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09c. The family $\sin \frac{nx}{2}$

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Essentially the only general device to prove that a collection $\{u_n\}$ of functions on $[a,b]$ is an orthogonal basis for $L^2[a,b]$ is to find a *self-adjoint compact* operator T on $L^2[a,b]$ such that the eigenvectors of T are the functions u_n.

The explicit Fourier-Dirichlet and Fejer arguments for ordinary Fourier series do not easily generalize. One family of natural extensions is to Sturm-Liouville boundary-value problems (1830s), but there already we see the necessity of the spectral theory of *self-adjoint compact operators* on Hilbert spaces, even though such ideas and proofs had to wait almost 60 years (1890s), for Bocher and Steklov, and then Hilbert and Schmidt and others a few years later.

As a non-trivial but relatively simple example, we prove that the functions $u_n(x) = \sin \frac{nx}{2}$ give an orthogonal basis of $L^2[0,2\pi]$. This does *not* follow in any easy way from the corresponding fact for $\{e^{inx}\}$ or for $\sin nx$ and $\cos nx$, despite some mythology to the contrary.

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1. Boundary value problem $u'' - \lambda u = f$, $u(0) = 0 = u(2\pi)$

Consider the boundary-value problem $u'' - \lambda u = f$ with $u(0) = u(2\pi) = 0$ on $[0,2\pi]$, and $\lambda \in \mathbb{C}$. Naively formulated, given a function f on $[0,2\pi]$, this asks to solve $u'' - \lambda u = f$ subject to the boundary conditions $u(0) = u(2\pi) = 0$. This formulation is naive because we have not specified what kind of *functions* f and u should be, apart from the apparent requirement that requiring vanishing of u at endpoints has sense to it, and that u admits derivatives of some sort.

The eigenvectors for $u \to u''$ are not hard to find: the differential equation $u'' = \lambda \cdot u$ has linearly independent solutions $u(x) = e^{\pm\sqrt{\lambda}\cdot x}$ for $\lambda \neq 0$, and $u(x) = ax + b$ for $\lambda = 0$. In fact, these are also the only *distributional* eigenvectors.

The Dirichlet boundary conditions (that is, vanishing at endpoints) exclude the $\lambda = 0$ eigenvectors. For $\lambda = s^2 \neq 0$, the requirement is to find constants A, B (not both 0) such that

$$Ae^{s\cdot 0} + Be^{-s\cdot 0} \;=\; 0 \;=\; Ae^{s\cdot 2\pi} + Be^{-s\cdot 2\pi}$$

Thus, $B = -A$, and $e^{s\cdot 2\pi} - e^{-s\cdot 2\pi} = 0$. Equivalently, $e^{2s\cdot 2\pi} = 1$. Thus, $s \in \frac{1}{2}\cdot\mathbb{Z}$. That is, $u_n(x) = \sin \frac{n}{2}x$ for $n = 1, 2, 3, \ldots$ are all the eigenvectors of $u \to u''$ meeting the Dirichlet boundary conditions, and have eigenvalues $= n^2/4$.

These eigenvectors are mutually orthogonal, either by direct computation, or by using an orthogonality property of eigenvectors for symmetric operators with different eigenvalues. The symmetry of $u \to u''$ via integration by parts needs the boundary conditions to succeed: with $v(0) = 0 = v(2\pi)$,

$$\langle u'', v\rangle \;=\; \int_0^{2\pi} u''(x)\cdot\overline{v(x)}\,dx \;=\; \big[u'(x)\overline{v(x)}\big]_0^{2\pi} - \int_0^{2\pi} u'(x)\cdot\overline{v'(x)}\,dx \;=\; -\int_0^{2\pi} u'(x)\cdot\overline{v'(x)}\,dx$$

Symmetrically, with $u(0) = 0 = u(2\pi)$, this is also equal to $\langle u, v''\rangle$. Thus, at least heuristically, on general principles the eigenvalues of $Tu = u''$ are real: for u a non-zero λ-eigenvector,

$$\lambda\langle u,u\rangle \;=\; \langle Tu,u\rangle \;=\; \langle u,Tu\rangle \;=\; \langle u,\lambda u\rangle \;=\; \overline{\lambda}\cdot\langle u,u\rangle$$

For $\lambda \neq \mu$ and corresponding eigenvalues u, v,

$$\lambda \cdot \langle u, v\rangle \;=\; \langle Tu, v\rangle \;=\; \langle u, Tv\rangle \;=\; \langle u, \mu v\rangle \;=\; \overline{\mu}\cdot\langle u, v\rangle \;=\; \mu\cdot\langle u, v\rangle$$

Thus, $\langle u, v\rangle = 0$. A potential problem in this proof of orthogonality is that $u \to u''$ is not exactly a map $L^2[0,2\pi] \to L^2[0,2\pi]$. We need not worry about this for the moment, since we can also prove the orthogonality by direct computation.

The immediate question is of *completeness*: are the functions $u_n(x) = \sin\frac{n}{2}x$ an orthogonal Hilbert space basis for $L^2[0,2\pi]$?

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2. *The Green's function*

In one dimension, as we are here, the idea is to solve the differential equation $\Delta u = f$ with boundary-value conditions $u(0) = 0 = u(2\pi)$ by finding a *Green's function* $G(x,y)$ on $[0,2\pi]\times[0,2\pi]$, that is, such that

$$\frac{d^2}{dx^2}\int_0^{2\pi} G(x,y)\,f(y)\;dy \;=\; f(x)$$

for f in $L^2[0,2\pi]$ and meeting the boundary conditions (in a suitable sense).

In particular, when $G(x,y)$ is in $L^2\big([0,2\pi]\times[0,2\pi]\big)$, then the operator

$$f \longrightarrow \int_0^{2\pi} G(x,y)\,f(y)\;dy$$

is *Hilbert-Schmidt*, hence *compact*. When also $G(y,x) = \overline{G(x,y)}$, the operator is *self-adjoint*, and the spectral theorem applies. That is, the topological closure of the algebraic span of the eigenvectors is the whole $L^2[0,2\pi]$. That is, the eigenvectors are *complete*.

The spectral theorem has further important parts: eigenspaces for distinct eigenvalues are mutually orthogonal, eigenspaces for non-zero eigenvalues are *finite-dimensional*, and there are only finitely-many eigenvalues larger than a given bound.

In the case at hand, following the general Sturm-Liouville prescription for Green's functions, we want $G(x,y)$ to be annihilated by $\frac{d^2}{dx^2}$ away from $0, y, 2\pi$, $G(0,y) = 0 = G(2\pi, y)$, and $G(x,y)$ is continuous at $x = y$. The last requirement is that the difference in slopes of $x \to G(x,y)$ to the left of y increases by 1 to the right of y. Here, the part of G in $0 < x < y$ is of the form $x \to ax$, in in $2\pi > x > y$ is $x \to b(2\pi - x)$, and the continuity and slope-difference conditions give

$$G(x,y) \;=\; \begin{cases} x\cdot(\frac{y}{2\pi} - 1) & (\text{in } 0 < x < y) \\ \\ y\cdot(\frac{x}{2\pi} - 1) & (\text{in } 2\pi > x > y) \end{cases}$$

From more modern viewpoint, the previous function satisfies

$$\frac{d^2}{dx^2}G(x,y) \;=\; \delta_{x-y} - \delta_0$$

And we should have extended $G(x,y)$ by 2π-periodicity, or, equivalently, look at $[0,2\pi]$ as parametrizing the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, to avoid counting the endpoints twice in computations.

Thus, assuming we can more the differential operator inside the integral, and writing integrals involving δ functions when we should really be writing *pairings* on Sobolev spaces $H^s(\mathbb{T})\times H^{-s}(\mathbb{T})$,

$$\frac{d^2}{dx^2}\int_{\mathbb{T}} G(x,y)\,f(y)\;dy \;=\; \int_{\mathbb{T}}\frac{d^2}{dx^2}G(x,y)\,f(y)\;dy \;=\; \int_{\mathbb{T}}(\delta_{x-y} - \delta_0)\,f(y)\;dy \;=\; f(x) - f(0)$$

For $f(0) = 0$ in a suitable sense, this shows that this $G(x,y)$ does solve the boundary-value problem.

The solution map is compact and self-adjoint, so there is an orthogonal basis of eigenfunctions. Applying d^2/dx^2 to an eigenvalue equation

$$\int_{\mathbb{T}} G(x,y)\, u(y)\, dy \;=\; \mu \cdot u(x)$$

gives

$$u - u(0) \;=\; \mu \cdot u''$$

With $u(0) = 0$, and taking $\lambda = \mu^{-1}$, we recover the differential equation explicitly solved earlier by elementary methods.

Thus, the functions $\sin \frac{nx}{2}$ give an orthogonal basis for $L^2[0, 2\pi]$. ///

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