(*March 30, 2020*)

# 09d. Fourier series and compact operators

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A discussion of Fourier series as a consequence of the spectral theorem for self-adjoint compact operators on Hilbert spaces is anachronistic. Nevertheless, it does offer some insights.

As we may know, essentially the only general device to prove that a collection $\{u_n\}$ of functions on an interval $[a, b]$ is an orthogonal basis for $L^2[a, b]$ is to find a *self-adjoint compact* operator $T$ on $L^2[a, b]$ such that the eigenvectors of $T$ are the functions $u_n$. The same applies to most Hilbert spaces of functions.

The explicit Fourier-Dirichlet and Fejer arguments for ordinary Fourier series do not easily generalize, for more than one reason. As we know, one family of natural extensions is to Sturm-Liouville boundary-value problems (1830s), where we already see the necessity of the spectral theory of *self-adjoint compact operators* on Hilbert spaces, even though such ideas and proofs had to wait almost 60 years (1890s), for Bocher and Steklov, and then Hilbert and Schmidt and others a few years later.

The simplest Green's function prescription for Sturm-Liouville problems does not immediately apply to Fourier series, that is, to the fact that exponentials $\{x \to e^{inx} : n \in \mathbb{Z}\}$, or trigonometric functions $\{1, \sin nx, \cos nx : n = 1, 2, 3, \ldots\}$, form orthogonal bases for $L^2[0, 2\pi]$. But a slightly abstracted version does, again using the spectral theory of compact self-adjoint operators and some distribution theory, proves that the exponentials or trigonometric functions give orthogonal bases.

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## 1. *Variant Green's function*

We want $k$ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\Delta k \;=\; \delta + (\text{something innocuous}) \qquad\qquad (\delta = \delta_{\mathbb{Z}} \in \mathcal{D}(\mathbb{T})^*)$$

where, ideally, the leftover is in $L^2(\mathbb{T})$ or better. Then we would make a Hilbert-Schmidt-Schwartz kernel

$$K(x, y) \;=\; k(x - y) \qquad\qquad (\text{for } x, y \in \mathbb{T})$$

Among other possibilities, since application of $\Delta$ to piecewise quadratic functions, up to a constant to be determined subsequently, we take $k(x) = -(x - \pi)^2$ on $[0, 2\pi]$ and extend by $2\pi\mathbb{Z}$-periodicity:

$$\Delta k(x) \;=\; \frac{d}{dx}(-2(x - \pi))(\text{at first on } [0, 2\pi], \text{ then periodicized}) \;=\; -2 + 4\pi\delta$$

since the jump on this sawtooth function is upward by $2\pi$. Thus, replacing $k(x)$ by $k(x)/4\pi$,

$$\Delta k \;=\; \delta - \frac{1}{2\pi}$$

Thus, with $K(x, y) = k(x - y)$, and

$$Tf(x) \;=\; \int_{\mathbb{T}} K(x, y)\, f(y)\, dy \qquad\qquad (\text{for suitable } f \in L^2(\mathbb{T}))$$

presumably moving the differential operator inside the integral via Gelfand-Pettis, and abusing notation toward the end,

$$\Delta T f(x) \;=\; \Delta \int_{\mathbb{T}} K(x,y)\, f(y)\; dy \;=\; \int_{\mathbb{T}} \Delta_x K(x,y)\, f(y)\; dy \;=\; \int_{\mathbb{T}} \Big(\delta(x-y) - \frac{1}{2\pi}\Big)\, f(y)\; dy \;=\; f(x) - \frac{1}{2\pi}\int_{\mathbb{T}} f$$

This computation is a proof when the integral is rewritten as a pairing among Sobolev spaces. This strongly suggests that the target $f \in L^2(\mathbb{T})$ should satisfy $\int_{\mathbb{T}} f = 0$.

So we consider the Hilbert space $V = \{u \in L^2(\mathbb{T}) : \int_{\mathbb{T}} u = 0\}$. The kernel $K(x,y)$ should be further adjusted, if necessary, so that $x \in K(x,y)$ is in $V$ for every $y$. Compute

$$\int_{\mathbb{T}} K(x,y)\; dx \;=\; \int_{\mathbb{T}} k(x-y)\; dx \;=\; \int_{\mathbb{T}} k(x)\; dx \;=\; \frac{-1}{4\pi}\int_0^{2\pi} (x-\pi)^2\; dx \;=\; \frac{-1}{4\pi}\cdot\frac{2\pi^3}{3} \;=\; -\frac{\pi^2}{6}$$

Thus, we should add $\frac{-\pi^2}{6}\cdot 2\pi$ to $K(x,y)$. This has no impact when $K(x,y)$ is integrated against $f \in V$.

Since $K(x,y)$ is continuous on $\mathbb{T}\times\mathbb{T}$, it is in $L^2(\mathbb{T}\times\mathbb{T})$, and gives a Hilbert-Schmidt operator. The function $k(x)$ itself is *even* and real-valued, so $K(x,y)$ is a hermitian kernel, and gives a self-adjoint compact operator. Thus, by the spectral theorem, its eigenvectors give an orthogonal basis for $V$.

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## 2. *Eigenfunctions*

The eigenfunction condition $Tu = \lambda \cdot u$ for $u \in V$ with $\lambda \neq 0$ implies

$$u(x) \;=\; \frac{1}{\lambda}\int_{\mathbb{T}} K(x,y)\, u(y)\; dy \qquad\qquad (\text{in } L^2(\mathbb{T}))$$

Applying $\Delta$ distributionally, abusing notation about pairings among Sobolev spaces as usual, by design

$$\Delta u(x) \;=\; \frac{1}{\lambda}\Delta\int_{\mathbb{T}} K(x,y)\, u(y)\; dy \;=\; \frac{1}{\lambda}\int_{\mathbb{T}} \Delta_x\, K(x,y)\, u(y)\; dy \;=\; \frac{1}{\lambda}\int_{\mathbb{T}} \delta(x-y)\, u(y)\; dy \;=\; \frac{1}{\lambda} u(x)$$

Thus, a $\lambda$-eigenfunction $u \in L^2(\mathbb{T})$ satisfies the distributional differential equation $u'' = \frac{1}{\lambda}u$. Lifting this back to $\mathbb{R}$ via the projection $\mathbb{R} \to \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, letting $\lambda = c^2$, the differential equation has two linearly independent solutions, $e^{\pm cx}$.

The requirement that a linear combination $u(x) = ae^{cx} + be^{-cx}$ is orthogonal to constants is

$$0 \;=\; \frac{ae^{2\pi c} - a}{c} + \frac{be^{-2\pi c} - b}{-c}$$

which gives

$$a\cdot(e^{2\pi c} - 1) \;=\; b\cdot(e^{-2\pi c} - 1)$$

The periodicity condition is

$$ae^{c(x+2\pi n)} + be^{-c(x+2\pi n)} \;=\; ae^{cx} + be^{-cx} \qquad\qquad (\text{for all } x \in \mathbb{R}, \text{ for all } n \in \mathbb{Z})$$

At $x = 0$ and $n = 1$, this is

$$ae^{2\pi c} + be^{-2\pi c} \;=\; a + b$$

or

$$a\cdot(e^{2\pi c} - 1) \;=\; -b\cdot(e^{-2\pi c} - 1)$$

For $a, b$ not both 0, the latter equation and the orthogonality condition give $e^{2\pi c} = 1$, and then $a, b$ can be arbitrary. Conversely, when $e^{2\pi c} = 1$, all the conditions are met. Thus, the non-zero eigenvalues are $-n^2$ with $n \in \mathbb{Z}$, with corresponding eigenspaces spanned by $e^{\pm inx}$, and these eigenvalues give an orthogonal basis for $V$.

(For $\lambda = 0$, apply $\Delta$ distributionally to $Tu = 0 \cdot u = 0$ to obtain $u(x) = 0$ for $u$ orthogonal to constants.)

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