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08k. Trace theorems

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http://www.math.umn.edu/˜garrett/m/real/notes_2019-20/08k_trace_theorems.pdf

The idea is that restriction $f|_Z$ of a function f in a Levi-Sobolev space $H^s(X)$ to a nice sub-space Z of X of codimension n gives $f|_Z \in H^{s-\frac{n}{2}-\varepsilon}(Z)$ for all $\varepsilon > 0$, if also $s - \frac{n}{2} > 0$.

This is often used in looking at boundary-conditions for partial differential equations on regions X in $\mathbb{R}^n$ with $Z = \partial X$. Proofs are simpler when X and $Z \subset X$ have more structure, so that spectral characterizations of Sobolev spaces can be used.

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1. *Very simple trace theorem*

[1.1] Theorem: For all $s > \frac{1}{2}$, for $f \in H^s(\mathbb{T}^2)$, the restriction $f|_Z$ to $Z = \mathbb{T} \times \{1\}$ satisfies

$$\Big| f|_Z \Big|_{H^{s-\frac{1}{2}-\varepsilon}} \ll_\varepsilon |f|_{H^s} \qquad \text{(implied constant independent of } f \text{, for all } \varepsilon > 0)$$

Proof: With a naive choice of indexing, we estimate the s^{th} Levi-Sobolev norm of the restriction of $f(x,y) = \sum_{m,n\in\mathbb{Z}^2} c_{m,n}\, e^{i(mx+ny)}$, using an auxiliary index t:

$$\Big| f|_Z \Big|^2_{H^s} = \sum_m \Big| \sum_n c_{mn} \Big|^2 \cdot (1+m^2)^s = \sum_m \Big| \sum_n \frac{1}{(1+n^2)^{\frac{t}{2}}} \cdot c_{mn} \cdot (1+n^2)^{\frac{t}{2}} \Big|^2 \cdot (1+m^2)^s$$

$$\le \sum_m \Big(\sum_n \frac{1}{(1+n^2)^t} \Big) \cdot \Big(\sum_n |c_{mn}|^2 \cdot (1+n^2)^t \Big) \cdot (1+m^2)^s$$

by Cauchy-Schwarz-Bunyakowsky. For the first inner sum to converge, we need $t > \frac{1}{2}$. Removing that finite constant,

$$\Big| f|_Z \Big|^2_{H^s} \ll_t \sum_{m,n} |c_{mn}|^2 \cdot (1+n^2)^t \cdot (1+m^2)^s$$

$$\le \sum_{m,n} |c_{mn}|^2 \cdot (1+m^2+n^2)^{s+t} = |f|^2_{H^{s+t}} \qquad (\text{with } t > \tfrac{1}{2} \text{ and } s \ge 0)$$

More precisely, the sequence of finite partial sums of f, restricted to Z, converge in the $H^s(Z)$-norm if the finite partial sums of f converge in the $H^{s+t}(\mathbb{T}^2)$-norm.

Now that we see how things go, let $t = \frac{1}{2} + \varepsilon$, and replace s by $s + \frac{1}{2} + \varepsilon$, to obtain the assertion of the theorem. ///

2. *Codimension one with circles*

Mimicking the previous argument:

[2.1] Theorem: For all $s > \frac{1}{2}$, for $f \in H^s(\mathbb{T}^n)$, the restriction $f|_Z$ to $Z = \mathbb{T}^{n-1} \times \{1\}$ satisfies

$$\Big| f|_Z \Big|_{H^{s-\frac{1}{2}-\varepsilon}} \ll_\varepsilon |f|_{H^s} \qquad \text{(implied constant independent of } f\text{, for } s - \tfrac{1}{2} - \varepsilon \ge 0)$$

Proof: Estimate the $s - \frac{1}{2} - \varepsilon$ Levi-Sobolev norm of the restriction of

$$f(x,y) = \sum_{(\xi,\ell)\in\mathbb{Z}^{n-1}\times\mathbb{Z}} c_{\xi,\ell}\, e^{i(\xi\cdot x + \ell m)}$$

as follows.

$$\Big| f|_Z \Big|^2_{H^{s-\frac{1}{2}-\varepsilon}} = \sum_\xi \Big| \sum_{\ell\in\mathbb{Z}} c_{\xi\ell} \Big|^2 \cdot (1+|\xi|^2)^{s-\frac{1}{2}-\varepsilon} = \sum_\xi \Big| \sum_{\ell\in\mathbb{Z}} \frac{1}{(1+\ell^2)^{\frac{1}{2}+\varepsilon}} \cdot c_{\xi\ell} \cdot (1+\ell^2)^{\frac{1}{2}+\varepsilon} \Big|^2 \cdot (1+|\xi|^2)^{s-\frac{1}{2}-\varepsilon}$$

$$\le \sum_\xi \Big(\sum_{\ell\in\mathbb{Z}} \frac{1}{(1+\ell^2)^{\frac{1}{2}+\varepsilon}} \Big) \cdot \Big(\sum_\ell |c_{\xi\ell}|^2 \cdot (1+\ell^2)^{\frac{1}{2}+\varepsilon} \Big) \cdot (1+|\xi|^2)^{s-\frac{1}{2}-\varepsilon}$$

by Cauchy-Schwarz-Bunyakowsky. The inner sum $\sum_{\ell\in\mathbb{Z}} \frac{1}{(1+\ell^2)^{\frac{1}{2}+\varepsilon}}$ converges. Removing that finite constant,

$$\Big| f|_Z \Big|^2_{H^{s-\frac{1}{2}-\varepsilon}} \ll_\varepsilon \sum_{\xi,\ell} |c_{\xi\ell}|^2 \cdot (1+\ell^2)^{\frac{1}{2}+\varepsilon} \cdot (1+|\xi|^2)^{s-\frac{1}{2}-\varepsilon}$$

For $s - \frac{1}{2} - \varepsilon \ge 0$, this is dominated by

$$\sum_{\xi,\ell} |c_{\xi\ell}|^2 \cdot (1+|\xi|^2+\ell^2)^{(\frac{1}{2}+\varepsilon)+(s-\frac{1}{2}-\varepsilon)} = \sum_{\xi,\ell} |c_{\xi\ell}|^2 \cdot (1+|\xi|^2+\ell^2)^s = |f|^2_{H^s} \qquad \text{(with } s - \tfrac{1}{2} - \varepsilon \ge 0)$$

More precisely, the sequence of finite partial sums of f, restricted to Z, converge in the $H^{s-\frac{1}{2}-\varepsilon}(Z)$-norm if the finite partial sums of f converge in the $H^s(\mathbb{T}^n)$-norm.

3. *Dependence on codimension*

Now with codimension m, rather than codimension 1:

[3.1] Theorem: For all $s > \frac{m}{2}$, for $f \in H^s(\mathbb{T}^n)$, the restriction $f|_Z$ to $Z = \mathbb{T}^{n-m} \times \{1\}$ satisfies

$$\Big| f|_Z \Big|_{H^{s-\frac{m}{2}-\varepsilon}} \ll_\varepsilon |f|_{H^s} \qquad \text{(implied constant independent of } f\text{, for } s - \tfrac{m}{2} - \varepsilon \ge 0)$$

Proof: This follows by the same argument, adapted by replacing the sum over $\ell \in \mathbb{Z}$ by $\ell \in \mathbb{Z}^m$. ///