Self-adjoint and everywhere-defined implies bounded

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This is a consequence of the uniform boundedness theorem, also called the Banach-Steinhaus theorem. In the simple case we use, the theorem is that, given a collection $\Lambda$ of linear maps $\lambda : V \to W$ of Hilbert spaces, if the image $\lambda(B) \subset W$ of the unit ball $B \subset V$ is bounded (with bound possibly depending on $\lambda$) for each individual $\lambda \in \Lambda$, then either there is a uniform bound for all $\lambda \in \Lambda$, or there is an individual $x$ such that $\sup_{\lambda} |\lambda(x)| = +\infty$.

We will apply this to a collection $\Lambda$ of functionals $\lambda : V \to \mathbb{C}$.

[0.1] Claim: Let $T : V \to V$ be a not-necessarily continuous/bounded linear map of a Hilbert space $V$ to itself, self-adjoint in the sense that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in V$. Then, in fact, $T$ is necessarily bounded, that is, is necessarily continuous.

[0.2] Remark: In other words, we cannot have unbounded, self-adjoint, everywhere-defined operators. The Axiom of Choice certainly would allow us to make everywhere-defined unbounded/not-continuous operators, but the result here shows that it is impossible to make such operators self-adjoint, no matter what we do.

Proof: Following Riesz-Fréchet, for $y \in V$, we have the usual linear functional $\lambda_y(x) = \langle x, y \rangle$. As usual, by Cauchy-Schwarz-Bunyakowsky, this functional is bounded (that is, continuous) (with norm at most $|y|$, and, in fact, exactly $|y|$).

Looking at the family of functionals $\Lambda = \{\lambda_{Ty} : |y| \leq 1\}$, using the self-adjointness of $T$, by Cauchy-Schwarz-Bunyakowsky,

$$\sup_{|y| \leq 1} |\lambda_{Ty}(x)| = \sup_{|y| \leq 1} |\langle x, Ty \rangle| = \sup_{|y| \leq 1} |\langle Tx, y \rangle| \leq \sup_{|y| \leq 1} \left( |Tx| \cdot |y| \right) = |Tx| < +\infty$$

(for each $x \in V$)

That is, $\sup_{\lambda \in \Lambda} |\lambda(x)| < \infty$ for every $x$. By uniform boundedness, there is a uniform bound $C$ such that $|\lambda_{Ty}(x)| \leq C$ for every $|y| \leq 1$. That is, $|\langle x, Ty \rangle| \leq C$ for every $|y| \leq 1$. In particular, $|\langle x, Tx \rangle| \leq C$ for every $|x| \leq 1$, and $T$ is bounded, hence, continuous.  

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