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10a. Schwartz kernel theorems, tensor products, nuclearity

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Hilbert-Schmidt operators $T : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)$ are exactly those continuous linear operators given by *kernels* $K(x, y)$ by [1]

$$Tf(y) \;=\; \int_{\mathbb{R}^m} K(x, y)\, f(x)\, dx$$

Here $K(x, y)$ is a *Schwartz kernels* $K(x, y) \in L^2(\mathbb{R}^{m+n})$. But most continuous linear maps $T : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)$ are *not* Hilbert-Schmidt, so do not have Schwartz kernels in $L^2(\mathbb{R}^{m+n})$. The obstacle is not just the non-compactness of $\mathbb{R}$, as most continuous $T : L^2(\mathbb{T}^m) \to L^2(\mathbb{T}^n)$ do *not* have kernels in this sense, either. That is, for most such T there is *no* $K(x, y) \in L^2(\mathbb{T}^{m+n})$ such that

$$Tf(y) \;=\; \int_{\mathbb{T}^m} K(x, y)\, f(x)\, dx$$

As it happens, enlarging the class of possible Schwartz kernels $K(x, y)$ on $\mathbb{R}^{m+n}$ to make every continuous linear map $L^2(\mathbb{T}^m) \to L^2(\mathbb{T}^n)$ have a kernel goes hand-in-hand with shrinking the source $L^2(\mathbb{T}^m)$ to test functions and enlarging the target $L^2(\mathbb{T}^n)$ to distributions.

Our interest here in tensor products and nuclear spaces is almost entirely aimed at conceptual proofs of Schwartz Kernel Theorems. Unsurprisingly, there is much more to be said, and there are many different viewpoints on these ideas, as partially attested-to by the bibliography.

Throughout, all topological vector spaces are *locally convex*, and $\mathrm{Hom}(X, Y)$ generally refers to *continuous* linear maps $X \to Y$, without necessarily committing to any of the several reasonable topologies on $\mathrm{Hom}(X, Y)$ itself.

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[1] Yes, this use of *kernel* is in conflict with the use of $\ker T$ for $T : X \to Y$ to denote $\ker T = \{x \in X : Tx = 0\}$. Nothing to be done about it, except possibly to prepend *Schwartz* or *integral* to the word in the present context. For that matter, *integral* does also have some unrelated algebraic senses, so perhaps *Schwartz kernel* is the best disambiguation.

1. *Concrete Schwartz' kernel theorems: statements*

Perhaps the simplest instance of a *Schwartz kernel theorem* is

[1.1] Theorem: Every continuous linear map $T : \mathcal{D}(\mathbb{T}^m) \to \mathcal{D}(\mathbb{T}^n)^*$ is given by a Schwartz kernel $K \in \mathcal{D}(\mathbb{T}^{m+n})^*$, by

$$(Tf)(\varphi) \;=\; K(f \otimes \varphi)$$

where $(f \otimes \varphi)(x, y) = f(x) \cdot \varphi(y)$ for $x \in \mathbb{T}^m$ and $y \in \mathbb{T}^n$. And conversely. *(Proof below.)*

In contrast to $\mathbb{T}^m$, where test functions and Schwartz functions and smooth functions and Sobolev spaces H^∞ are all the same, the non-compactness of $\mathbb{R}$ causes a bifurcation: test functions and distributions, or Schwartz functions and tempered distributions. Both the following are tangible instances of a *Schwartz kernel theorem*:

[1.2] Theorem: Every continuous linear map $T : \mathcal{D}(\mathbb{R}^m) \to \mathcal{D}(\mathbb{R}^n)^*$ is given by a Schwartz kernel $K \in \mathcal{D}(\mathbb{R}^{m+n})^*$, by

$$(Tf)(\varphi) \;=\; K(f \otimes \varphi)$$

where $(f \otimes \varphi)(x, y) = f(x) \cdot \varphi(y)$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. And conversely. *(Proof below.)*

[1.3] Theorem: Every continuous linear map $T : \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}(\mathbb{R}^n)^*$ is given by a Schwartz kernel $K \in \mathscr{S}(\mathbb{R}^{m+n})^*$, by

$$(Tf)(\varphi) \;=\; K(f \otimes \varphi)$$

where $(f \otimes \varphi)(x, y) = f(x) \cdot \varphi(y)$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. And conversely. *(Proof postponed.)*

The proofs depend on existence of categorically genuine *tensor products* of suitable topological vector spaces, such as spaces of test functions and Schwartz functions, examples of *nuclear* spaces, clarified below.

The proof of nuclearity of $\mathcal{D}(\mathbb{R}^n)$ is more complicated than for $\mathcal{D}(\mathbb{T}^n)$, and proof for $\mathscr{S}(\mathbb{R}^n)$ yet more complicated, as it happens.

[1.4] Remark: Continuity of maps to spaces of distributions depends on topologies on those spaces of distributions. It is reasonable to presume that these topologies are the *weak dual* (also called *weak-** topologies. Indeed, the statements of the theorems are correct with that assumption. In fact, the statements of the theorems are still correct with any of a range of *stronger* topologies on distributions, as clarified below.

2. *Examples of Schwartz kernels*

[... iou ...]

$\delta(x - y)$, Hilbert transform, Fourier, ... traces ... !?!?!

3. *Adjunctions of* Hom *and* $\otimes$

An *adjunction* is a natural isomorphism between two related spaces of homomorphisms, of the form

$$\mathrm{Hom}_{\mathfrak{C}}(LX, Y) \;\approx\; \mathrm{Hom}_{\mathfrak{D}}(X, RY) \qquad \text{(for all objects } X \in \mathfrak{D} \text{ and } Y \in \mathfrak{C})$$

where $L : \mathfrak{D} \to \mathfrak{C}$ and $R : \mathfrak{C} \to \mathfrak{D}$ are functors. As suggested by the notation, L is the *left adjoint* and R is the *right adjoint*.

For $\mathfrak{C}, \mathfrak{D}$ both the category of abelian groups, for example, a basic adjunction is [2]

$$\mathrm{Hom}(A \otimes B, C) \;\approx\; \mathrm{Hom}(A, \mathrm{Hom}(B, C)) \qquad \text{(for abelian groups } A, B, C\text{)}$$

by

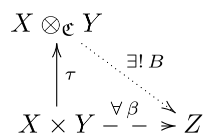
$$\Phi \to \varphi_\Phi \quad \text{with} \quad \varphi_\Phi(a)(b) = \Phi(a \otimes b) \qquad \text{and} \qquad \Phi_\varphi \longleftarrow \varphi \quad \text{with} \quad \Phi_\varphi(a \otimes b) = \varphi(a)(b)$$

Quantifying over A, C, for fixed B, this asserts that the functor $LA = A \otimes B$ is *left adjoint* to $RC = \mathrm{Hom}(B, C)$, and $\mathrm{Hom}(B, -)$ is *right adjoint* to $(-) \otimes B$.

One direction of the isomorphism is easy, namely, $\Phi \to \varphi_\Phi$ with $\varphi_\Phi(a)(b) = \Phi(a \otimes b)$. The other direction of the isomorphism, $\Phi_\varphi \longleftarrow \varphi$, needs properties of the tensor product. Specifically, $\beta_\varphi \longleftarrow \varphi$ with bilinear form $\beta_\varphi(a \times b) = (\varphi(a))(b)$ makes immediate sense, but we need *representability* of this map, in the sense that all *bilinear* maps $\beta : A \times B \to C$ should produce *linear* maps Φ_φ from an object $A \otimes B$ not depending on C or φ.

Suitable forms of such an adjunction will prove an abstract form of a Schwartz kernel theorem, below.

In a category $\mathfrak{C}$ whose objects that admit linear maps, a tensor product $X \otimes_{\mathfrak{C}} Y \in \mathfrak{C}$ (if it exists!) is an object with a fixed bilinear map $\tau : X \times Y \to X \otimes_{\mathfrak{C}} Y$ such that, for every bilinear $X \times Y \longrightarrow Z$, there is a unique linear $B : X \otimes_{\mathfrak{C}} Y \longrightarrow Z$ giving a commutative diagram



Usually, the bilinear map $X \times Y \to X \otimes_{\mathfrak{C}} Y$ is not explicitly named, but/and the image of $x \times y$ in $X \otimes_{\mathfrak{C}} Y$ is denoted $x \otimes y$. This is exactly the meaning of the symbols $x \otimes y$. Existence of the tensor product asserts that for given bilinear map $\beta(x \times y)$ on $X \times Y$ there is exactly one bilinear map B with $B(x \otimes y) = \beta(x \times y)$.

Proof of Schwartz kernel theorems requires existence of genuine it tensor products for suitable objects in an appropriate category of *topological vector spaces*. For $\mathbb{C}$-vectorspaces *without* topologies, with the usual (algebraic) tensor product of $\mathbb{C}$-vector spaces, the adjunction is

$$\mathrm{Hom}_{\mathbb{C}}(A \otimes_{\mathbb{C}} B, C) \;\approx\; \mathrm{Hom}_{\mathbb{C}}(A, \mathrm{Hom}_k(B, C))$$

and the special case $C = k$ gives

$$(A \otimes_k B)^* \;=\; \mathrm{Hom}_k(A \otimes_k B, k) \;\approx\; \mathrm{Hom}_k(A, B^*) \qquad (k\text{-vectorspaces } A, B, C)$$

That is, maps from A to B^* are given by *kernels* in $(A \otimes B)^*$. The validity of this adjunction for suitable *topological* vector spaces, and existence of genuine tensor products, requires more. As a cautionary point, we recall in an appendix the demonstration that infinite-dimensional Hilbert spaces do *not* have tensor products, despite constructions that may appear to produce them.

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4. *Topologies on* $\mathrm{Hom}(Y, Z)$

Spaces of continuous linear maps $\mathrm{Hom}(Y, Z)$ on topological vector spaces Y, Z, and dual spaces Y^* and $(X \otimes Y)^*$, are unambiguously defined as sets or vectorspaces without prescribing a topology for $\mathrm{Hom}(Y, Z)$. There are several reasonable topologies:

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[2] Such isomorphisms have a long history. In a homological setting, they arise in Cartan-Eilenberg in the early 1950's. In computation theory and logic, it was in H. Curry's 1930 work (from 1924 work of M. Schönfinkel), and was visible in G. Frege's 1895 thesis.

A *weak* topology on $\text{Hom}(Y,Z)$ is the *finite-to-open* topology, which has a basis at 0 given by sets of the form

$$U_{S,N} = \{S \in \text{Hom}(Y,Z) : TS \subset N\} \qquad \text{(for } \textit{finite } S \subset Y \text{ and open } N \ni 0 \text{ in } Z\text{)}$$

A *strong* topology on $\text{Hom}(Y,Z)$ is the *bounded-to-open* topology, which has a basis at 0 given by sets of the form

$$U_{S,N} = \{T \in \text{Hom}(Y,Z) : TS \subset N\} \qquad \text{(for } \textit{bounded } S \subset Y \text{ and open } N \ni 0 \text{ in } Z\text{)}$$

With $Z = \mathbb{C}$ and Banach space Y, the bounded-to-open topology gives the *Banach space* topology on the dual Y^*. There is also an intermediate *compact-to-open* topology, with basis at 0

$$U_{S,N} = \{T \in \text{Hom}(Y,Z) : TS \subset N\} \qquad \text{(for } \textit{compact } S \subset Y \text{ and open } N \ni 0 \text{ in } Z\text{)}$$

These topologies are given by respective families of seminorms:

$$\nu_{S,N}(T) = \inf\{t > 0 : TS \subset tN\} \qquad \text{(for } \textit{finite } S \subset Y \text{ and balanced convex open } N \ni 0 \text{ in } Z\text{)}$$
$$\nu_{S,N}(T) = \inf\{t > 0 : TS \subset tN\} \qquad \text{(for } \textit{bounded } S \subset Y \text{ and balanced convex open } N \ni 0 \text{ in } Z\text{)}$$
$$\nu_{S,N}(T) = \inf\{t > 0 : TS \subset tN\} \qquad \text{(for } \textit{compact } S \subset Y \text{ and balanced convex open } N \ni 0 \text{ in } Z\text{)}$$

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5. *Continuity conditions on bilinear maps*

In the description of a categorically genuine *tensor product* of *topological* vector spaces X, Y, there is an ambiguity about the *continuity* requirement on bilinear maps: *joint* continuity, or mere *separate* continuity?

Correspondences between bilinear maps and iterated homomorphisms hold more broadly than existence of genuine tensor products.

For the following, associate to bilinear $\beta : X \times Y \to Z$ linear $B_\beta \in \text{Hom}(X, \text{Hom}(Y,Z))$ by $B_\beta(x)(y) = \beta(x \times y)$, and, oppositely, $\beta_B(x \times y) = B(x)(y)$, as in the adjunction above.

[5.1] Claim: Continuity of $y \to \beta(x_o \times y)$ in $y \in Y$ for a fixed $x_o \in X$ is equivalent to $B(x_o) \in \text{Hom}(Y,Z)$.

Proof: This claim is nearly tautologous. Note that there is no reference to a topology on $\text{Hom}(Y,Z)$. Indeed, $B(x_o)(y) = \beta(x_o \times y)$. ///

[5.2] Claim: Continuity of $x \to \beta(x \times y_o)$ in $x \in X$ for every fixed $y_o \in Y$ is equivalent to $B \in \text{Hom}(X, \text{Hom}(Y,Z))$ with the weak *finite-to-open* topology on $\text{Hom}(Y,Z)$.

Proof: Although $\text{Hom}(Y,Z)$ has a topology here, there is no reference to a topology on $\text{Hom}(X, \text{Hom}(Y,Z))$. Using the seminorms $\nu_{S,N}$ above, defining the weak finite-to-open topology on $\text{Hom}(Y,Z)$, it suffices to take $S = \{y_o\}$. Let $N \subset Z$ be a balanced convex open. Let $\varepsilon > 0$. To make $\nu_{S,N}(B(x)) < \varepsilon$ is to make $|B(x)(y_o)| < \varepsilon$, which is to make $|\beta(x \times y_o)| < \varepsilon$. Since $x \to \beta(x \times y_o)$ is continuous in x, there is an open neighborhood $U \ni 0$ in X such that $x \in U$ satisfies the desired inequality. Reversing this argument gives the converse. ///

[5.3] Claim: *Joint* continuity of β implies $B \in \text{Hom}(X, \text{Hom}(Y,Z))$ with the strong *bounded-to-open* topology on $\text{Hom}(Y,Z)$.

Proof: Joint continuity of β includes joint continuity at $(0,0)$. Thus, given open $U \ni 0$ in Z, there are convex, balanced opens $0 \in N_1 \subset X$ and $0 \in N_2 \subset Y$ such that $\beta(N_1 \times N_2) \subset U$. Given bounded $S \subset Y$, let $t > 0$ be large enough so that $tN_2 \supset S$. For $x \in t^{-1}N_1$,

$$B(x)(S) = \beta(x \times S) \subset \beta(t^{-1}N_1 \times tN_2) = \beta(N_1 \times N_2) \subset U$$

That is, $B(x)$ is continuous at 0 with respect to the seminorms for the bounded-to-open topology. ///

6. *Some ambiguity removed*

As noted in the previous section, there are at least two types of possible continuity requirements on bilinear maps, *joint* continuity and *separate* continuity, and this generally would give two different specifications of tensor products (if they exist at all):

One tensor product $X \otimes_{\text{sep}} Y$ is a topological vector space and *separately* continuous bilinear map $\tau_{\text{sep}} : X \times Y \longrightarrow X \otimes_{\text{sep}} Y$ such that, for every *separately* bilinear $\beta : X \times Y \longrightarrow Z$, there is a unique continuous linear $B : X \otimes_{\text{sep}} Y \longrightarrow Z$ fitting into the commutative diagram

$$\begin{array}{ccc} X \otimes_{\text{sep}} Y & & \\ {\scriptstyle \tau_{\text{sep}}} \uparrow & \overset{\exists!\, B}{\searrow} & \\ X \times Y & \overset{\forall\, \beta}{\dashrightarrow} & Z \end{array}$$

An opposite specification: a topological vector space $X \otimes_{\text{jnt}} Y$ and *jointly* continuous bilinear map $\tau_{\text{jnt}} : X \times Y \longrightarrow X \otimes_{\text{jnt}} Y$ such that, for every *jointly* continuous bilinear $\beta : X \times Y \longrightarrow Z$, there is a unique continuous linear $B : X \otimes_{\text{jnt}} Y \longrightarrow Z$ fitting into the commutative diagram

$$\begin{array}{ccc} X \otimes_{\text{jnt}} Y & & \\ {\scriptstyle \tau_{\text{jnt}}} \uparrow & \overset{\exists!\, B}{\searrow} & \\ X \times Y & \overset{\forall\, \beta}{\dashrightarrow} & Z \end{array}$$

Fortunately, for Fréchet spaces part of this ambiguity is absent:

[6.1] Theorem: For Fréchet spaces X, Y and locally convex Z, a *separately* continuous bilinear map $X \times Y \to Z$ is necessarily *jointly* continuous. *(Proof in an appendix.)*

Let $\text{Bil}_{\text{jnt}}(X \times Y, Z)$ be the jointly continuous bilinear maps, and $\text{Bil}_{\text{sep}}(X \times Y, Z)$ the separately continuous bilinear maps.

[6.2] Corollary: Let $\text{Hom}_{\text{wk}}(Y, Z)$ be $\text{Hom}(Y, Z)$ with the (weak) finite-to-open topology, and let $\text{Hom}_{\text{str}}(Y, Z)$ be $\text{Hom}(Y, Z)$ with the (strong) bounded-to-open topology. For Fréchet X, Y and locally convex Z, the natural injection $\text{Hom}(X, \text{Hom}_{\text{str}}(Y, Z)) \to \text{Hom}(X, \text{Hom}_{\text{wk}}(Y, Z))$ is an *isomorphism*, and $\text{Bil}_{\text{jnt}}(X \times Y, Z) \to \text{Hom}_{\text{str}}(Y, Z)$ is an isomorphism.

Proof: We have

$$\begin{array}{ccc} \text{Hom}(X, \text{Hom}_{\text{wk}}(Y, Z)) & \overset{\approx}{\longleftarrow} & \text{Bil}_{\text{sep}}(X \times Y, Z) \\ {\scriptstyle \text{inj}} \uparrow & & \uparrow {\scriptstyle \approx} \\ \text{Hom}(X, \text{Hom}_{\text{str}}(Y, Z)) & \underset{\text{inj}}{\longleftarrow} & \text{Bil}_{\text{jnt}}(X \times Y, Z) \end{array}$$

The vertical map on the right is an isomorphism by the theorem above. Thus, the lower and left injections are *bijections*. Indeed, for *any* topology $\text{Hom}_{??}(Y, Z)$ intermediate between $\text{Hom}_{\text{str}}(X, Y)$ and $\text{Hom}_{\text{wk}}(Y, Z)$, the analogous injections

$$\text{Hom}(X, \text{Hom}_{\text{str}}(Y, Z)) \longrightarrow \text{Hom}(X, \text{Hom}_{??}(Y, Z)) \longrightarrow \text{Hom}(X, \text{Hom}_{\text{wk}}(Y, Z))$$

must be bijections. ///

[6.3] Corollary: For Fréchet spaces X, Y, if either $X \otimes_{\text{sep}} Y$ or $X \otimes_{\text{jnt}} Y$ exists (in some reasonable category), then the other exists, and the two are isomorphic.

Proof: Suppose $X \otimes_{\text{sep}} Y$ exists. By the theorem, τ_{sep} is actually jointly continuous. A given *jointly* continuous $\beta : X \times Y \to Z$ is *separately* continuous, so there is unique continuous linear $B : X \otimes_{\text{sep}} Y \to Z$ giving the desired commutative diagram. Thus, $X \otimes_{\text{sep}} Y$ and τ_{sep} fit the characterization of $X \otimes_{\text{jnt}} Y$ and τ_{jnt}.

Oppositely, assume $X \otimes_{\text{jnt}} Y$ exists. Given *separately* continuous $\beta : X \times Y \to Z$, by the theorem β is *jointly* continuous, so uniquely factors through $X \otimes_{\text{jnt}} Y$, and $X \otimes_{\text{jnt}} Y$ and τ_{jnt} fit the characterization of $X \otimes_{\text{sep}} Y$ and τ_{sep}.

Based on the latter remarks, we recall the usual inevitable categorical argument of the isomorphism of the two tensor products, when both exist:

When both exist, the (separately, but also jointly) continuous bilinear map $X \times Y \to X \otimes_{\text{sep}} Y$ induces a unique compatible linear map $B_1 : X \otimes_{\text{jnt}} Y \to X \otimes_{\text{sep}} Y$. Symmetrically, since $X \times Y \to X \otimes_{\text{sep}} Y$ is actually jointly continuous (by the theorem), it induces a unique compatible map B_2 in the other direction. Thus, $B_2 \circ B_1$ and $B_1 \circ B_2$ are continuous linear self-maps of $X \otimes_{\text{jnt}} Y$ and $X \otimes_{\text{sep}} Y$ commuting with the bilinear maps $X \times Y$ to them. The uniqueness aspect of the universal mapping property shows that both $B_2 \circ B_1$ and $B_1 \circ B_2$ must be the identity, so B_1, B_2 are mutual inverses. ///

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7. *Nuclear Fréchet spaces*

What remains is *existence* of tensor products, at least for certain Fréchet spaces.

Roughly, the intention of *nuclear spaces* is that they should admit genuine *tensor products*, aiming at an abstract Schwartz Kernel Theorem.

Enlarging the class of possible Schwartz kernels $K(x, y)$ sufficiently so that every continuous $L^2(\mathbb{T}^m) \to L^2(\mathbb{T}^n)$ has such a kernel turns requires a larger family of topological vector spaces than Hilbert spaces or Banach spaces, so that *some* of them do have *tensor products*.

Countable projective limits of Hilbert spaces with Hilbert-Schmidt transition maps constitute the simplest class of *nuclear spaces*: they admit *tensor products*, as we see below. Countable limits of Hilbert spaces are also Fréchet, so these are *nuclear Fréchet* spaces.

The simplest natural example of such a space is the Levi-Sobolev space $H^\infty(\mathbb{T}^n)$ on a product $\mathbb{T}^n$ of circles $\mathbb{T} = S^1$, where the simplest Rellich compactness and Sobolev imbedding give the requisite Hilbert-Schmidt property.

[7.1] $V \otimes_{\text{HS}} W$ is not a categorical tensor product To be clear, again, the Hilbert space $V \otimes_{\text{HS}} W$ is *not* a categorical tensor product of (infinite-dimensional) Hilbert spaces V, W. In particular, although the bilinear map $V \times W \to V \otimes_{\text{HS}} W$ is continuous, there are continuous bilinear $\beta : V \times W \to X$ to Hilbert spaces H which do *not* factor through any continuous linear map $B : V \otimes_{\text{HS}} W \to X$.

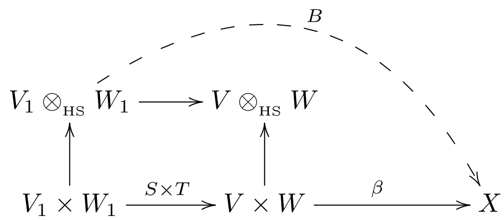
The case $W = V^*$ and $X = \mathbb{C}$, with $\beta(v, \lambda) = \lambda(v)$ already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting v_i be an orthonormal basis for V and $\lambda_i(v) = \langle v, v_i \rangle$ an orthonormal basis for V^*, necessarily

$$B\Big(\sum_{ij} c_{ij}\, v_i \otimes \lambda_j\Big) \;=\; \sum_{ij} c_{ij} \beta(v_i, \lambda_j) \;=\; \sum_i c_{ii} \qquad\qquad (???)$$

However, $\sum_i \frac{1}{i} v_i \otimes \lambda_i$ is in $V \otimes_{\text{HS}} V^*$, but the alleged value of B is impossible. In effect, the obstacle is that there are Hilbert-Schmidt maps which are not of trace class.

[7.2] Approaching tensor products and nuclear spaces Let V, W, V_1, W_1, X be Hilbert spaces with Hilbert-Schmidt maps $S : V_1 \to V$ and $T : W_1 \to W$. We claim that for any (jointly) continuous

$\beta : V \times W \to X$, there is a unique continuous $B : V_1 \otimes_{\text{HS}} W_1 \to X$ giving a commutative diagram



In fact, $B : V_1 \otimes_{\text{HS}} W_1 \to X$ is *Hilbert-Schmidt*. As the diagram suggests, $V \otimes_{\text{HS}} W$ is bypassed, playing no role.

Proof: Once the assertion is formulated, the argument is the only thing it can be: The continuity of β gives a constant C such that $|\beta(v,w)| \le C \cdot |v| \cdot |w|$, for all $v \in V$, $w \in W$. Then

$$|\beta(Sv, Tw)| \;\le\; C \cdot |Sv| \cdot |Tv|$$

Squaring and summing over orthonormal bases v_i and w_j,

$$\sum_{ij} |\beta(Sv_i, Tw_j)|^2 \;\le\; C \cdot \sum_{ij} |Sv_i|^2 \cdot |Tw_j|^2 \;=\; C \cdot |S|^2_{\text{HS}} \cdot |T|^2_{\text{HS}} \;<\; \infty$$

The only possible definition is

$$B(\sum_{ij} c_{ij}\, v_i \otimes w_j) \;=\; \sum_{ij} c_{ij}\, \beta(Sv_i, Tw_j)$$

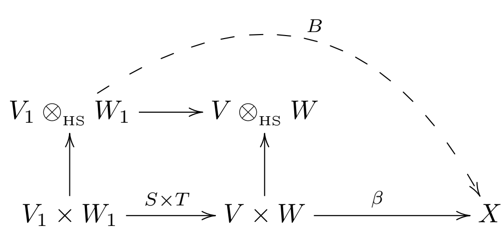
Then Cauchy-Schwarz-Bunyakowsky

$$\sum_{ij} \big|c_{ij}\, \beta(Sv_i, Tw_j)\big|^2 \;\le\; \sum_{ij} \big|c_{ij}\big|^2 \;\cdot\; \sum_{ij} \big|\beta(Sv_i, Tw_j)\big|^2 \;\le\; \sum_{ij} \big|c_{ij}\big|^2 \;\cdot\; \Big(C \cdot |S|^2_{\text{HS}} \cdot |T|^2_{\text{HS}}\Big)$$

shows that $B : V_1 \otimes W_1 \to X$ is Hilbert-Schmidt. ///

The following is the starting point for proving that various classes of spaces admit full tensor products:

[7.3] Theorem: Let V, W, V_1, W_1 be Hilbert spaces with Hilbert-Schmidt maps $S : V_1 \to V$ and $T : W_1 \to W$, and X a normed space. For (jointly) continuous $\beta : V \times W \to X$, there is a unique continuous $B : V_1 \otimes_{\text{HS}} W_1 \to X$ giving a commutative diagram



Of course, B must be given by the formula

$$B(\sum_{ij} c_{ij}\, v_i \otimes w_j) \;=\; \sum_{ij} c_{ij}\, \beta(Sv_i, Tw_j)$$

with orthonormal bases v_i, w_i for V_1, W_1.

Proof: This argument is a variant of the previous. Again, the continuity of β gives a constant C such that $|\beta(v,w)| \le C\cdot|v|\cdot|w|$. Of course, we have no choice but to define $B : V_1 \otimes_{\rm HS} W_1 \to X$ by the given expression. The point is to prove continuity:

$$\big|B(\sum_{ij} c_{ij}\, v_i \otimes w_j)\big|_X \;\le\; \sum_{ij} |c_{ij}| \cdot |\beta(Sv_i, Tw_j)|_X \;\le\; \Big(\sum_{ij} |c_{ij}|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{ij} |\beta(Sv_i \times Tw_j)|_X^2\Big)^{\frac{1}{2}}$$

$$\le\; \Big(\sum_{ij} |c_{ij}|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{ij} C^2 \cdot |Sv_i|^2 \cdot |Tw_j)|^2\Big)^{\frac{1}{2}} \;=\; \Big|\sum_{ij} c_{ij}\, v_i \otimes w_j)\Big|_{V_1 \otimes_{\rm HS} W_1} \cdot\; C \cdot |S|_{\rm HS} \cdot |T|_{\rm HS}$$

Thus, $\beta \circ (S \times T)$ factors (uniquely) through $V_1 \otimes_{\rm HS} W_1$. ///

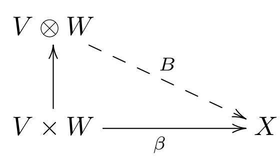
[7.4] A class of nuclear Fréchet spaces

We take the basic *nuclear Fréchet spaces* to be countable limits of Hilbert spaces with *Hilbert-Schmidt* transition maps.

That is, for a countable collection of Hilbert spaces $V_0, V_1, V_2, \ldots$ with *Hilbert-Schmidt* maps $\varphi_i : V_i \to V_{i-1}$, the limit $V = \lim_i V_i$ in the category of locally convex topological vector spaces is a *nuclear Fréchet space.* [3]

Let $\mathfrak{C}$ be the category of locally convex topological vector spaces. Every locally convex topology can be given by a family of seminorms. This expresses the topological vector space as a limit of normed spaces (by collapsing vector spaces so that seminorms become norms). Thus, every locally convex topological vector space is a subspace of a limit (possibly with a complicated indexing set) of normed spaces.

[7.5] Theorem: *Nuclear Fréchet spaces admit tensor products* in $\mathfrak{C}$. That is, for nuclear Fréchet spaces $V = \lim_i V_i$ and $W = \lim W_i$ there is a nuclear Fréchet space $V \otimes W$ and (jointly) continuous bilinear $V \times W \to V \otimes W$ such that, given a jointly continuous bilinear map $\beta : V \times W \to X$ of nuclear spaces V, W to locally convex X, there is a unique continuous linear map $B : V \otimes W \to X$ giving a commutative diagram



In particular, $V \otimes W \approx \lim_i V_i \otimes_{\rm HS} W_i$.

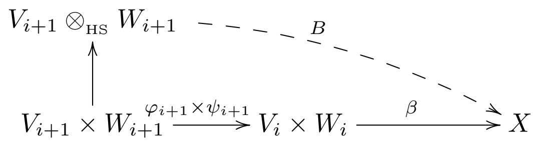
Proof: As will be seen at the end of this proof, the defining property of limits reduces to the case that X is itself a normed space. Let $\varphi_i : V_i \to V_{i-1}$ and $\psi_i : W_i \to W_{i-1}$ be the transition maps. First, we claim that, for large-enough index i, the bilinear map $\beta : V \times W \to X$ factors through $V_i \times W_i$. Indeed, the topologies on V and W are such that, given $\varepsilon_o > 0$, there are indices i, j and open neighborhoods of zero $E \subset V_i$, $F \subset W_j$ such that $\beta(E \times F) \subset \varepsilon_o$-ball at 0 in X. Since β is $\mathbb{C}$-bilinear, for *any* $\varepsilon > 0$,

$$\beta(\frac{\varepsilon}{\varepsilon_o} E \times F) \quad\subset\quad \varepsilon\text{-ball at 0 in } X$$

That is, β is already continuous in the $V_i \times W_j$ topology. Replace i, j by their maximum, so $i = j$.

[3] The new aspect is the nuclearity, not the Fréchet-ness: an arbitrary *countable* limit of Hilbert spaces is (provably) Fréchet, since an arbitrary countable limit of *Fréchet* spaces is Fréchet.

The theorem of the previous section shows that the only *possible* B fitting into the diagram

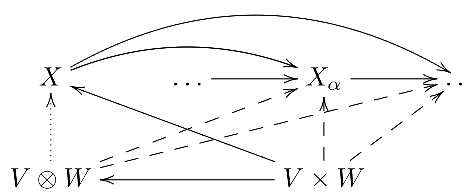


is indeed *continuous*. Thus, the categorical tensor product is the limit of the Hilbert-Schmidt completions of the algebraic tensor products of the limitands:

$$(\lim_i V_i) \otimes (\lim_j W_j) \;=\; \lim_i \big(V_i \otimes_{\mathrm{HS}} W_i\big)$$

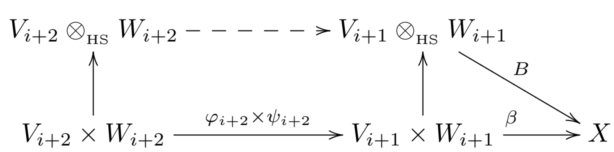
The transition maps in this limit are Hilbert-Schmidt, so the limit is again nuclear Fréchet.

As remarked at the beginning of the proof, the general case follows from the basic characterization of limits: for $X = \lim_\alpha X_\alpha$ with X_α normed, a continuous bilinear map $V \otimes W \to X$ is exactly a compatible family of maps $V \otimes W \to X_\alpha$. To obtain this compatible family, observe that a continuous bilinear $V \times W \to X$ composed with projections $X \to X_\alpha$ gives a compatible family of continuous bilinear maps $V \times W \to X_\alpha$. These induce compatible linear maps $V \otimes W \to X_\alpha$, as in the commutative diagram



The linear maps $V \otimes W \to X_\alpha$ induce a unique continuous linear $V \otimes W \to X$.

When X is a Hilbert space, in fact B is Hilbert-Schmidt. Applying the same argument with X replaced by $V_{i+1} \otimes_{\mathrm{HS}} W_{i+1}$ shows that the dotted map in



is Hilbert-Schmidt. Thus, $\lim_i (V_i \otimes_{\mathrm{HS}} W_i)$ is again nuclear Fréchet. ///

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8. *Adjunction: Schwartz Kernel Theorem for nuclear Fréchet spaces*

First, we prove a form of the adjunction for topological vector spaces.

Let X, Y be nuclear Fréchet space of the form $X = \lim X_i$ and $Y = \lim Y_i$ with Hilbert spaces X_i, Y_i and Hilbert-Schmidt transition maps $X_i \to X_{i-1}$ and $Y_i \to Y_{i-1}$. Let Z be locally convex.

[8.1] Theorem: Giving $\mathrm{Hom}(Y, Z)$ any topology as fine as the (weak) finite-to-open topology and no finer than the (strong) bounded-to-open topology, we have an isomorphism of $\mathbb{C}$-vectorspaces

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \;\approx\; \mathrm{Hom}(X \otimes Y, Z)$$

Proof: The map from Φ in $\mathrm{Hom}(X \otimes Y, Z)$ to φ_Φ in $\mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ first sends Φ to the jointly continuous bilinear form $\beta_\Phi(x \times y) = \Phi(x \otimes y)$ by composing with the canonical jointly continuous

$X \times Y \to X \otimes Y$. From the earlier discussion of continuity conditions, the continuity in Y for fixed $x_o \in X$ is equivalent to $\varphi_\Phi(x_o) \in \mathrm{Hom}(Y,Z)$. The separate continuity in X for fixed y_o is equivalent to $\varphi_\Phi \in \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$ with weak finite-to-open topology on $\mathrm{Hom}(Y,Z)$.

On the other hand, φ in $\mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$ with weak finite-to-open topology on $\mathrm{Hom}(Y,Z)$ gives separately continuous bilinear form $\beta_\varphi(x \times y) = \varphi(x)(y)$. The joint continuity of separately continuous bilinear maps for X, Y Fréchet, and *existence* of the tensor product $X \otimes Y$, gives the continuous linear $\Phi_\varphi \in \mathrm{Hom}(X \otimes Y, Z)$.

As earlier, $X \otimes Y$ is independent of topology on $\mathrm{Hom}(Y,Z)$ in the indicated range, due to the joint continuity of separately continuous bilinear maps for X, Y Fréchet. ///

Taking $Z = \mathbb{C}$, we have

[8.2] Corollary: Giving Y^* any topology as fine as the (weak) finite-to-open topology and no finer than the (strong) bounded-to-open topology, we have an isomorphism of $\mathbb{C}$-vectorspaces

$$\mathrm{Hom}_{\mathfrak{C}}(X, Y^*) \approx (X \otimes_{\mathfrak{C}} Y)^*$$

[8.3] Remark: We can also strengthen the assertion to refer to topologies on $\mathrm{Hom}(X, Y^*)$ and $(X \otimes_{\mathfrak{C}} Y)^*$.

---

9. $\mathcal{D}(\mathbb{T}^n)$ *is nuclear Fréchet*

Let $\mathbb{T}$ be the circle $\mathbb{R}/2\pi\mathbb{Z}$. In terms of Fourier series, for $s \geq 0$ the s^{th} L^2 Levi-Sobolev space on $\mathbb{T}^m$ is

$$H^s(\mathbb{T}^m) = \{\sum_\xi c_\xi \, e^{i\xi\cdot x} \in L^2(\mathbb{T}^m) \;:\; \sum_\xi |c_\xi|^2 \cdot (1+|\xi|^2)^s < \infty\}$$

The Levi-Sobolev imbedding theorem asserts that

$$H^{k+\frac{m}{2}+\varepsilon}(\mathbb{T}^m) \subset C^k(\mathbb{T}^m) \qquad \text{(for all } \varepsilon > 0)$$

Thus,

$$C^\infty(\mathbb{T}^m) = H^{+\infty}(\mathbb{T}^m) = \lim_s H^s(\mathbb{T}^m) \approx \lim \Big(\ldots \to H^2(\mathbb{T}^m) \to H^1(\mathbb{T}^m) \to H^0(\mathbb{T}^m) \Big)$$

We recall a form of Rellich's compactness lemma:

[9.1] Theorem: For $s > t$, $H^s(\mathbb{T}^n) \to H^t(\mathbb{T}^n)$ is Hilbert-Schmidt for $s > t + \frac{n}{2}$.

Proof: [... iou ...] ///

[9.2] Corollary: $H^\infty(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$ is nuclear Fréchet. ///

---

10. $\mathcal{D}(\mathbb{T}^m) \otimes \mathcal{D}(\mathbb{T}^n) \approx \mathcal{D}(\mathbb{T}^{m+n})$

[10.1] Claim:

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \approx H^{+\infty}(\mathbb{T}^{m+n})$$

induced from the natural

$$(\varphi \otimes \psi)(x,y) = \varphi(x)\,\psi(y) \qquad (\varphi \in H^{+\infty}(\mathbb{T}^m),\ \psi \in H^{+\infty}(\mathbb{T}^n),\ x \in \mathbb{T}^m,\ y \in \mathbb{T}^n)$$

Indeed, our construction of this tensor product is

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \;=\; \lim_s \Big(H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \Big)$$

The inequalities

$$(1+|\xi|^2+|\eta|^2)^2 \;\ge\; (1+|\xi|^2)(1+|\eta|^2) \;\ge\; 1+|\xi|^2+|\eta|^2 \qquad (\text{for } \xi \in \mathbb{Z}^m,\ \eta \in \mathbb{Z}^n)$$

give

$$H^{2s}\mathbb{T}^{m+n}) \;\subset\; H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \;\subset\; H^s(\mathbb{T}^{m+n}) \qquad (\text{for } s \ge 0)$$

The limit only depends on cofinal sublimits, so, indeed,

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \;\approx\; H^{+\infty}(\mathbb{T}^{m+n})$$

///

Thus,

$$\text{Hom}(\mathcal{D}(\mathbb{T}^m), \mathcal{D}(\mathbb{T}^n)^*) \;\approx\; \text{Hom}(\mathcal{D}(\mathbb{T}^m) \otimes \mathcal{D}(\mathbb{T}^n), \mathbb{C}) \;\approx\; \text{Hom}(\mathcal{D}(\mathbb{T}^{m+n}), \mathbb{C}) \;=\; \mathcal{D}(\mathbb{T}^{m+n})^*$$

This completes the proof of a concrete Schwartz kernel theorem for $\mathcal{D}(\mathbb{T}^n)$, namely,

$$\text{Hom}(\mathcal{D}(\mathbb{T}^m), \mathcal{D}(\mathbb{T}^n)^*) \;\approx\; \mathcal{D}(\mathbb{T}^{m+n})^*$$

as asserted in the first section. ///

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11. *Nuclear LF-spaces*

Although the statement of the Schwartz kernel theorem for test functions on $\mathbb{R}^n$ is identical in form to that for $\mathcal{D}(\mathbb{T}^n)$, the proof *must* be somewhat different, because $\mathcal{D}(\mathbb{R}^n)$ is not a Fréchet space. It is an LF-space, that is, a strict colimit (also called strict inductive limit) of Fréchet spaces. Thus, proof of existence of tensor products must be somewhat different.

We expect to prove that $\mathcal{D}(\mathbb{R}^n)$ is *nuclear* because it is a (locally convex) strict colimit of the nuclear Fréchet spaces $\mathcal{D}(K)$, where K runs through any reasonable set of compact subsets of $\mathbb{R}^n$. Since cofinal (co)limits give the same outcome, we can take K ranging through cubes $K_N = [-N, N]^n$. Each $\mathcal{D}(K_N)$ imbeds in $\mathcal{D}(\mathbb{T}^n)$ as a closed subspace, so is nuclear Fréchet.

For simplicity, we only consider index sets $\{1, 2, 3, \ldots\}$ for colimits.

As a temporary notation, justified by the theorem, for nuclear Fréchet X_i and Y_i, write

$$(\text{colim} X_i) \otimes (\text{colim} Y_i) \;=\; \text{colim}(X_i \otimes Y_i)$$

Here, for $X_i = \lim_j (X_i)_j$ for Hilbert spaces $(X_i)_j$ with Hilbert-Schmidt transition maps, and similarly for Y_i, we have seen that a tensor product constructed as

$$X_i \otimes Y_i \;=\; \lim_j \big((X_i)_j \otimes_{\text{HS}} (Y_i)_j \big)$$

fulfills all requirements of a genuine tensor product.

[11.1] Theorem: Let X_i, Y_i be nuclear Fréchet, for $i = 1, 2, 3, \ldots$. Assume that the underlying *sets* of (locally convex) $X = \text{colim} X_i$ and $Y = \text{colim} Y_i$ are the ascending unions of the underlying sets of X_i and Y_i.

Then colimits $X = \mathrm{colim} X_i$ and $Y = \mathrm{colim} Y_i$ are *nuclear*, in the sense that separately continuous $X \times Y \to Z$ for locally convex Z uniquely factors through continuous linear maps from $\mathrm{colim}(X_i \otimes Y_i)$, with a canonical separately continuous $X \times Y \to \mathrm{colim}(X_i \otimes Y_i)$.

[11.2] Remark: The conclusion of the theorem justifies declaring

$$(\mathrm{colim}_i X_i) \otimes (\mathrm{colim} Y_i) \;=\; \mathrm{colim}(X_i \otimes Y_i)$$

Further, since we have seen that $X_i \otimes Y_i$ is still nuclear Fréchet, the tensor product $(\mathrm{colim}_i X_i) \otimes (\mathrm{colim} Y_i) = \mathrm{colim}(X_i \otimes Y_i)$ is itself still in the same class of nuclear spaces as $\mathrm{colim} X_i$ and $\mathrm{colim} Y_i$.

[11.3] Remark: The assumption on underlying sets applies at least to *strict* colimits X, Y of nuclear Fréchet spaces X_i, Y_i, and also to some other situations.

Proof: First, we claim that a compatible family of separately continuous bilinear maps $X_i \times Y_i \to Z$ gives a separately continuous $\mathrm{colim} X_i \times \mathrm{colim} Y_i \to Z$. For all $x_o \in X$, in fact x_o lies in some limitand X_{i_o}. For all $j \geq i_o$, $\{x_o\} \times Y_j \to Z$ is continuous linear on Y_j. This gives a continuous linear map on the colimit. And similarly with the roles of X, Y reversed. This verifies the claim.

To prove that a separately continuous $\mathrm{colim} X_i \times \mathrm{colim} Y_i \to Z$ gives continuous linear $\mathrm{colim}(X_i \otimes Y_i) \to Z$, observe that a family of jointly continuous $X_i \times Y_i \to Z$ with X_i, Y_i nuclear Fréchet gives a family of continuous $X_i \otimes Y_i \to Z$. The characterization of colimit gives continuous $\mathrm{colim}(X_i \otimes Y_i) \to Z$. ///

[11.4] Corollary: Strict colimits of nuclear Fréchet spaces are nuclear. ///

We may call such strict colimits *nuclear LF-spaces*.

---

12. *Schwartz kernel theorem for nuclear LF-spaces*

We verify that the tensor product, whose existence was confirmed in the previous section, fits into the desired adjunction. As in the nuclear Fréchet case earlier, this proof is really a variation on the proof that $\mathrm{colim}(X_i \otimes Y_i)$ satisfies the requirements for a tensor product.

[12.1] Theorem: Let $X = \mathrm{colim} X_i$ and $Y = \mathrm{colim} Y_i$ be colimits of nuclear Fréchet spaces X_i, Y_i such that the underlying sets of X, Y are the ascending unions of the limitands. Put

$$X \otimes Y \;=\; \mathrm{colim}(X_i \otimes Y_i)$$

For every locally convex Z, we have a natural isomorphism of $\mathbb{C}$-vectorspaces

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \;\approx\; \mathrm{Hom}(X \otimes Y, Z)$$

where $\mathrm{Hom}(Y, Z)$ is given the (weak) finite-to-open topology.

Proof: As earlier, we have no choice about the right-to-left map when restricted to the algebraic tensor product, and factoring through bilinear maps:

$$\mathrm{Hom}(X \otimes_{\mathrm{alg}} Y, Z) \;\longrightarrow\; \mathrm{Bil}(X, \mathrm{Hom}(Y, Z)) \;\longrightarrow\; \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

is completely specified by $\Phi \to \beta_\Phi \to \varphi_\Phi$, given by $\beta_\Phi(x \times y) = \Phi(x \otimes y)$, and then $\varphi_\Phi(x)(y) = \beta_\Phi(x \times y)$. From the earlier discussion of continuity properties of bilinear forms, since β_Φ is (at least) separately continuous in Y, $\varphi_\Phi(x)$ is in the *set* $\mathrm{Hom}(Y, Z)$, and because it is separately continuous in X, φ_Φ is in $\mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ when the latter is given the weak topology. These implications can be run in reverse, and this part of the argument does not depend on the nuclearity of the limitands.

The critical point is the extendability of $\Phi_\varphi(x \otimes y) = \varphi(x)(y)$ for $\varphi \in \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ from $\mathrm{Hom}(X \otimes_{\mathrm{alg}} Y, Z)$ to $\mathrm{Hom}(X \otimes Y, Z)$. This uses the nuclearity of the limitands, demonstrated in the previous theorem. ///

[12.2] **Corollary:** For X, Y nuclear LF-spaces, $\mathrm{Hom}(X, Y^*) \approx (X \otimes Y)^*$, where Y^* has the weak dual topology. ///

---

13. $\mathcal{D}(\mathbb{R}^n)$ *is a nuclear LF-space*

By definition, $\mathcal{D}(\mathbb{R}^n)$ is a strict colimit $\mathrm{colim}_N \mathcal{D}(K_N)$ where (for example) K_N is the cube

$$K_N = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \le N\}$$

[13.1] **Claim:** $\mathcal{D}(K_N)$ is nuclear Fréchet.

Proof: Via the quotient map $q : \mathbb{R}^n \to \mathbb{R}^n/(2N \cdot \mathbb{Z}^n) \approx \mathbb{T}^n$, the function space $\mathcal{D}(K_N)$ is exactly the pull-back of the closed subspace of $\mathcal{D}(\mathbb{R}^n/(2N \cdot \mathbb{Z}^n)$ consisting of functions vanishing to infinite order at (images of) points $x = (x_1, \ldots, x_n)$ with at least one $x_i \in 2N \cdot \mathbb{Z}$. The following unsurprising claim implies that $\mathcal{D}(K_N)$ is indeed nuclear Fréchet. ///

[13.2] **Claim:** Closed subspaces of nuclear Fréchet spaces are nuclear Fréchet.

Proof: Let $X = \lim X_i$ be nuclear Fréchet, with Hilbert spaces X_i and Hilbert-Schmidt transition maps. Let Y be a closed subspace of X. There is a natural inclusion $X \subset \prod X_i$ as a closed subspace, by identifying X as elements $x = (x_1, x_2, \ldots)$ of the product such that $x_i \to x_{i+1}$ under all transition maps. This requirement is indeed a closed condition.

Let Y_i be the Hilbert-space completion of the image of Y under the projection $X \to X_i$. The natural restriction $Y_i \to Y_{i-1}$ of the Hilbert-Schmidt $X_i \to X_{i-1}$ is Hilbert-Schmidt. By construction, Y is the limit of the Y_i (among other possibiities), so is nuclear Fréchet. ///

[13.3] **Corollary:** $\mathrm{Hom}(\mathcal{D}(\mathbb{R}^m), \mathcal{D}(\mathbb{R}^n)^*) \approx (\mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n))^*$ ///

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14. $\mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n) \approx \mathcal{D}(\mathbb{R}^{m+n})$

Of course it is completely unsurprising that something like the following theorem should hold, with suitable $\otimes$ and suitable sense of equality or isomorphism. Thus, the true issues are about correct notions of $\otimes$ and equality. And, then, what are the proof mechanisms?

[14.1] **Theorem:** $\mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^{m+n})$, where the tensor product is that of nuclear LF-spaces.

Proof: Some technical preparations are required, and then a significant point is the invocation of the *existence* of the tensor product of nuclear LF-spaces.

Under reasonable further hypotheses on a compact Hausdorff space X with positive regular Borel measure, Gelfand-Pettis integrals of continuous, compactly-supported functions f on X, with values in a quasi-complete locally convex topological vector space V, can be approximated by Riemann-like sums.

[14.2] **Claim:** Given a countable decomposition $X = \bigcup_n X_n$ with closed sets X_n, with $\mathrm{meas}\,(X_m \cap X_n) = 0$ for $m \ne n$, for every continuous V-valued function f on X,

$$\int_X f = \sum_n \int_{X_n} f$$

Proof: In this context, with fixed subsets X_n, we must show that the right-hand side, a countable sum of Gelfand-Pettis integrals of f on the subsets X_n, is a Gelfand-Pettis integral of f on X. Indeed, for $\lambda \in V^*$,

$$\lambda\Big(\sum_n \int_{X_n} f\Big) \;=\; \sum_n \lambda\Big(\int_{X_n} f\Big) \;=\; \sum_n \int_{X_n} \lambda\circ f \;=\; \int_X \lambda\circ f$$

from the countable additivity of scalar-valued integration. The last integral is $\lambda(\int_X f)$. This holds for all λ, so Hahn-Banach gives the desired equality. ///

[14.3] Claim: Given open $N \ni 0$ in V, suppose that the X_n are small enough so that $f(x) - f(y) \in N$ for all $x, y \in X_n$. Then, for any set of choices $x_n \in X_n$,

$$\int_X f \;-\; \sum_n \text{meas}\,(X_n)\cdot f(x_n) \;\in\; 2\cdot \text{meas}\,(X)\cdot N$$

Proof: The closure $\overline{E}$ of $E \subset V$ is the intersection of sets $E + N$ for open $N \ni 0$, so $\overline{E} \subset E + N$ for any single such N. For v in the closure of the convex hull of $f(X_n)$, take $y_1, \ldots, y_k \in X_n$ and $t_1, \ldots, t_k \geq 0$ such that $\sum_j t_j = 1$ and such that

$$v \;\in\; \sum_{j=1}^k t_j\cdot f(y_j) \;+\; N$$

Then

$$v - f(x_n) \;\in\; \sum_{j=1}^k t_j\cdot (f(y_j) - f(x_n)) + N \;=\; \sum_{j=1}^k t_j\cdot N \;+\; N \;=\; 2\cdot N$$

That is,

$$\text{closure of the convex hull of } f(X_n) \;\subset\; f(x_n) + 2N$$

Thus,

$$\int_{X_n} f \;\in\; \text{meas}\,(X_n)\cdot \big(f(x_n) + 2N\big)$$

Adding up,

$$\int_X f \;=\; \sum_n \int_{X_n} f \;\in\; \sum_n \text{meas}\,(X_n)\cdot \big(f(x_n) + 2N\big) \;=\; \sum_n \text{meas}\,(X_n)\cdot f(x_n) \;+\; 2\cdot \text{meas}\,(X)\cdot N$$

///

This finishes the general preparations. Now:

[14.4] Claim: The natural image of the *algebraic* tensor product $\mathcal{D}(\mathbb{R}^m) \otimes_{\text{alg}} \mathcal{D}(\mathbb{R}^n)$ is *sequentially dense* in $\mathcal{D}(\mathbb{R}^{m+n})$.

Proof: Let $u_i = \varphi_i \otimes \psi_i$ be an approximate identity with $\varphi_i \in \mathcal{D}(\mathbb{R}^m)$ and $\varphi_i \in \mathcal{D}(\mathbb{R}^n)$. For $f \in \mathcal{D}(\mathbb{R}^{m+n})$, the basic estimate on Gelfand-Pettis integrals shows that with the translation action of $\mathbb{R}^{m+n}$ on $\mathcal{D}(\mathbb{R}^{m+n})$, with the associated integral action, $u_i \cdot f \to f$, where

$$(u_i\cdot f)(x,y) \;=\; \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} \varphi_i(\xi)\,\psi_i(\eta)\,f(x+\xi, y+\eta)\,d\xi\,d\eta$$

Replacing ξ by $\xi - x$ and η by $\eta - y$ in the integral, letting $f^\vee(\xi,\eta) = f(-\xi,-\eta)$, the integral becomes the integral operator action of $f^\vee$ on u_i:

$$u_i\cdot f \;=\; f^\vee\cdot u_i$$

As above, the latter integral can be approximated arbitrarily well in $\mathcal{D}(\mathbb{R}^{m+n})$ by finite sums

$$\sum_j c_j f^\vee(z_j)\, T_{z_j} u_i \;=\; \sum_j c_j f^\vee(z_j)\, T_{z_j}(\varphi_i \otimes \psi_i) \;\in\; \mathcal{D}(\mathbb{R}^m) \otimes_{\text{alg}} \mathcal{D}(\mathbb{R}^n)$$

where T_{z_j} is translation by $z_j \in \mathbb{R}^{m+n}$. ///

Resuming the main proof: the bilinear map $\mathcal{D}(\mathbb{R}^m) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^{m+n}$ given by $\varphi \times \psi \to \Phi_{\varphi,\psi}$ with $\Phi_{\varphi,\psi}(x \oplus y) = \varphi(x) \cdot \psi(y)$ is at least *separately* continuous, so uniquely factors through $\mathcal{D}(\mathbb{R}^m) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n)$.

Both $\mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^{m+n}$ are quasi-complete, with the purely algebraic tensor product $\mathcal{D}(\mathbb{R}^m) \otimes_{\text{alg}} \mathcal{D}(\mathbb{R}^n)$ (or its image) sequentially dense in both.

Thus, the induced map

$$\mathcal{D}(\mathbb{R}^m) \otimes \mathcal{D}(\mathbb{R}^n) \;\longrightarrow\; \mathcal{D}(\mathbb{R}^{m+n})$$

must be a topological isomorphism. ///

This completes a proof of a Schwartz kernel theorem for test functions on Euclidean spaces.

15. *Appendix: joint continuity of bilinear maps*

Joint continuity of *separately* continuous bilinear maps on Hilbert spaces, is an easy corollary of Baire category. The result extends to Fréchet spaces with a little more work. First:

[15.1] Claim: A bilinear map $\beta : X \times Y \to Z$ on Hilbert spaces X, Y, Z, continuous in each variable *separately*, is *jointly* continuous.

Proof: Fix a neighborhood N of 0 in Z. Take sequences $x_n \to x_o$ in X and $y_n \to y_o$ in Y. For each $x \in X$, by continuity in Y, $\beta(x, y_n) \to \beta(x, y_o)$. Thus, for each $x \in X$, the set of values $\beta(x, y_n)$ is *bounded* in Z. The linear functionals $x \to \beta(x, y_n)$ are *equicontinuous*, by Banach-Steinhaus, so there is a neighborhood U of 0 in X so that $b_n(U) \subset N$ for all n. In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) \;=\; \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

we have $x_n - x_o \in U$ for large n, and $\beta(x_n - x_o, y_o) \in N$. Also, by continuity in Y, $\beta(x_o, y_n - y_o) \in N$ for large n. Thus, $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$, proving *sequential* continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

For a more general result, we recall some preparatory ideas:

A set E of continuous linear maps from a topological vectorspace X to Y is *equicontinuous* when, for every neighborhood U of 0 in Y, there is a neighborhood N of 0 in X so that $T(N) \subset U$ for every $T \in E$.

[15.2] Claim: Let V be a strict colimit of a locally convex closed subspaces V_i. Let Y be a locally convex topological vectorspace. A set E of continuous linear maps from V to Y is *equicontinuous* if and only if for each index i the collection $E|_{V_i} = \{T|_{V_i} : T \in E\}$ of restrictions is equicontinuous.

Proof: Given a neighborhood U of 0 in Y, shrink U if necessary so that U is convex and balanced. For each index i, let N_i be a convex, balanced neighborhood of 0 in V_i so that $TN_i \subset U$ for all $T \in E$. Let N be the convex hull of the union of the N_i in the locally convex coproduct of the V_i. By the convexity of N, still $TN \subset U$ for all $T \in E$. By the construction of the coproduct topology as the *diamond topology*, N is an open neighborhood of 0 in the coproduct. Hence the image of N in the colimit, a quotient of the coproduct, is a neighborhood of 0. This gives the equicontinuity of E. The other direction of the implication is easy. ///

Next, we need

[15.3] Claim: *Banach-Steinhaus/uniform boundedness* Let X be a Fréchet space or LF-space and Y an arbitrary topological vector space. A set E of linear maps $X \to Y$, such that every set $Ex = \{Tx : T \in E\}$ of pointwise values is *bounded* in Y, is *equicontinuous.*

Proof: First consider X Fréchet. Given a neighborhood U of 0 in Y, let $A = \bigcap_{T\in E} T^{-1}\overline{U}$. By assumption, $\bigcup_n nA = X$. By the Baire category theorem, the complete metric space X is not a countable union of nowhere dense subsets, so at least one of the closed sets nA has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, A itself has non-empty interior, containing some $x + N$ for a neighborhood N of 0 and $x \in A$. For every $T \in E$,

$$TN \;\subset\; T\{a - x : a \in A\} \;\subset\; \{u_1 - u_2 \;:\; u_1, u_2 \in \overline{U}\} \;=\; \overline{U} - \overline{U}$$

By continuity of addition and scalar multiplication in Y, given an open neighborhood U_o of 0, there is U such that $\overline{U} - \overline{U} \subset U_o$. Thus, $TN \subset U_o$ for every $T \in E$, and E is equicontinuous.

For $X = \bigcup_i X_i$ an LF-space, this argument already shows that E restricted to each X_i is equicontinuous. From the previous claim, this gives equicontinuity on the strict colimit. ///

A corollary of Banach-Steinhaus:

[15.4] Corollary: A *separately* continuous bilinear map $\beta : X \times Y \to Z$ from Fréchet spaces X, Y to an arbitrary topological vector space Z is *jointly* continuous.

Proof: Fix an open $N \ni 0$ in Z. Let $x_n \to x_o$ in X and $y_n \to y_o$ in Y. For each $x \in X$, by continuity in Y, $\beta(x, y_n) \to \beta(x, y_o)$. Thus, for each $x \in X$, the set of values $\beta(x, y_n)$ is *bounded* in Z. By Banach-Steinhaus, the linear functionals $x \to \beta(x, y_n)$ are *equicontinuous*, so there is an open $U \ni 0$ in X so that $b_n(U) \subset N$ for all n. In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) \;=\; \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

$x_n - x_o \in U$ for large n, and $\beta(x_n - x_o, y_o) \in N$. Similarly, by continuity in Y, $\beta(x_o, y_n - y_o) \in N$ for large n. Thus, $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$, proving *sequential* continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

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16. *Appendix: convex hulls*

For convenience and perspective about expression of Gelfand-Pettis integrals as limits of finite sums, we review some basic points.

[16.1] Claim: For a subset E of a locally convex topological vector space V, and for continuous linear $f : V \to W$ to another locally convex topological vector space W, the closure of the convex hull of $f(E)$ is the closure of the convex hull of $f(\overline{E})$, where $\overline{E}$ is the topological closure of E.

Proof: First, recall that the closure $\overline{E}$ of a subset E of a topological vector space is the intersection of all $E + U$ where U runs over opens containing 0. Thus, the closure of the convex hull of $f(\overline{E})$ is $\bigcap f(\overline{E}) + U$ with $0 \in U \subset W$.

The convex hull of $f(\overline{E})$ is the collection of finite convex combinations $\sum_{i=1}^n t_i f(v_i)$ with $v_i \in \overline{E}$. For each v_i, let $x_{i,\alpha}$ be a net such that $\lim_\alpha x_{i,\alpha} = v_i$. The continuity of f assures that $\lim_\alpha f(x_{i,\alpha}) = f(v_i)$. Thus, there is α_i such that $f(v_i) \in f(x_{i,\alpha_i}) + U$. Then

$$\sum_{i=1}^n t_i f(v_i) \;\in\; \sum_{i=1}^n t_i \cdot \big(f(x_{i,\alpha_i}) + U\big) \;=\; \Big(\sum_{i=1}^n t_i f(x_{i,\alpha_i}\Big) + U$$

This holds for every $U \ni 0$. ///

Unsurprisingly:

[16.2] Claim: A compact subset E of a locally convex topological vector space V is *bounded*.

Proof: In this context, *boundedness* is that, given an open $U \ni 0$, there is t_o such that for all $z \in \mathbb{C}$ with $|z| \ge t_o$ we have $z \cdot U \supset E$. Without loss of generality, shrink U so that it is *balanced*, in the sense that $z \cdot U \subset U$ for all $|z| \le 1$.

First, for an individual point v, continuity at $z = 0$ of scalar multiplication $v \to z \cdot v$ gives that, for every open $U \ni 0$, there is $\delta > 0$ such that $z \cdot v \in U + 0 \cdot v = U$ for all $|z| < \delta$. Then $|z| > 1/\delta$ gives $v \in z \cdot U$.

For each $v \in E$, let t_v be such that $v \in z \cdot U$ for all $|z| \ge t_v$. Then $E \subset \bigcup_{v\in E} t_v \cdot U$, and by compactness there are finitely-many $t_1, \ldots, t_n$ such that $E \subset t_1 U \cup \ldots \cup t_n U$. By balancedness of U, the latter union is contained in $t_o U$ with t_o the maximum of $t_1, \ldots, t_n$. Again by balancedness, $t_o U \subset zU$ for all $|z| \ge t_o$. ///

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17. *Appendix: Hilbert-Schmidt operators*

For convenience, we recall some features of Hilbert-Schmidt operators.

[17.1] Prototype: integral operators

For $K(x, y)$ in $C^o\big([a, b] \times [a, b]\big)$, define $T : L^2[a, b] \to L^2[a, b]$ by

$$Tf(y) \;=\; \int_a^b K(x, y)\, f(x)\, dx$$

The function K is the *integral kernel*, or *Schwartz kernel* of T. Approximating K by *finite* linear combinations of 0-or-1-valued functions shows T is a uniform operator norm limit of finite-rank operators, so is *compact*. The *Hilbert-Schmidt* operators include such operators, where the integral kernel $K(x, y)$ is allowed to be in $L^2\big([a, b] \times [a, b]\big)$.

[17.2] Hilbert-Schmidt norm on $V \otimes_{\text{alg}} W$

In the category of Hilbert spaces and continuous linear maps, there is *no* tensor product in the categorical sense, as demonstrated in an appendix.

Without claiming anything about genuine tensor products in any category of topological vector spaces, the *algebraic* tensor product $X \otimes_{\text{alg}} Y$ of two Hilbert spaces has a hermitian inner product $\langle, \rangle_{\text{HS}}$ determined by

$$\langle x \otimes y, x' \otimes y' \rangle_{\text{HS}} \;=\; \langle x, x' \rangle\, \langle y, y' \rangle$$

Let $X \otimes_{\text{HS}} Y$ be the completion with respect to the corresponding norm $|v|_{\text{HS}} = \langle v, v \rangle_{\text{HS}}^{1/2}$

$$X \otimes_{\text{HS}} Y \;=\; |\cdot|_{\text{HS}}\text{-completion of } X \otimes_{\text{alg}} Y$$

This completion is a Hilbert space. Unfortunately, it is not a genuine tensor product of X, Y, when both are infinite-dimensional, in effect because not every continuous linear map $X \to Y^*$ is Hilbert-Schmidt.

[17.3] Hilbert-Schmidt operators

For Hilbert spaces V, W the finite-rank [4] continuous linear maps $T : V \to W$ can be identified with the algebraic tensor product $V^* \otimes_{\text{alg}} W$, by [5]

$$(\lambda \otimes w)(v) \;=\; \lambda(v) \cdot w$$

The space of *Hilbert-Schmidt operators* $V \to W$ is the completion of the space $V^* \otimes_{\text{alg}} W$ of finite-rank operators, with respect to the *Hilbert-Schmidt norm* $|\cdot|_{\text{HS}}$ on $V^* \otimes_{\text{alg}} W$. For example,

$$|\lambda \otimes w + \lambda' \otimes w'|^2_{\text{HS}} \;=\; \langle \lambda \otimes w + \lambda' \otimes w', \lambda \otimes w + \lambda' \otimes w' \rangle$$

$$= \; \langle \lambda \otimes w, \lambda \otimes w \rangle + \langle \lambda \otimes w, \lambda' \otimes w' \rangle + \langle \lambda' \otimes w', \lambda \otimes w \rangle + \langle \lambda' \otimes w', \lambda' \otimes w' \rangle$$

$$= \; |\lambda|^2 |w|^2 + \langle \lambda, \lambda' \rangle \langle w, w' \rangle + \langle \lambda', \lambda \rangle \langle w', w \rangle + |\lambda'|^2 |w'|^2$$

When $\lambda \perp \lambda'$ or $w \perp w'$, the monomials $\lambda \otimes w$ and $\lambda' \otimes w'$ are orthogonal, and

$$|\lambda \otimes w + \lambda' \otimes w'|^2_{\text{HS}} \;=\; |\lambda|^2 |w|^2 + |\lambda'|^2 |w'|^2$$

That is, the space $\text{Hom}_{\text{HS}}(V, W)$ of Hilbert-Schmidt operators $V \to W$ is the *closure* of the space of finite-rank maps $V \to W$, in the space of all continuous linear maps $V \to W$, under the Hilbert-Schmidt norm. By construction, $\text{Hom}_{\text{HS}}(V, W)$ is a Hilbert space.

[17.4] Expressions for Hilbert-Schmidt norm, adjoints

The Hilbert-Schmidt norm of finite-rank $T : V \to W$ can be computed from any choice of orthonormal basis v_i for V, by

$$|T|^2_{\text{HS}} \;=\; \sum_i |Tv_i|^2 \qquad\qquad \text{(at least for finite-rank } T\text{)}$$

Thus, taking a limit, the same formula computes the Hilbert-Schmidt norm of T known to be Hilbert-Schmidt. Similarly, for two Hilbert-Schmidt operators $S, T : V \to W$,

$$\langle S, T \rangle_{\text{HS}} \;=\; \sum_i \langle Sv_i, Tv_i \rangle \qquad\qquad \text{(for any orthonormal basis } v_i\text{)}$$

The Hilbert-Schmidt norm $|\cdot|_{\text{HS}}$ dominates the *uniform operator norm* $|\cdot|_{\text{op}}$: given $\varepsilon > 0$, take $|v_1| \le 1$ with $|Tv_1|^2 + \varepsilon > |T|^2_o p$. Choose $v_2, v_3, \ldots$ so that $v_1, v_2, \ldots$ is an orthonormal basis. Then

$$|T|^2_{\text{op}} \;\le\; |Tv_1|^2 + \varepsilon \;\le\; \varepsilon + \sum_n |Tv_n|^2 \;=\; \varepsilon + |T|^2_{\text{HS}}$$

This holds for every $\varepsilon > 0$, so $|T|^2_{\text{op}} \le |T|^2_{\text{HS}}$. Thus, Hilbert-Schmidt limits are operator-norm limits, and Hilbert-Schmidt limits of finite-rank operators are *compact*.

Adjoints $T^* : W \to V$ of Hilbert-Schmidt operators $T : V \to W$ are Hilbert-Schmidt, since for an orthonormal basis w_j of W

$$\sum_i |Tv_i|^2 \;=\; \sum_{ij} |\langle Tv_i, w_j \rangle|^2 \;=\; \sum_{ij} |\langle v_i, T^* w_j \rangle|^2 \;=\; \sum_j |T^* w_j|^2$$

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[4] As usual a *finite-rank* linear map $T : V \to W$ is one with finite-dimensional image.

[5] Proof of this identification: on one hand, a map coming from $V^* \otimes_{\text{alg}} W$ is a *finite* sum $\sum_i \lambda_i \otimes w_i$, so certainly has finite-dimensional image. On the other hand, given $T : V \to W$ with finite-dimensional image, take $v_1, \ldots, v_n$ be an orthonormal basis for the orthogonal complement $(\ker T)^\perp$ of $\ker T$. Define $\lambda_i \in V^*$ by $\lambda_i(v) = \langle v, v_i \rangle$. Then $T \sim \sum_i \lambda_i \otimes Tv_i$ is in $V^* \otimes W$. The second part of the argument uses the completeness of V.

[17.5] Criterion for Hilbert-Schmidt operators

We claim that a continuous linear map $T : V \to W$ with Hilbert space V is Hilbert-Schmidt if for some orthonormal basis v_i of V

$$\sum_i |Tv_i|^2 \;<\; \infty$$

and then (as above) that sum computes $|T|^2_{\text{HS}}$. Indeed, given that inequality, letting $\lambda_i(v) = \langle v, v_i \rangle$, T is Hilbert-Schmidt because it is the Hilbert-Schmidt limit of the finite-rank operators

$$T_n \;=\; \sum_{i=1}^{n} \lambda_i \otimes Tv_i$$

[17.6] Composition of Hilbert-Schmidt operators with continuous operators

Post-composing: for Hilbert-Schmidt $T : V \to W$ and continuous $S : W \to X$, the composite $S \circ T : V \to X$ is Hilbert-Schmidt, because for an orthonormal basis v_i of V,

$$\sum_i |S \circ Tv_i|^2 \;\le\; \sum_i |S|^2_{\text{op}} \cdot |Tv_i|^2 \;=\; |S|_{\text{op}} \cdot |T|^2_{\text{HS}} \qquad \text{(with operator norm } |S|_{\text{op}} = \sup_{|v|\le 1} |Sv|\text{)}$$

Pre-composing: for continuous $S : X \to V$ with Hilbert X and orthonormal basis x_j of X, since adjoints of Hilbert-Schmidt are Hilbert-Schmidt,

$$T \circ S \;=\; (S^* \circ T^*)^* \;=\; (\text{Hilbert-Schmidt})^* \;=\; \text{Hilbert-Schmidt}$$

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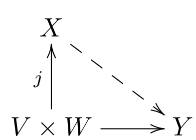
18. *Appendix: non-existence of tensor products of Hilbert spaces*

Tensor products of infinite-dimensional Hilbert spaces do not exist.

That is, for infinite-dimensional Hilbert spaces V, W, **there is no** Hilbert space X and continuous bilinear map $j : V \times W \longrightarrow X$ such that, for every continuous bilinear $V \times W \longrightarrow Y$ to a Hilbert space Y, there is a unique continuous linear $X \longrightarrow Y$ fitting into the commutative diagram



That is, *there is no tensor product in the category of Hilbert spaces and continuous linear maps.*

Yes, it *is* possible to put an inner product on the *algebraic* tensor product $V \otimes_{\text{alg}} W$, by

$$\langle v \otimes w,\, v' \otimes w' \rangle \;=\; \langle v, v' \rangle \cdot \langle w, w' \rangle$$

and extending. The *completion* $V \otimes_{\text{HS}} W$ of $V \otimes_{\text{alg}} W$ with respect to the associated norm, is a Hilbert space, identifiable with *Hilbert-Schmidt* operators $V \longrightarrow W^*$. *However*, this Hilbert space fails to have the universal property in the categorical characterization of tensor product, as we see below. This Hilbert space H is important in its own right, but is widely misunderstood as being a tensor product in the categorical sense.

The *non-existence* of tensor products of infinite-dimensional Hilbert spaces is important *in practice*, not only as a cautionary tale [6] about naive category theory, insofar as it leads to Grothendieck's idea of *nuclear spaces*, which *do* admit tensor products.

Proof: First, we review the point that the Hilbert-Schmidt tensor product $H = V \otimes_{\text{HS}} W$ is *not* a Hilbert-space tensor product, although it is a Hilbert space. For simplicity, suppose that V, W are *separable*, in the sense of having countable Hilbert-space bases.

Choice of such bases allows an identification of W with the continuous linear Hilbert space dual V^* of V. Then we have the continuous bilinear map $V \times V^* \longrightarrow \mathbb{C}$ by $v \times \lambda \longrightarrow \lambda(v)$. The *algebraic* tensor product $V \otimes_{\text{alg}} W$ injects to $H = V \otimes_{\text{HS}} V^*$, and the image is identifiable with the *finite-rank* maps $V \longrightarrow V$. The linear map $T : H \longrightarrow \mathbb{C}$ induced on the image of $V \otimes_{\text{alg}} V^*$ is *trace*. If $H = V \otimes_{\text{HS}} V^*$ were a Hilbert-space tensor product, the trace map would extend continuously to it from finite-rank operators. However, there are many Hilbert-Schmidt operators that are not of trace class. For example, letting e_i be an orthonormal basis, the element

$$\sum_n \frac{1}{n} \cdot e_n \otimes e_n \quad \in \quad V \otimes_{\text{HS}} V^*$$

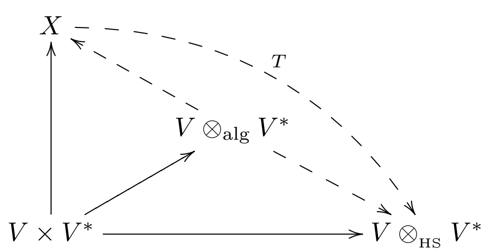
does not have a finite trace, since $\sum_{n \le N} 1/n \sim \log N$. In other words, the difficulty is that

$$T\Big(\sum_{a \le n \le b} \frac{1}{n} \cdot e_n \otimes e_n \Big) \;=\; \sum_{a \le n \le b} \frac{1}{n} \cdot T(e_n \otimes e_n) \;=\; \sum_{a \le n \le b} \frac{1}{n}$$

Thus, the partial sums of $\sum_n \frac{1}{n}\, e_n \otimes e_n$ form a Cauchy sequence, but the values of T on the partial sums go to $+\infty$. Thus, the Hilbert-Schmidt tensor product cannot be a Hilbert-space tensor product.

Now we show that no *other* Hilbert space can be a tensor product, by comparing to the Hilbert-Schmidt tensor product.

Let $V \times W \longrightarrow X$ be a purported Hilbert-space tensor product, and, again, let W be the dual of V, without loss of generality. By assumption, the continuous bilinear injection $V \times V^* \longrightarrow V \otimes_{\text{HS}} V^*$ induces a unique continuous linear map $T : X \longrightarrow H$ fitting into a commutative diagram



The linear map $V \otimes_{\text{alg}} V^* \longrightarrow V \otimes_{\text{HS}} V^*$ is injective, since $V \otimes_{\text{HS}} V^*$ is a completion of $V \otimes_{\text{alg}} V^*$. Thus, unsurprisingly, $V \otimes_{\text{alg}} V^* \longrightarrow X$ is necessarily injective. The uniqueness of the linear induced maps implies that the image of $V \otimes_{\text{alg}} V^*$ is *dense* in X. Also, $T : X \longrightarrow V \otimes_{\text{HS}} V^*$ is the identity on the copies of $V \otimes_{\text{alg}} V^*$ imbedded in X and $V \otimes_{\text{HS}} V^*$. Let $T^* : V \otimes_{\text{HS}} V^* \longrightarrow X$ be the adjoint of T, defined by

$$\langle x, T^* y \rangle_X \;=\; \langle Tx, y \rangle_{V \otimes_{\text{HS}} V^*}$$

[6] Many of us are not accustomed to worry about *existence* of objects defined by universal mapping properties, because we proved their existence by set-theoretic *constructions* of them, long before becoming aware of mapping-property characterizations. Much as naive set theory does not lead to paradoxes without effort, naive category theory's recharacterization of objects close to prior experience rarely describes non-existent objects. Nevertheless, the present example is genuine.

On the imbedded copies of $V \otimes_{\rm alg} V^*$

$$\langle v\otimes\lambda, T^*(w\otimes\mu)\rangle_X \;=\; \langle T(v\otimes\lambda), w\otimes\mu\rangle_{V\otimes_{\rm HS}V^*} \;=\; \langle v\otimes\lambda, w\otimes\mu\rangle_{V\otimes_{\rm HS}V^*} \qquad \text{(for } v,w\in V \text{ and } \lambda,\mu\in V^*\text{)}$$

Given $v \in V$ and $\lambda \in V^*$, the orthogonal complement $(v\otimes\lambda)^\perp$ is the closure of the span of monomials $v'\otimes\lambda'$ where *either* $v' \perp v$ *or* $\lambda' \perp \lambda$. For such $v'\otimes\lambda'$,

$$0 \;=\; \langle v'\otimes\lambda', v\otimes\lambda\rangle_H \;=\; \langle T(v'\otimes\lambda'), v\otimes\lambda\rangle_H \;=\; \langle v'\otimes\lambda', T^*(v\otimes\lambda)\rangle_X$$

Thus, for any monomial $v\otimes\lambda$, the image $T^*(v\otimes\lambda)$ is a scalar multiple of $v\otimes\lambda$. The same is true of monomials $(v+w)\otimes(\lambda+\mu)$. Taking v, w linearly independent and λ, μ linearly independent and expanding shows that the scalars do not depend on v, λ. Thus, T^* is a scalar on $V\otimes_{\rm alg} V^*$.

That is, there is a (necessarily real) constant C such that

$$C\cdot\langle v\otimes\lambda, w\otimes\mu\rangle_X \;=\; \langle v\otimes\lambda, T^*(w\otimes\mu)\rangle_X \;=\; \langle T(v\otimes\lambda), w\otimes\mu\rangle_{V\otimes_{\rm HS}V^*} \;=\; \langle v\otimes\lambda, w\otimes\mu\rangle_{V\otimes_{\rm HS}V^*}$$

since T identifies the imbedded copies of $V\otimes_{\rm alg} V^*$. That is, up to the constant C, the inner products from X and $V\otimes_{\rm HS} V^*$ restrict to the same hermitian form on $V\otimes_{\rm alg} V^*$. Thus, any putative tensor product X differs from $V\otimes_{\rm HS} V^*$ only by scaling. However, we saw that the natural pairing $V\times V^* \longrightarrow \mathbb{C}$ does not factor through a continuous linear map $V\otimes_{\rm HS} V^* \longrightarrow \mathbb{C}$, because there exist Hilbert-Schmidt maps not of trace class.

Thus, there is no tensor product of infinite-dimensional Hilbert spaces. ///

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