cdot\frac{1}{|x|^s}$ Again, similarly, $\widehat{v}_s$ is a constant multiple of v_{-1-s}. Since v_s is *odd*, we use an odd Schwartz function, $\varphi(x) = xe^{-\pi x^2}$, which has $\widehat{\varphi} = -i\varphi$:

$$\widehat{v}_s(\varphi) \;=\; v_s(\widehat{\varphi}) \;=\; u_s(-i\varphi) \;=\; -i\int_{\mathbb{R}} \mathrm{sgn}(x)\cdot|x|^s\,xe^{-\pi x^2}\,dx \;=\; -i\int_0^\infty x^{\frac{s+2}{2}}\,e^{-\pi x}\,\frac{dx}{x} \;=\; \pi^{-\frac{s+2}{2}}\,\Gamma(\frac{s+2}{2})$$

and

$$v_{-1-s}(\varphi) \;=\; -i\pi^{-\frac{(-s-1)+2}{2}}\,\Gamma(\frac{(-1-s)+2}{2}) \;=\; -i\pi^{\frac{s-1}{2}}\,\Gamma(\frac{-s+1}{2})$$

so

$$\widehat{v}_s \;=\; \frac{\pi^{-\frac{s+2}{2}}\,\Gamma(\frac{s+2}{2})}{-i\pi^{\frac{s-1}{2}}\,\Gamma(\frac{-s+1}{2})}\cdot v_{-1-s}$$

The unique fixed point of $s \to -1-s$ is $s = -1/2$, and

$$\widehat{u}_{\frac{1}{2}} \;=\; u_{\frac{1}{2}} \qquad\qquad \widehat{v}_{\frac{1}{2}} \;=\; iv_{\frac{1}{2}}$$

[2.5] Remark: To prove a (weaker) uniqueness result only for *tempered* distributions, we can use Fourier transforms to show that at poles, the distribution w_s on $\mathcal{D}_0$ from the proof of the existence-and-uniqueness theorem above, does not extend to a tempered distribution with the same parity and homogeneity. That is, we want to (re-) prove that the corresponding derivative of δ is the only *tempered* distribution of the corresponding parity and homogeneity.

Any tempered distribution of fixed parity and degree of homogeneity s with $\mathrm{Re}\,(s) > -1$ restricts to a scalar multiple of u_s or v_s on $\mathcal{D}_0$, and has a unique extension to $\mathcal{D}_+$, by the argument of the theorem of the previous section. For $\mathrm{Re}\,(s) < -1$, the Fourier transform of such a tempered distribution has the same parity, and homogeneity $-1-s$, from the same considerations as above. On the other hand, for $\mathrm{Re}\,(s) \le -1$,

$$\mathrm{Re}\,(-1-s) \;=\; -1 - \mathrm{Re}\,(s) \;\ge\; -1-(-1) \;=\; 0 \;>\; -1$$

Thus, by the simpler uniqueness argument for $\mathrm{Re}\,(s) > -1$, the Fourier transform of such a distribution is u_{-1-s} or v_{-1-s}, up to constant multiples. ///

The integrals in the following claim are *Frullani integrals.* The heuristic for this evaluation is clear: differentiate the integral with respect to t, and use the fundamental theorem of calculus. The obvious potential problem is justification.

[2.6] Claim: For $t > 0$, at least for Schwartz f,

$$\int_0^\infty \frac{f(tx)-f(x)}{x}\,dx \;=\; f(0)\cdot\log t \;=\; \delta(f)\cdot\log t$$

Proof: Let $F_t(f)$ be that integral. By changing variables in the integral, for $r > 0$ we have the homogeneity $F_t(f\circ r) \;=\; F_t(f)$. By uniqueness, F_t is a linear combination of the *odd* distribution v_{-1} and the *even* distribution δ with that same homogeneity. Odd Schwartz functions are necessarily of the form $xf(x)$, and

$$F_t(xf(x)) \;=\; \int_0^\infty \frac{(tx)f(tx)-xf(x)}{x}\,dx \;=\; \int_0^\infty tf(tx)-f(x)\,dx \;=\; \int_0^\infty tf(tx)\,dx - \int_0^\infty f(x)\,dx$$

$$=\; \int_0^\infty f(x)\,dx - \int_0^\infty f(x)\,dx \;=\; 0$$

by changing variables in the integral. Thus, F_t is *even*, and by uniqueness is a constant multiple, probably depending on t, of δ.

Let f be a Schwartz function such that $f(x) = e^{-x}$ for $x \ge 0$. Then

$$f \;\longrightarrow\; \int_0^\infty x^s\cdot(f(tx)-f(x))\,\frac{dx}{x}$$

is a holomorphic distribution-valued function of s on $\mathrm{Re}(s) > -1$, whose value at $s = 0$ is $F_t(f)$. For $\mathrm{Re}(s) > 0$, we can separate $f(tx)$ and $f(x)$:

$$\int_0^\infty x^s \cdot (f(tx) - f(x)) \, \frac{dx}{x} \;=\; \int_0^\infty x^s \cdot f(tx) \, \frac{dx}{x} - \int_0^\infty x^s \cdot f(x) \, \frac{dx}{x}$$

$$=\; (t^{-s} - 1) \int_0^\infty x^s \cdot e^{-x} \, \frac{dx}{x} \;=\; (t^{-s} - 1) \cdot \Gamma(s)$$

Near $s = 0$ this is

$$(t^{-s} - 1) \cdot \Gamma(s) \;=\; (e^{s \log t} - 1) \cdot \Gamma(s) \;=\; (s \log t + O(s^2)) \cdot \Big(\frac{1}{s} + O(1)\Big) \;=\; \log t + O(s)$$

Thus, the limit as $s \to 0^+$ is $\log t$, and $F_t = \delta \cdot \log t$. The identity principle for vector-valued holomorphic functions gives the asserted identity at $s = 0$. ///

[2.7] Remark: The previous identity also gives a specific assertion of continuity of the functional given by the integral: for Schwartz f,

$$\Big| \int_0^\infty \frac{f(tx) - f(x)}{x} \, dx \Big| \;\le\; |f(0)| \cdot |\log t|$$

Thus we have an extension-by-continuity to the completion of Schwartz functions with respect to the metric $f \to |f(0)|$. However, we also want the integral to converge. Thus, we might take the completion of Schwartz functions with respect to sup norm on $[0, \infty)$, namely, continuous functions on $[0, \infty)$ going to 0 at infinity.

[2.8] Remark: As M. Riesz (1938) showed a few years later, Hadamard's (1932) finite-part functional derived from $1/|x|^{3/2}$ by some *ad hoc* manipulations was in fact the correct meromorphic continuation of a linear combination of $1/|x|^s$ and/or $\mathrm{sgn}(x)/|x|^s$ to the point $s = 3/2$.

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3. *Rotation-invariant distributions supported at* $\{0\}$

In contrast to $\mathbb{R}$ where the rotational symmetries are just *parity*, on $\mathbb{R}^n$ with $n > 1$ these symmetries are potentially more complicated, especially with $n \ge 3$, where the rotational symmetry group $SO(n, \mathbb{R})$ is non-abelian. To momentarily skirt these issues, in this and the immediate sequel we consider only rotation-*invariant* distributions on $\mathbb{R}^n$, rather than allowing equivariance by other representations of $SO(n, \mathbb{R})$.

[3.1] Theorem: The rototation-invariant distributions on $\mathbb{R}^n$ supported at $\{0\}$ are finite linear combinations of δ, $\Delta\delta$, $\Delta^2\delta$, $\Delta^3\delta$, ...

Proof: [... iou ...] ///

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4. *Distributions* $|x|^s$ *on* $\mathbb{R}^n$

Certainly in $\mathrm{Re}(s) > -n$, where $u_s(x) = |x|^s$ is locally integrable, it is rotationally invariant and has the obvious homogeneity. Let E be the Euler operator

$$E \;=\; \sum_{j=1}^n x_j \, \frac{\partial}{\partial x_j}$$

Euler's identity gives $E|x|^s = s \cdot |x|^s$.

[4.1] Theorem: Up to constant multiples, for all $s \in \mathbb{C}$ there is a unique rotation-invariant distribution satisfying $Eu = s \cdot u$. For $s \neq -n, -n-2, -n-4, \ldots$, the meromorphic continuation of (integrate-against) u_s is such a distribution. For s among $-n, -n-2, -n-4, \ldots$, the distributions are δ, $\Delta\delta$, $\Delta^2\delta$, ...

The proof will occupy this section.

[4.2] Differentiation identity and meromorphic continuation

For $\mathrm{Re}(s) \geq 2$, the function $u_s(x) = |x|^s$ on $\mathbb{R}^n$ is twice-continuously-differentiable. In particular, with the usual Euclidean Laplacian Δ,

$$\Delta |x|^s \;=\; \sum_{i=1}^n \frac{\partial}{\partial x_i} \frac{s}{2} \cdot 2x_i \cdot (|x|^2)^{\frac{s}{2}-1} \;=\; \sum_{i=1}^n \Big(\frac{s}{2} \cdot 2 \cdot (|x|^2)^{\frac{s}{2}-1} + \frac{s}{2}(\frac{s}{2}-1) \cdot (2x_i)^2 \cdot (|x|^2)^{\frac{s}{2}-2} \Big)$$

$$=\; ns \cdot |x|^{s-2} + s(s-2)|x|^{s-2} \;=\; s(s+n-2) \cdot |x|^{s-2}$$

Visibly, the case $n = 2$ is anomalous, because the two linear factors become identical.

The identity $\Delta u_s = s(s+n-2) \cdot u_{s-2}$ at first holds for $\mathrm{Re}(s) \geq 2$, as an equality of continuous functions. At the same time, u_s analytically continues as an $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-valued function of s, therefore as a tempered distribution-valued function of s, to $\mathrm{Re}(s) > -n$. Thus, Δu_s exists as tempered distribution at least on $\mathrm{Re}(s) > -n$. Rewrite the identity as

$$u_{s-2} \;=\; \frac{\Delta u_s}{s(s+n-2)}$$

and replace s by $s+2$:

$$u_s \;=\; \frac{\Delta u_{s+2}}{(s+2)(s+n)}$$

This expression makes sense of u_s as tempered distribution on $\mathrm{Re}(s) > -n-2$ except for possible poles at $s = -n$ and $s = -2$. For $n > 2$, in fact there is no pole at $s = -2$, because u_{-2} is locally integrable. Indeed, $\Delta u_0 = 0$, so $\Delta u_{s+2}/(s+2)$ is holomorphic at $s = -2$.

Repeating the differentiation:

$$u_s \;=\; \frac{\Delta u_{s+2}/(s+2)}{s+n} \;=\; \frac{\Delta^2 u_{s+4}/(s+2)(s+4)}{(s+n)(s+n-2)}$$

The factors $(s+2)(s+4)$ are indeed cancelled by the vanishing of $\Delta^2 u_2$ and $\Delta^2 u_4$, leaving possible poles at $s = -n, -n-2$. Continuing, u_s extends to a meromorphic tempered-distribution-valued function on $\mathbb{C}$, with poles at most at $s = -n, -n-2, -n-4, \ldots$

[4.3] Regularization and $\mathrm{Res}_{s=-n} u_s \;=\;$ const $\times\, \delta$

With $n > 2$, the *first* (rightmost) pole of u_s, at $s = -n$, is a multiple of Dirac δ at 0, seen as follows. Indeed, *locally* away from $x = 0$, we have the vanishing $\Delta u_{2-n}(x) = 0$, showing that the support of Δu_{2-n} is $\{0\}$, as expected.

With Gaussian $\gamma(x) = e^{-|x|^2}$, given Schwartz function f, the difference $f - f(0) \cdot \gamma$ vanishes at 0, so the integral for

$$u_s\big(f(x) - f(0) \cdot \gamma(x)\big) \;=\; \int_{\mathbb{R}^n} |x|^s \cdot \big(f(x) - f(0) \cdot \gamma(x)\big)\, dx$$

is absolutely convergent for $\mathrm{Re}(s) > -n-1$, and is a holomorphic function of s in that half-plane. The identity principle assures that this analytic continuation correctly evaluates u_s on $f - f(0) \cdot \gamma$. In particular, there is no pole at $s = -n$. Thus,

$$\big(\mathrm{Res}_{s=-n} u_s\big)(f) \;=\; \big(\mathrm{Res}_{s=-n} u_s\big)(f - f(0) \cdot \gamma) + f(0) \cdot \big(\mathrm{Res}_{s=-n} u_s\big)(\gamma) \;=\; 0 + f(0) \cdot \big(\mathrm{Res}_{s=-n} u_s\big)(\gamma)$$

Since $f(0) = \delta(f)$, the residue is a constant multiple of δ, with constant

$$\mathrm{Res}_{s=-n} \int_{\mathbb{R}^n} |x|^s\, e^{-|x|^2}\, dx \;=\; |S^{n-1}| \cdot \mathrm{Res}_{s=-n} \int_0^\infty t^s\, e^{-t^2}\, t^{n-1}\, dt \;=\; |S^{n-1}| \cdot \tfrac{1}{2} \mathrm{Res}_{s=-n} \int_0^\infty t^{\frac{s+n}{2}}\, e^{-t}\, \frac{dt}{t}$$

$$= |S^{n-1}| \cdot \tfrac{1}{2}\mathrm{Res}_{s=-n} \int_0^\infty t^{\frac{s+n}{2}}\, e^{-t}\, \frac{dt}{t} \;=\; |S^{n-1}| \cdot \tfrac{1}{2}\mathrm{Res}_{s=-n} \cdot \Gamma(\frac{s+n}{2})$$

$$= |S^{n-1}| \cdot \tfrac{1}{2}\mathrm{Res}_{s=-n} \frac{2}{s+n} \;=\; |S^{n-1}| \;=\; \text{natural measure of } S^{n-1}$$

[4.4] Existence and uniqueness Let $\mathscr{S}^{\,\mathrm{sph}}$ be the space of rotationally invariant Schwartz functions. Let $\mathscr{S}_0^{\mathrm{sph}}$ be the subspace of those functions which vanish to infinite order at 0, that is, have all partial derivatives vanishing at 0. The very short exact sequence

$$0 \longrightarrow \mathcal{D}_0^{\mathrm{sph}} \longrightarrow \mathcal{D}^{\,\mathrm{sph}}$$

(using Hahn-Banach) gives

$$0 \longrightarrow X \longrightarrow (\mathcal{D}^{\,\mathrm{sph}})^* \longrightarrow (\mathcal{D}_0^{\mathrm{sph}})^* \longrightarrow 0$$

where X is the space of rotationally invariant distributions supported at 0. From above, X consists of finite linear combinations of δ, $\Delta\delta$, $\Delta^2\delta$, $\ldots$, so

$$0 \longrightarrow \bigoplus_k \mathbb{C}\cdot\Delta^k\delta \longrightarrow (\mathcal{D}^{\,\mathrm{sph}})^* \longrightarrow (\mathcal{D}_0^{\mathrm{sph}})^* \longrightarrow 0$$

The operator $T = T_s = E - s$ stabilizes all the spaces involved, giving a complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_k \mathbb{C}\cdot\Delta^k\delta & \longrightarrow & (\mathcal{D}^{\,\mathrm{sph}})^* & \longrightarrow & (\mathcal{D}_0^{\mathrm{sph}})^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_s} & & \downarrow{\scriptstyle T_s} & & \downarrow{\scriptstyle T_s} & & \\
0 & \longrightarrow & \bigoplus_k \mathbb{C}\cdot\Delta^k\delta & \longrightarrow & (\mathcal{D}^{\,\mathrm{sph}})^* & \longrightarrow & (\mathcal{D}_0^{\mathrm{sph}})^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From a complex of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the Snake Lemma gives a long exact sequence

$$0 \longrightarrow \ker T|_A \longrightarrow \ker T|_B \longrightarrow \ker T|_C \xrightarrow{\;\eta\;} \frac{A}{TA} \longrightarrow \frac{B}{TB} \longrightarrow \frac{C}{TC} \longrightarrow 0$$

with *connecting homomorphism* η, whose details we do not need. Here, we have an element of $\ker T|_C$, and we ask whether it is an image of an element from $\ker T|_B$, and, if so, whether it is uniquely so.

Existence of an extention from is equivalent to surjectivity of $\ker T|_B \to \ker T|_C$, which is implied by $A/TA = \{0\}$, without any information about the connecting homomorphism η. We know the distributions supported on $\{0\}$, in particular the rotationally invariant ones. Since $\Delta^k\delta$ is homogeneous of degree $-n-2k$,

$$E(\Delta^k\delta) \;=\; -n-2k\cdot\Delta^k\delta$$

By direct calculation, T_s is surjective on $A = \bigoplus_k \mathbb{C}\cdot\Delta^k\delta$ except for $s = -n, -n-2, -n-4, \ldots$, since

$$T_s\Big(\sum_{0\le k\in\mathbb{Z}} c_k\cdot\Delta^k\delta\Big) \;=\; \sum_{0\le k\in\mathbb{Z}} c_k\cdot\Big((-n-2k)-s\Big)\cdot\Delta^k\delta$$

The sums are finite, and by hypothesis the values eigenvalues $(n+2k)(2k-2)-s(s+n-2)$ are non-zero unless $s=-n-2k$ so the map T_s is invertible away from those values.

Uniqueness of the extension follows from $\ker T|_A = \{0\}$. In the present circumstance, triviality of this kernel is that there is no rotationally invariant distribution supported at 0 and annihilated by $T = T_s$. As it happens, this is the same condition as that for existence.

We should show that (up to scalar multiples) the restriction of (integration against) $|x|^s$ is the only rotation-invariant, homogeneous-degree-s functional on $\mathscr{S}_0^{\,\mathrm{sph}}$. In essence, this is because the rotation group combined with positive dilations is *transitive* on $\mathbb{R}^n - \{0\}$. *[... iou ...]*

[4.5] Remark: To disambiguate, we should show that $(\mathscr{S}^{\,\mathrm{sph}})* \approx (\mathscr{S}^*)^{\,\mathrm{sph}}$, and $(S_0^*)^{\,\mathrm{sph}} = (S_0^{\mathrm{sph}})^*$. *[... iou ...]*

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5. *Fourier transforms, Euler operator, homogeneity*

[5.1] Fourier transform and homogeneity For $u\in\mathcal{D}^*$ positive of homogeneous of degree s in the sense above, we have $Eu = s\cdot u$, with Euler operator E. Eigenfunctions for the Euler operator behave well under Fourier transform: with $Eu = s\cdot u$,

$$s\cdot\widehat{u} \;=\; \widehat{s\cdot u} \;=\; \widehat{Eu} \;=\; \sum_{j=1}^n \Big(x_j\frac{\partial}{\partial x_j}u\Big)^{\widehat{}} \;=\; \sum_{j=1}^n -\frac{\partial}{\partial x_j}(x_j\cdot\widehat{u}) \;=\; -\sum_{j=1}^n\Big(1\cdot\widehat{u} + x_j\frac{\partial}{\partial x_j}\widehat{u}\Big)$$

Thus,

$$E\,\widehat{u} \;=\; -(s+n)\cdot\widehat{u}$$

Thus, positive homogeneity of degree s is converted to positive homogeneity of degree $-(s+n)$. In particular, by the uniqueness theorem,

$$\widehat{\frac{1}{|x|^s}} \;=\; (\text{constant})\times\frac{1}{|x|^{n-s}}$$

Since the two functions are locally integrable for $0<\mathrm{Re}(s)<n$, this makes literal sense for s in that range. More generally, some sort of extension/regularization is required.

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6. *Green's functions on $\mathbb{R}^n$ with $n\ge 3$*

It is essentially elementary that $\frac{d^2}{dx^2}|x| = 2\delta$ on $\mathbb{R}$. On $\mathbb{R}^n$ with $n>1$ it is less obvious how to solve $\Delta u = \delta$.

[6.1] Solving $\Delta u = \delta$ by Fourier transform From $\Delta u = \delta$, we have $-4\pi^2 r^2\cdot\widehat{u} = 1$. For $n\ge 3$, $1/r^2$ is locally integrable, so $\widehat{u} = -1/4\pi^2 r^2$ is a tempered distribution, by integration against it. From just above, the Fourier transform of $1/r^2$ is a constant multiple of $1/r^{n-2}$, which is again locally integrable.

[6.2] Solving $\Delta u = \delta$ by meromorphic continuation The distribution-valued function $(s+n)u_s$ takes value $\mathrm{Res}_{s=-n} u_s$ at $s=-n$. By the identity principle, the equality

$$\Delta u_{s+2} \;=\; (s+2)\cdot(s+n)u_s$$

also holds at $s=-n$, so

$$\Delta\,\frac{1}{|x|^{n-2}} \;=\; \Delta u_{-n+2} \;=\; (-n+2)\cdot|S^{n-1}|\cdot\delta \qquad\qquad \text{(distributionally)}$$

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7. *Rotation-equivariant distributions on $\mathbb{R}^2$ supported at $\{0\}$*

As a more complicated example than the one-dimensional case, but less complicated than considering more complicated homogeneous distributions than rotation-invariant ones in $\mathbb{R}^n$, the two-dimensional case admits considerable simplification. In part, the simplification is due to the abelian-ness of the rotation group, but also due to the convenient existence of complex numbers, whose multiplication gives a convenient expression of both rotation and dilation. As in the rotation-invariant case in $\mathbb{R}^n$, we need an auxiliary classification result.

A usual, let $z=x+iy$ and $\overline{z}=x-iy$. Let

$$\partial \;=\; \frac{\partial}{\partial z} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\Big) \qquad\qquad \overline{\partial} \;=\; \frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)$$

A complex number $\mu\in\mathbb{C}$ with $|\mu|=1$ gives a *rotation* of $\mathbb{C}\approx\mathbb{R}^2$ by (complex) multiplication. Using coordinates $z=x+iy$ on $\mathbb{C}$ as usual, a function f on $\mathbb{C}$ is *rotation equivariant* by $\mu\to\mu^n$ (with $n\in\mathbb{Z}$) when

$$f(\mu\cdot z) \;=\; \mu^n\cdot f(z) \qquad\qquad \text{(for all } |\mu|=1 \text{ and } z\in\mathbb{C})$$

Write $f\circ\mu$ for the function $z\to f(\mu\cdot z)$. A *distribution* u on $\mathbb{C}\approx\mathbb{R}^2$ is *equivariant* by $\mu\to\mu^n$ when

$$u(\varphi\circ\mu^{-1}) \;=\; \mu^n\cdot u(\varphi) \qquad\qquad \text{(for all } |\mu|=1 \text{ and } \varphi\in\mathcal{D})$$

This convention is consistent with the rotation-equivariance requirement on *functions*.

[7.1] Theorem: Given $n\in\mathbb{Z}$, the distributions u supported at $\{0\}$ and rotation equivariant by $\mu\to\mu^n$ are exactly the finite linear combinations of

$$u \;=\; \partial^k\overline{\partial}^\ell\delta \qquad\qquad \text{(with } 0\le k,\ell\in\mathbb{Z} \text{ and } -k+\ell=n)$$

Proof: We convert the problem to one about rotation-equivariant *polynomials* via Fourier transform. We re-express Fourier transform in terms of $z,\overline{z}$, and another complex variable w and $\overline{w}$:

$$\widehat{f}(w) \;=\; \int_{\mathbb{C}} e^{-2\pi i\cdot\mathrm{Re}\,(z\overline{w})} f(z)\,dx\,dy \;=\; \int_{\mathbb{C}} e^{-\pi i\cdot(z\overline{w}+\overline{z}w)} f(z)\,dx\,dy$$

Integrating by parts,

$$\widehat{\partial f} \;=\; \pi i\cdot\overline{w}\cdot\widehat{f} \qquad\qquad \widehat{\overline{\partial} f} \;=\; \pi i\cdot w\cdot\widehat{f}$$

Thus, with $0\le m,n\in\mathbb{Z}$, noting that $\widehat{\delta}=1$,

$$\widehat{\partial^m\overline{\partial}^n\delta} \;=\; (\pi i)^{m+n}\cdot\overline{w}^m\cdot w^n$$

We check that Fourier transform preserves rotation equivariance:

$$\widehat{f\circ\mu}(w) \;=\; \int_{\mathbb{C}} e^{-\pi i\cdot(z\overline{w}+\overline{z}w)} f(\mu\cdot z)\,dx\,dy \;=\; \int_{\mathbb{C}} e^{-\pi i\cdot(\frac{z}{\mu}\overline{w}+\frac{\overline{z}}{\mu^{-1}}w)} f(z)\,dx\,dy \;=\; \widehat{f}(\mu\cdot w)$$

The monomials $\overline{w}^k w^\ell$ are linearly independent, so for a linear combination to have a specified rotation equivariance requires that every monomial have that equivariance. The monomial $\overline{w}^k w^\ell$ has rotation equivariance $\mu\to\mu^{-k+\ell}$. ///

8. *Distributions* $(z/|z|)^n \cdot |z|^s$ *on* $\mathbb{R}^2 \approx \mathbb{C}$

Let

$$u_{n,s}(z) \;=\; \Big(\frac{z}{|z|}\Big)^n \cdot |z|^s \qquad\qquad (\text{for } n \in \mathbb{Z} \text{ and } s \in \mathbb{C})$$

For $\mathrm{Re}(s) > -2$, this function is locally integrable, so gives a (tempered) distribution.

It is convenient to describe *rotations* on $\mathbb{R}^2 \approx \mathbb{C}$ in terms of multiplication (in $\mathbb{C}$) by $\mu \in \mathbb{C}$ with $|\mu| = 1$. The behavior of $u_{n,s}$ under rotation by μ is

$$u_{n,s}(\mu \cdot z) \;=\; \Big(\frac{\mu \cdot z}{|\mu \cdot z|}\Big)^n \cdot |\mu \cdot z|^s \;=\; \mu^n \cdot \Big(\frac{z}{|z|}\Big)^n \cdot |z|^s \;=\; \mu^n \cdot u_{n,s}(z)$$

[8.1] Theorem: For all $n \in \mathbb{Z}$ and $s \in \mathbb{C}$, there is a unique $\mu \to \mu^n$ rotation equivariant, positive-homogeneous degree s distribution on $\mathbb{C} \approx \mathbb{R}^2$. For $s \neq -2-|n|, -2-|n|-2, -2-|n|-4, \ldots$, this distribution is (the meromorphic continuation of) $u_{n,s}$. At the point $-2-|n|-2k$, the distribution is a suitable derivative of δ. Namely, $\partial^k \overline{\partial}^\ell \delta$ has positive-homogeneity degree $-2-k-\ell$, and rotation equivariance by $\mu \to \mu^{-k+\ell}$.

Proof: [... iou ...] ///