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03a. Discrete Fubini-Tonelli

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Fubini-Tonelli, about *changing order of summation*, and/or *rearrangements* of infinite sums, can be addressed prior to measure-and-integration. The proof techniques also illustrate why Lebesgue's dominated convergence theorem and monotone convergence theorems are correct, again, prior to measure-and-integration.

Of course, an optimist would presume or hope that interchange of limits is justified under mild-and-usually-met hypotheses. And, of course, it is not surprising that things can fail when pushed beyond natural operating limitations. Here, we confirm some expected and good outcomes, under mild, reasonable hypotheses.

For example, for $\{a_{ij} : i,j = 1,2,\ldots\}$ a doubly-indexed set of *non-negative* real numbers, we will prove that the two different obvious *iterated sums* are equal:

$$\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} \;=\; \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}$$

where $+\infty$ is allowed as a possible value of inner sums and/or of the whole, in the sense that the two *implied limits* can be interchanged:

$$\lim_{M\to+\infty} \sum_{i=1}^M \Big(\lim_{N\to+\infty} \sum_{j=1}^N a_{ij}\Big) \;=\; \lim_{N\to+\infty} \sum_{j=1}^N \Big(\lim_{M\to+\infty} \sum_{i=1}^M a_{ij}\Big) \qquad\qquad \text{(allowing value } +\infty)$$

Because finite sums can be interchanged with limits, this is equivalent to

$$\lim_{M\to+\infty} \lim_{N\to+\infty} \sum_{i\le M,\, j\le N} a_{ij} \;=\; \lim_{N\to+\infty} \lim_{M\to+\infty} \sum_{i\le M,\, j\le N} a_{ij}$$

On another hand, to attribute a different sense to the double sum, for *any* nested sequence $\Phi_1 \subset \Phi_2 \subset \ldots$ of finite subsets of $S = \{(i,j) : i,j = 1,2,\ldots\}$ such that $\bigcup \Phi_n = S$, we will prove that also

$$\lim_n \sum_{(i,j)\in\Phi_n} a_{ij} \;=\; \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} \;=\; \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij} \qquad\qquad \text{(allowing value } +\infty)$$

Sums over finite sets are unambiguous, so this gives one way to characterize a doubly infinite sum, somewhat more complicated than the limit of a sequence. This also includes the assertion that arbitrary *rearrangements* of such sums give the same value.

We will also prove more-general assertions about double sums, that including both limits of finite subsums, and the iterated sums. The subsums $A_i = \sum_j a_{ij}$ and $B_j = \sum_i a_{ij}$ are *not* finite subsums, so this has non-trivial content.

For *complex* a_{ij}, under the condition $\sum_{ij} |a_{ij}| < +\infty$ as sup of finite subset sums, or under the equivalent condition $\sum_{i=1}^\infty \sum_{j=1}^\infty |a_{ij}| < \infty$, or under the equivalent condition $\sum_{i=1}^\infty \sum_{j=1}^\infty |a_{ij}| < \infty$, again

$$\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} \;=\; \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij} \;=\; \sum_{i,j} a_{ij}$$

where the last expression is a limit over finite subsums, for example.

In this context, we prove that, for a sequence a_n of complex numbers, if $\sum_n |a_n| < +\infty$, then arbitrary *rearrangements* give the same sum $\sum_n a_n$. That is, for any re-indexing a_{i_n} with bijection $n \leftrightarrow i_n$ of $\{1, 2, 3, \ldots\}$,

$$\sum_{n=1}^{\infty} a_{i_n} \;=\; \sum_{n=1}^{\infty} a_n$$

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1. *Limits of finite subsums: non-negative case*

Let S be a non-empty set of non-negative real numbers. Let Φ be the directed set [1] of *finite* subsets of S, ordered by inclusion. Let

$$\sigma(S) \;=\; \sup_{F\in\Phi} \sum_{a\in F} a \;\leq\; +\infty$$

[1.1] Theorem: For any *cofinal*[2] subset Φ' of Φ,

$$\sup_{F\in\Phi} \sum_{a\in F} a \;=\; \sup_{F\in\Phi'} \sum_{a\in F} a \;=\; \lim_{F\in\Phi'} \sum_{a\in F} a \qquad\qquad \text{(allowing value } +\infty)$$

[1.2] Remark: Existence of a limit $L < \infty$ over the directed Φ is that, for every $\varepsilon > 0$ there is $F \in \Phi$ such that, for every $F \supset F_o$, with $F \in \Phi$,

$$\Big| L - \sum_{a\in F} a \Big| \;<\; \varepsilon$$

A limit $+\infty$ requires that, for every $C > 0$, there is $F_o \in \Phi$ such that, for every $F \supset F_o$, with $F \in P$,

$$\Big| \sum_{a\in F} a \Big| \;>\; C$$

Proof: Sups of non-empty sets of real numbers always exist, when the value $+\infty$ is allowed. Likewise, for Φ' a subset of the set Φ of finite subsets of S,

$$\sup_{F\in\Phi'} \sum_{a\in F} \;\leq\; \sup_{F\in\Phi} \sum_{a\in F}$$

To show the opposite inequality when Φ' is *cofinal*, first, suppose $\sigma(S) = \sup_{F\in\Phi} a$ is finite. Let $F_o \in \Phi$ be such that $\sum_{a\in F_o} a > \sup_{F\in\Phi} a - \varepsilon$. Cofinality gives $F' \in \Phi'$ such that $F' \supset F_o$, so

$$\sum_{a\in F'} a \;\geq\; \sum_{a\in F_o} a \;>\; \sigma(S) - \varepsilon$$

Thus, $\sup_{F\in\Phi'} \sum_{a\in F} \geq \sigma(S)$, giving equality.

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[1] A directed set Φ is a partially ordered set, with inequality relation $<$ or $\leq$, such that for any two elements x, y, there is z such that $x \leq z$ and $y \leq z$.

[2] P' is *cofinal* in P when, for every $\Phi \in P$, there is $\Phi' \in P'$ such that $\Phi' \supset \Phi$.

Similarly, with $\sigma(S) = +\infty$, given $C > 0$, take $F_o \in \Phi$ such that $\sum_{a \in F_o} a > C$. Cofinality gives $F' \in \Phi'$ such that $F' \supset F_o$, so

$$\sum_{a \in F'} a \;\ge\; \sum_{a \in F_o} a \;>\; C$$

Thus, $\sup_{F \in \Phi'} \sum_{a \in F} = +\infty = \sigma(S)$, giving equality.

To show that the sup is actually the limit, in a slightly more general sense than *sequential* limits, just translate the previous discussion slightly, as follows. Let $L = \sup_{F \in \Phi'} \sum_{a \in F} a$. For finite L, for $\varepsilon > 0$, choose $F_o \in \Phi'$ so that $\sum_{a \in F_o} a > L - \varepsilon$. Since all the a_{ij} are non-negative, for every $F \in \Phi'$ with $F \supset F_o$,

$$\sum_{a \in F} a > L \;\ge\; \sum_{a \in F_o} a \;>\; L - \varepsilon$$

The analogous monotonicity also treats the case $L = +\infty$. ///

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2. *Limits of finite subsums: signed and complex case*

Let S be a set of complex numbers, such that $\sum_{a \in S} |a| < +\infty$, in the sense proven in previous sections to be well-defined. Let Φ be the directed set of finite subsets of S.

[2.1] Theorem: $\lim_{F \in \Phi} \sum_{a \in F} a$ exists.

[2.2] Corollary: For any cofinal subset Φ' of Φ, $\lim_{F \in \Phi'} \sum_{a \in F} a$ exists and is equal to $\lim_{F \in \Phi} \sum_{a \in F} a$. *(Proof below.)*

[2.3] Remark: The corollary generalizes the idea that absolutely convergent sums can be rearranged arbitrarily without changing their value.

Proof: Existence of this limit, in terms of Cauchy nets, is that, given $\varepsilon > 0$, there is $F_o \in \Phi$ such that, for all $F_1 \supset F_o$ and $F_2 \supset F_o$,

$$\Big| \sum_{a \in F_1} a - \sum_{a \in F_2} a \Big| \;<\; \varepsilon$$

Since $\lim_{F \in \Phi} \sum_{a \in F} |a|$ exists, $\sum_{a \in F} |a|$ is a Cauchy net: given $\varepsilon > 0$, there is F_o such that, for all $F_1 \supset F_o$ and $F_2 \supset F_o$,

$$\Big| \sum_{a \in F_1} |a| - \sum_{a \in F_2} |a| \Big| \;<\; \varepsilon$$

In particular,

$$\sum_{a \in F_j - F_o} |a| \;=\; \Big| \sum_{a \in F_j} |a| - \sum_{a \in F_o} |a| \Big| \;<\; \varepsilon \qquad\qquad (\text{for } j = 1, 2)$$

Thus, by the triangle inequality,

$$\Big| \sum_{a \in F_1} a - \sum_{a \in F_2} a \Big| \;=\; \Big| \sum_{a \in F_1} a - \sum_{a \in F_o} a + \sum_{a \in F_o} a - \sum_{a \in F_2} a \Big| \;\le\; \Big| \sum_{a \in F_1} a - \sum_{a \in F_o} a \Big| + \Big| \sum_{a \in F_o} a - \sum_{a \in F_2} a \Big|$$

$$= \Big| \sum_{a \in F_1 - F_o} a \Big| + \Big| \sum_{a \in F_2 - F_o} a \Big| \;\le\; \sum_{a \in F_1 - F_o} |a| + \sum_{a \in F_2 - F_o} |a| \;<\; 2\varepsilon$$

This proves $\sum_{a \in F} a$ is a Cauchy net in $\mathbb{C}$. ///

For the corollary:

Proof: Let $L = \lim_{F\in\Phi} \sum_{a\in F} a$. For a cofinal subset Φ' of Φ, given $\varepsilon > 0$, let $F_o \in \Phi$ be such that $\left|L - \sum_{a\in F} a\right| < \varepsilon$ for every $F \in \Phi$ with $F \supset F_o$. Since Φ' is cofinal in Φ, there is $F'_o \in \Phi'$ with $F'_o \supset F_o$. Then for every $F \in \Phi'$ with $F \supset F'_o$, since also $F \supset F_o$, we have the desired inequality

$$\left|L - \lim_{F\in\Phi'} \sum_{a\in F} a\right.$$

Thus, $\lim_{F\in\Phi'} \sum_{a\in F} a = L$, as claimed. ///

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3. *Discrete Fubini-Tonelli: non-negative case*

[3.1] Theorem: Let $S = \{a_{ij} : 1 \le i, j < \infty\}$ be a set of non-negative real numbers, and Φ the directed set of its finite subsets, under inclusion. Then

$$\sum_i \sum_j a_{ij} \;=\; \sum_j \sum_i a_{ij} \;=\; \lim_{F\in\Phi} \sum_{(i,j)\in F} a_{ij}$$

Proof: Let $A_i = \sum_j a_{ij} \le +\infty$. To show that $\lim_M \sum_{i\le M} A_i = \sigma(S)$, treat a few cases. First, if some A_i is $+\infty$, then $\sigma\{a_{ij} : 1 \le j < +\infty\} = +\infty$. Since $R \subset S$ implies $\sigma(R) \le \sigma(S)$, we have $\sigma(S) = +\infty$.

Next, suppose every A_i is finite, but $\sum_i A_i = +\infty$. Given $C > 0$, take M sufficient large such that $\sum_{i\le M} A_i > C$. Given $\varepsilon > 0$, for each $i \le M$, take N_i such that

$$\sum_{j\le N_i} a_{ij} \;>\; \sum_j a_{ij} - \frac{\varepsilon}{2^i}$$

With $F = \{a_{ij} : i \le M,\, j \le N_i\}$,

$$\sum_{a\in F} a \;=\; \sum_{i\le M,\, j\le N_i} a_{ij} \;>\; \sum_{i\le M} \Big(\sum_j a_{ij} - \frac{\varepsilon}{2^i}\Big) \;=\; \sum_{i\le M} \Big(A_i - \frac{\varepsilon}{2^i}\Big) \;>\; C - \varepsilon \sum_{i\le M} \frac{1}{2^i} \;>\; C - \varepsilon$$

Thus, $\sigma(\{a_{ij}\}) = +\infty$.

Very similarly, with $\sum_i A_i = L < \infty$, take $\varepsilon > 0$, and M sufficient large such that $\sum_{i\le M} A_i > L - \varepsilon$. For each $i \le M$, take N_i such that

$$\sum_{j\le N_i} a_{ij} \;>\; \sum_j a_{ij} - \frac{\varepsilon}{2^i}$$

With $F = \{a_{ij} : i \le M,\, j \le N_i\}$,

$$\sum_{a\in F} a \;=\; \sum_{i\le M,\, j\le N_i} a_{ij} \;>\; \sum_{i\le M} \Big(\sum_j a_{ij} - \frac{\varepsilon}{2^i}\Big) \;=\; \sum_{i\le M} \Big(A_i - \frac{\varepsilon}{2^i}\Big) \;>\; L - \varepsilon \sum_{i\le M} \frac{1}{2^i} \;>\; L - \varepsilon$$

Thus, $\sigma(\{a_{ij}\}) \ge L$.

In the opposite direction, first suppose that $\sigma\{a_{ij}\} = +\infty$. Given $C > 0$, take M, N large enough so that $\sum_{i\le M,\, j\le N} a_{ij} > C$. Then

$$\sum_{i\le M} A_i \;\ge\; \sum_{i\le M} \sum_{j\le N} a_{ij} \;>\; C$$

So $\sum_i A_i = +\infty$. Similarly, with $\sigma\{a_{ij}\} = L < +\infty$, given $\varepsilon > 0$, take M, N large enough so that $\sum_{i\le M,\, j\le N} a_{ij} > L - \varepsilon$. Then

$$\sum_{i\le M} A_i \;\ge\; \sum_{i\le M}\sum_{j\le N} a_{ij} \;>\; L-\varepsilon$$

so $\sum_i A_i \ge L$. ///

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4. *Discrete Fubini-Tonelli: signed and complex case*

Let $S = \{a_{ij} : 1 \le i, j < \infty\}$ be a set of complex numbers, and Φ the directed set of its finite subsets.

[4.1] Claim: If $\sum_{ij} |a_{ij}| < +\infty$, then

$$\lim_{M\to\infty}\lim_{N\to\infty} \sum_{i\le M,\, j\le N} a_{ij} \;=\; \lim_{N\to\infty}\lim_{M\to\infty} \sum_{i\le M,\, j\le N} a_{ij} \;=\; \lim_{F\in\Phi} \sum_{(i,j)\in F} a_{ij}$$

Proof: As in the non-negative real situation, the hypotheses assure that the infinite subsums $A_i = \sum_j a_{ij}$ and $B_j = \sum_i a_{ij}$ are approximated by finite subsums, and (absolutely) convergent. Further, both iterated sums are equal to the limit sum over $F \in \Phi$.

First, to see that the infinite subsums are approximated by finite subsums, given $\varepsilon > 0$, convergence of $\sum_j |a_{ij}|$ assures that, for every $\varepsilon > 0$, there is j_o such that, for all $j_1, j_2 \ge j_o$,

$$\Big|\sum_{j\le j_1} |a_{ij}| \;-\; \sum_{j\le j_2} |a_{ij}|\Big| \;<\; \varepsilon$$

In particular, this applies to j_o and any $j_1 \ge j_o$:

$$\Big|\sum_{j_o<j\le j_1} |a_{ij}|\Big| \;=\; \Big|\sum_{j\le j_1} |a_{ij}| \;-\; \sum_{j\le j_o} |a_{ij}|\Big| \;<\; \varepsilon$$

Thus, for such $\varepsilon, j_o, j_1, j_2$,

$$\Big|\sum_{j\le j_1} a_{ij} \;-\; \sum_{j\le j_2} a_{ij}\Big| \;\le\; \Big|\sum_{j\le j_1} a_{ij} \;-\; \sum_{j\le j_o} a_{ij}\Big| + \Big|\sum_{j\le j_2} a_{ij} \;-\; \sum_{j\le j_o} a_{ij}\Big|$$

$$= \Big|\sum_{j_o<j\le j_1} a_{ij}\Big| + \Big|\sum_{j_o<j\le j_2} a_{ij}\Big| \;\le\; \sum_{j_o<j\le j_1} |a_{ij}| + \sum_{j_o<j\le j_2} |a_{ij}| \;<\; 2\varepsilon$$

Thus, $\sum_{j\le M} a_{ij}$ is Cauchy, therefore convergent. Thus, $A_i = \sum_j a_{ij}$ is (absolutely) convergent, and nicely approximable by its finite subsums.

Next, we show that the sequence $\sum_{i\le M} A_i$ converges to $\lim_{F\in\Phi} \sum_{a\in F} a$. Given $\varepsilon > 0$, let M_o, N_o be large enough so that $\sum_{i\le M,\, j\le N} a_{ij}$ is within ε of the limit L of the finite subsums for every $M \ge M_o$ and $N \ge N_o$. For each $i \le M_o$, let j_i be large enough so that $\sum_{j\le j_i} a_{ij}$ is within $\varepsilon/2^i$ of A_i. Let $N'_o = \max_{i\le M_o} j_i$. Then

$$\Big|L - \sum_{i\le M_o} A_i\Big| \;<\; \varepsilon + \Big|\sum_{i\le M_o,\, j\le N'_o} a_{ij} - \sum_{i\le M_o} A_i\Big| \;\le\; \varepsilon + \sum_{i\le M_o} \Big|\sum_{j\le N'_o} a_{ij} - A_i\Big| \;<\; \varepsilon + \sum_{i\le M_o} \frac{\varepsilon}{2^i} \;<\; 2\varepsilon$$

Thus, $\sum_{i\le M} A_i$ converges to L. ///
