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Representation theory of finite abelian groups

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1. *Simultaneous eigenvectors for finite abelian groups*

For a *single* linear operator T on a complex vector space V, and for a complex number λ, the λ-*eigenspace* V_λ of T on V is

$$V_\lambda \;=\; \{v \in V \;:\; Tv = \lambda \cdot v\}$$

[1.1] Unitarizability, diagonalizability of finite-order operators Various natural hypotheses assure that V is a direct sum of T-eigenspaces, that is, that T is *diagonalizable*, as opposed to having any Jordan blocks. Here, a natural hypothesis is that T is of *finite order*, that is, $T^n = 1$ for some $1 \le n \in \mathbb{Z}$. With this hypothesis, we can construct an complex-hermitian *inner product* $\langle , \rangle$ on V such that T is *unitary* in the sense that

$$\langle Tv, Tw\rangle \;=\; \langle v, w\rangle \qquad\qquad (\text{for all } v, w \in V)$$

Take an *arbitrary* hermitian inner-product $\langle , \rangle_o$, and create a T-invariant one by *averaging*:

$$\langle v, w\rangle \;=\; \sum_{\ell=1}^{n} \langle T^\ell v,\, T^\ell w\rangle_o$$

The averaging process does not disturb positive-definiteness or hermitian-ness. The T-invariance, that is, unitariness of T, is easy, using $T^n = 1$,

$$\langle Tv, Tw\rangle \;=\; \sum_{\ell=1}^{n} \langle T^\ell Tv,\, T^\ell Tw\rangle_o \;=\; \sum_{\ell=2}^{n+1} \langle T^\ell v,\, T^\ell w\rangle_o \;=\; \sum_{\ell=2}^{n} \langle T^\ell v,\, T^\ell w\rangle_o \;+\; \langle T^{n+1} v,\, T^{n+1} w\rangle_o$$

$$=\; \sum_{\ell=2}^{n} \langle T^\ell v,\, T^\ell w\rangle_o \;+\; \langle Tv,\, Tw\rangle_o \;=\; \sum_{\ell=1}^{n} \langle T^\ell v,\, T^\ell w\rangle_o \;=\; \langle v, w\rangle$$

Then the *spectral theorem for unitary operators* implies that T is diagonalizable, that is, that V is an *orthogonal* direct sum of T-eigenspaces:

$$V \;=\; \bigoplus_\lambda V_\lambda \qquad\qquad (\text{with an } \textit{orthogonal} \text{ direct sum})$$

[1.2] Commuting operators Another linear operator S *commuting* with T *stabilizes* each T-eigenspace V_λ: for $v \in V_\lambda$:

$$T(Sv) \;=\; (TS)v \;=\; (ST)v \;=\; S(Tv) \;=\; S(\lambda v) \;=\; \lambda \cdot Sv$$

since the linearity of S is that S commutes with scalar multiplication.

[1.3] Finite abelian groups of operators We want to prove that a finite abelian group G of operators on a finite-dimensional complex vectorspace V is *simultaneously diagonalizable.* That is, we claim that V is a direct sum of *simultaneous eigenspaces* for all operators in G.

In this situation, the notion of *eigenvalue* must be a little more complicated than individual numbers: for *each* $g \in G$, there must be a number $\lambda_g \in \mathbb{C}$. That is, an *eigenvalue* is really a *map* $g \to \lambda_g$ from G to $\mathbb{C}$. In this context, for two eigenvalues λ, μ to be *distinct* means that $\lambda_g \neq \mu_g$ for *some* $g \in G$ (not necessarily for *all* $g \in G$).

Further, if there is a non-zero eigenvector v for a given collection of eigenvalues $g \to \lambda_g$, then $g \to \lambda_g$ is a *group homomorphism* from G to $\mathbb{C}^\times$:

$$\lambda_{gh} \cdot v \;=\; (gh) \cdot v \;=\; g(hv) \;=\; g(\lambda_h \cdot v) \;=\; \lambda_h \cdot gv \;=\; \lambda_h \lambda_g \cdot v \;=\; \lambda_g \lambda_h \cdot v$$

That is, $g \to \lambda_g$ is a *character* of G. Let $\widehat{G}$ denote the collection of all characters of G:

$$\widehat{G} \;=\; \{\text{group homomorphisms } \chi : G \to \mathbb{C}^\times\}$$

and for a character χ let the χ-*eigenspace* in V be

$$V_\chi \;=\; \{v \in V \;:\; gv = \chi(g) \cdot v, \text{ for all } g \in G\}$$

Thus, our claim is that for a finite abelian group G of linear operators on finite-dimensional complex vector space V,

$$V \;=\; \bigoplus_{\chi \in \widehat{G}} V_\chi$$

As with a single operator generating a finite group of operators, we can make a hermitian inner product on V by *averaging* over G a given, arbitrary hermitian inner product $\langle , \rangle_o$: put

$$\langle v, w \rangle \;=\; \sum_{g \in G} \langle gv, gw \rangle \qquad\qquad (\text{for } v, w \in V)$$

The G-invariance follows by changing variables in the summation: for $h \in G$,

$$\langle hv, hw \rangle \;=\; \sum_{g \in G} \langle ghv, ghw \rangle \;=\; \sum_{g \in G} \langle gv, gw \rangle \qquad\qquad (\text{replacing } g \text{ by } gh^{-1})$$

Thus, all the operators in G are *unitary* with respect to $\langle , \rangle$.

The proof: suppose we have the direct sum decomposition for vector spaces of dimension $< n$. Let V be of dimension n. First, a silly case: if all operators $g \in G$ are *scalar*, then *every* vector is a simultaneous eigenvector for all the operators in G, and we are done. So now consider the (serious) case that *not* all operators in G are scalar. Let $g \in G$ be a non-scalar operator. By the spectral theorem for unitary operators, V has an orthogonal decomposition into eigenspaces for g, implicitly with different eigenvalues. Since g is non-scalar, every one of these eigenspaces has dimension $< n$. By induction, and by the fact that the operators all commute, each such eigenspace decomposes as an orthogonal direct sum of *simultaneous* eigenspaces for G. Thus, the whole space V is an orthogonal direct sum of simultaneous eigenspaces. This completes the induction.

2. *Cancellation lemma, orthogonality of distinct characters*

Let G be a finite group, not necessarily abelian. The *cancellation lemma* is

[2.0.1] Lemma: For a non-trivial group homomorphism $\sigma : G \to \mathbb{C}^\times$,

$$\sum_{g \in G} \sigma(g) \;=\; 0$$

Proof: Since σ is not identically 1, there is $g_o \in G$ such that $\sigma(g_o) \neq 1$. Then

$$\sum_{g \in G} \sigma(g) \;=\; \sum_{g \in G} \sigma(g_o g) \;=\; \sum_{g \in G} \sigma(g_o)\,\sigma(g) \;=\; \sigma(g_o) \sum_{g \in G} \sigma(g)$$

by replacing g by $g_o g$ in the sum, using the fact that left multiplication by g_o is a bijection of G to itself. Subtracting,

$$(1 - \sigma(g_o)) \cdot \sum_{g \in G} \sigma(g) \;=\; 0$$

Since $\sigma(g_o) \neq 1$, necessarily the sum is 0. ///

[2.0.2] Corollary: Let $\sigma \neq \tau$ be group homomorphisms $G \to \mathbb{C}^\times$. Then

$$\sum_{g \in G} \sigma(g)\,\overline{\tau}(g) \;=\; 0$$

Proof: Since G is finite, there is N such that $g^N = e$ for every $g \in G$. Thus,

$$\tau(g)^N \;=\; \tau(g^N) \;=\; \tau(e) \;=\; 1$$

Thus, $\tau(g)$ is a root of unity, and $|\tau(g)| = 1$. In particular, $\overline{\tau}(g) = \tau(g)^{-1}$. Then $\sigma\overline{\tau} = \sigma\tau^{-1}$ is a character of G, and is not the trivial character. The cancellation lemma gives the vanishing. ///

[2.0.3] Remark: Despite the simplicity of the arguments above, the cancellation and orthogonality devices are remarkably useful in applications.

3. *Representations of finite abelian groups*

It is useful to consider a slight shift of viewpoint. Instead of having a finite abelian group G of linear automorphisms of a fixed complex vector space V, we might fix a finite abelian group G, and consider various group homomorphisms $\rho : G \to \mathrm{Aut}_{\mathbb{C}} V$ for various complex vectorspaces V. The pair ρ, V is a *representation* of G.

Although it would be most strictly correct to write the action of $g \in G$ on $v \in V$ via ρ as $\rho(g)(v)$, context should be sufficient to allow writing simply $g \cdot v$ or gv.

Similarly, it is common to write V_ρ for V to emphasize the role played by ρ, or to use other notational devices, but, in fact, context should make dependencies clear.

The collection $\widehat{G}$ of characters $G \to \mathbb{C}^\times$ makes sense for arbitrary groups G, with or without realizing them as subgroups of linear automorphisms of vector spaces.

For finite abelian G, the image $\rho(G)$ of G in the group of automorphisms of V is still finite abelian, and, as a corollary of the result for finite abelian subgroups of $\mathrm{Aut}_{\mathbb{C}} V$,

$$V = \bigoplus_{\chi \in \widehat{G}} V_\chi$$

where

$$V_\chi = \{v \in V : g \cdot v = \chi(g) \cdot v, \text{ for all } g \in G\}$$

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4. *Fourier expansions on finite abelian groups*

Let G be a finite abelian group, and $L^2(G)$ the complex vectorspace of complex-valued functions [1] on G, with *inner product*[2]

$$\langle f, \varphi \rangle = \sum_{x \in G} f(x)\,\overline{\varphi}(x)$$

Again, a *character* χ on a group is a group homomorphism [3]

$$\chi : G \longrightarrow \mathbb{C}^\times \qquad \text{(group homomorphism)}$$

and $\widehat{G}$ is the collection of characters $\chi : G \to \mathbb{C}^\times$. For f a complex-valued function on G, the *Fourier transform* $\widehat{f}$ of f is the function on $\widehat{G}$ defined by

$$\widehat{f}(\chi) = \langle f, \chi \rangle \qquad (\text{for } \chi \in \widehat{G})$$

The *Fourier expansion* or *Fourier series* of f is

$$f \sim \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\, \chi$$

[4.0.1] Theorem: On a finite abelian group, the Fourier expansion of a complex-valued function f *represents* f, in the sense that, for every $g \in G$,

$$f(g) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\, \chi(g)$$

The elements of $\widehat{G}$ form an orthogonal basis for $L^2(G)$. In particular, the Fourier coefficients are *unique.*

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[1] In general, for a space X with some sort of *integral* on it, the notation $L^2(X)$ means functions f so that $\int_X |f|^2 < \infty$. On *finite* sets integrals become sums, possibly weighted, and this finiteness condition becomes *vacuous.* Nevertheless, it is good to use this notation as a reminder of the larger context.

[2] The notation $L^2(G)$ is meant to suggest the presence of the *inner product* on this space of functions. On a general space X with an integral, the iner product is $\langle f_1, f_2 \rangle = \int_X f_1 \overline{f}_2$.

[3] The term *character* has different meanings in different contexts. The simplest sense is a group homomorphism to $\mathbb{C}^\times$. However, an equally important use is for the *trace* of a group homomorphism $\rho : G \to GL_n(k)$ from G to invertible n-by-n matrices with entries in a field k. In the latter sense,

$$(\text{character of } \rho)(g) = \text{trace}\Big(\rho(g)\Big)$$

For *infinite*-dimensional representations, further complications appear. Context is always necessary to know which sense is intended.

[4.0.2] Remark: What are we *not* doing? The theorem asserts nothing directly about the collection $\widehat{G}$ of characters of G. Its proof uses no information about these characters. Its proof uses nothing about the structure theorem for finite abelian groups. All that is used is a *spectral theorem*.

The proof is in the following paragraphs.

[4.1] Translation action on functions

The distinguishing feature of functions on a *group* is that the group acts on itself by right or left multiplication (or whatever the group operation is called), thereby moving around the *functions on it*.

The group operation in G will be written *multiplicatively*, not *additively*, to fit better with other notational conventions.

The group G *acts* on the vector space $L^2(G)$ of functions on itself by *translation*: for $g \in G$, the *translate* $T_g f$ of a function f by g is the function on G defined by [4]

$$(T_g f)(x) \;=\; f(xg) \qquad\qquad \text{(for function } f\text{, and } x, g \in G)$$

The maps-on-function T_g are vectorspace endomorphisms of the vectorspace of functions on G:

$$\begin{cases} T_g(f_1+f_2)(x) \;=\; (f_1+f_2)(xg) \;=\; f_1(xg)+f_2(xg) \;=\; T_g f_1(x) + T_g f_2(x) & \text{(additivity)} \\ T_g(c\cdot f)(x) \;=\; (c\cdot f)(gx) \;=\; c\big(f(gx)\big) \;=\; \big(c\cdot (T_g f)\big)(x) & \text{(scalar } c) \end{cases}$$

To reduce clutter, the action of $g \in G$ on functions f may be written simply gf or $g \cdot f$. The *associativity* property

$$(gh)f \;=\; g(hf) \qquad\qquad \text{(for } g, h \in G\text{, function } f)$$

comes from the associativity of the group operation itself:

$$\big((gh)f\big)(x) \;=\; f(x(gh)) \;=\; f((xg)h) \;=\; (hf)(xg) \;=\; \big(g(hf)\big)(x)$$

The associativity property is equivalent to the assertion that the map $g \to T_g$ is a *group homomorphism* from G to $\mathbb{C}$-linear automorphisms of $L^2(G)$ (and that the identity element of g acts trivially).

Since $g \to T_g$ is a group homomorphism, the abelian-ness of G implies that the linear maps T_g, T_h commute: since $gh = hg$,

$$T_g \circ T_h \;=\; T_{gh} \;=\; T_{hg} \;=\; T_h \circ T_g \qquad\qquad \text{(for all } g, h \in G)$$

Since G is finite, there is a positive integer N such that, for all $g \in G$, $g^N = e \in G$. Thus,

$$\chi(g)^N \;=\; \chi(g^N) \;=\; \chi(e) \;=\; 1 \qquad\qquad \text{(for any } \chi \in \widehat{G})$$

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[4] For *non-abelian* groups G, there are two translation actions, namely, *left* and *right*

$$\begin{cases} T_g^{\text{right}} f(x) \;=\; f(xg) \\ T_g^{\text{left}} f(x) \;=\; f(g^{-1}x) \end{cases}$$

The inverse in the left translation is for *associativity*

$$T_{gh}^{\text{left}} f \;=\; T_g\Big(T_h f\Big)$$

For abelian groups, the two translation actions become essentially the same thing, insofar as either alone gives all the information that the two together could give. Also, for abelian groups, the inverse in the definition of the left translation action loses some of its significance, since for *abelian* groups $g \to g^{-1}$ is a group automorphism.

That is, the values of χ lie on the unit circle in $\mathbb{C}^\times$, so $|\chi(g)| = 1$. In particular, χ is *unitary* in the sense that

$$\chi(g)^{-1} \;=\; \overline{\chi(g)}$$

We claim that the linear operators T_g are also *unitary*, in the sense that

$$\langle T_g f, T_g F\rangle \;=\; \langle f, F\rangle \qquad\qquad \text{(for } g \in G\text{, functions } f, F\text{)}$$

To prove this, compute directly:

$$\langle T_g f, T_g F\rangle \;=\; \sum_{h\in G} (T_g f)(h)\,\overline{(T_g F)(h)} \;=\; \sum_{h\in G} f(hg)\,\overline{F(hg)}$$

Change variables in the sum, by replacing h by hg^{-1}. Here the fact that G is a *group* is used: g^{-1} exists, and is closed under the group law:

$$\sum_{h\in G} f(hg)\,\overline{F(hg)} \;=\; \sum_{h\in G} f(h)\,\overline{F(h)} \;=\; \langle f, F\rangle$$

proving the unitarity.

From above, in the case that $V = L^2(G)$,

$$L^2(G) \;=\; \bigoplus_{\chi\in\widehat{G}} L^2(G)_\chi$$

We will show that each $L^2(G)_\chi$ is one-dimensional, spanned by χ itself.

On one hand, every $\chi \in \widehat{G}$ is a complex-valued function on G, so is in $L^2(G)$. Indeed, $\chi \in L^2(G)_\chi$:

$$(T_g\chi)(h) \;=\; \chi(hg) \;=\; \chi(h)\,\chi(g) \;=\; \chi(g)\,\chi(h) \qquad\qquad \text{(since } \mathbb{C}^\times \text{ is abelian)}$$

On the other hand, $L^2(G)_\chi$ is *exactly* scalar multiples $\mathbb{C}\cdot\chi$ of χ: for $f \in L^2(G)_\chi$,

$$f(g) \;=\; f(e\cdot g) \;=\; (T_g f)(e) \;=\; \chi(g)\cdot f(e) \;=\; f(e)\cdot\chi(g) \qquad\qquad \text{(identity } e \in G\text{)}$$

That is,

$$f \;=\; f(e)\cdot\chi \qquad\qquad \text{(for } f \in L^2(G)_\chi\text{)}$$

By the orthogonality of V_χ and V_τ for distinct χ, τ, the characters are an *orthogonal basis* for $L^2(G)$. Their lengths are readily determined, using the earlier-noted *unitariness* $\overline{\chi} = \chi^{-1}$:

$$\langle\chi,\chi\rangle \;=\; \sum_{g\in G} \chi(g)\cdot\overline{\chi}(g) \;=\; \sum_{g\in G} \chi(g)\cdot\chi(g)^{-1} \;=\; \sum_{g\in G} 1 \;=\; |G|$$

Any $f \in L^2(G)$ is a linear combination of orthogogonal basis elements e_i

$$f \;=\; \sum_i \frac{\langle f, e_i\rangle\cdot e_i}{\langle e_i, e_i\rangle}$$

Using the orthogonal basis $\chi \in \widehat{G}$,

$$f \;=\; \sum_{\chi\in\widehat{G}} \frac{\langle f,\chi\rangle\cdot\chi}{\langle\chi,\chi\rangle} \;=\; \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \langle f,\chi\rangle\cdot\chi$$

This is an equality of functions on the finite set G, and $\widehat{f}(\chi)$ is defined to be $\langle f, \chi\rangle$, so

$$f(g) \;=\; \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \langle f,\chi\rangle\cdot\chi(g) \;=\; \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \widehat{f}(\chi)\cdot\chi(g) \qquad \text{(for all } g\in G)$$

This proves the representability of functions on finite abelian groups by their Fourier series. ///

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5. *Appendix: spectral theorem for unitary operators*

Let V be a finite-dimensional complex vector space with a hermitian inner product $\langle,\rangle$. A linear map $T: V \to V$ is *unitary* if it preserves the inner product, in the sense that

$$\langle Tv, Tw\rangle \;=\; \langle v, w\rangle \qquad \text{(for all } v, w\in V)$$

Thus, the *adjoint* T^* of a unitary operator T has the property

$$\langle v, w\rangle \;=\; \langle Tv, Tw\rangle \;=\; \langle v, T^*Tw\rangle$$

Subtracting, $\langle v, T^*Tw - w\rangle = 0$ for all v, so $T^*Tw = w$ for all $w \in V$. That is, unitary T is *invertible*, and $T^* = T^{-1}$. This also shows that $T^*T = TT^*$.

The *inverse* of a unitary operator is unitary, since

$$\langle T^{-1}v, T^{-1}w\rangle \;=\; \langle T^*v, T^{-1}w\rangle \;=\; \langle v, TT^{-1}w\rangle \;=\; \langle v, w\rangle$$

Eigenvalues λ of a unitary operator T are of absolute value 1, since for a λ-eigenvector v

$$\lambda\overline{\lambda}\langle v, v\rangle \;=\; \langle \lambda v, \lambda v\rangle \;=\; \langle Tv, Tv\rangle \;=\; \langle v, v\rangle$$

In particular, eigenvalues λ are non-zero, and $\lambda^{-1} = \overline{\lambda}$.

Given $\lambda \in \mathbb{C}$, let

$$\lambda\text{-eigenspace of } T \;=\; V_\lambda \;=\; \{v\in V \;:\; Tv = \lambda\cdot v\}$$

[5.0.1] Theorem: The vectorspace is an *orthogonal* direct sum

$$V \;=\; \bigoplus_\lambda V_\lambda \qquad \text{(eigenspaces of unitary } T)$$

Proof: We grant ourselves the more elementary fact that, because V is finite-dimensional and $\mathbb{C}$ is algebraically closed, there is at least *one* one eigenvalue λ and non-zero eigenvector v for T. Thus, the λ-eigenspace V_λ is not $\{0\}$.

Now the unitariness is used, to set up an induction on dimension. We claim that T stabilizes the orthogonal complement

$$V_\lambda^\perp \;=\; \{w\in V \;:\; \langle w, v\rangle = 0 \text{ for all } v\in V_\lambda\}$$

Indeed, for w in that orthogonal complement and $v \in V_\lambda$,

$$\langle Tw, v\rangle \;=\; \langle w, T^*v\rangle \;=\; \langle w, T^{-1}v\rangle \;=\; \langle w, \lambda^{-1}v\rangle \;=\; \lambda\langle w, v\rangle \;=\; 0 \qquad \text{(for all } v\in V_\lambda)$$

The restriction of a unitary operator T to a T-stable subspace is obviously still unitary. By induction on the dimension of the vectorspace, $V_\lambda^\perp$ is an orthogonal direct sum of T-eigenspaces: $V_\lambda^\perp = \bigoplus_\mu V'_\mu$. Then

$$V \;=\; V_\lambda \oplus \bigoplus_\mu V'_\mu$$

is the orthogonal direct sum decomposition of the whole space. ///
