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Unitary representations of topological groups

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We develop basic properties of unitary Hilbert space representations of topological groups. The groups G are *locally-compact*, *Hausdorff*, and *countably based*. This is a slight revision of my handwritten 1992 notes, which were greatly influenced by [Robert 1983]. Nothing here is new, apart from details of presentation. See the brief notes on chronology at the end.

One purpose is to isolate techniques and results in representation theory which do not depend upon additional structure of the groups. Much can be done in the representation theory of *compact* groups without anything more than the compactness. Similarly, the discrete decomposition of $L^2(\Gamma\backslash G)$ for compact quotients $\Gamma\backslash G$ depends upon nothing more than compactness. Schur orthogonality and inner product relations for *discrete series* representations inside *regular* representations can be discussed without further hypotheses.

The purely topological treatment of compact groups shows a degree of commonality between subsequent treatments of Lie groups and of p-adic groups, whose rich details might otherwise obscure the simplicity of *some* of their properties.

We briefly review Haar measure, and prove basic things about invariant measures on quotients $H\backslash G$, where this notation refers to a quotient on the left, consisting of cosets Hg.

We briefly consider Gelfand-Pettis integrals for continuous compactly-supported vector valued functions with values in Hilbert spaces.

We emphasize *discretely occurring* representations, neglecting (continuous) *Hilbert integrals* of representations. Treatment of these *discrete series* representations suffices for compact groups.

Though it entails some complications, we pay attention to closed central subgroups Z of groups G, and distinguish spaces of functions on G by their behavior under Z. This has a cost in proofs and in notation. However, these minor complications are genuine, already arising with $GL(2,\mathbb{R})$ and $GL(2,\mathbb{Q}_p)$, which have discrete series representations modulo their centers, but not otherwise.

1. *Definitions: unitary representations, irreducibility*

Always G is a group with a *locally compact Hausdorff* topology, with continuous multiplication and inversion. Further, G has a *countable basis*.

Let V be a complex Hilbert space with inner product $\langle , \rangle$ and associated norm $| \, |$. A continuous $\mathbb{C}$-linear map $T : V \to V$ is *unitary* when it is invertible and

$$\langle Tv, Tw \rangle \;=\; \langle v, w \rangle \qquad \text{(for all } v, w \in V\text{)}$$

For finite-dimensional V existence of the inverse *follows* from the preservation of the inner product. Also, in general, it in fact suffices to make the simpler demand that

$$\langle Tv, Tv \rangle \;=\; \langle v, v \rangle \qquad \text{(for all } v \in V\text{)}$$

since the more general condition can easily be shown to follow from the simpler one, by considering $v \pm w$ and $v \pm iw$.

It is immediate that a product of two unitary operators is unitary, as is the inverse of a unitary operator, so the collection of all unitary operators on V forms a *group*.

A *unitary representation* of G on V is a group homomorphism

$$\pi : G \to \{ \text{ unitary operators on } V\}$$

with the *continuity property*

$$g \to \pi(g)v \qquad \text{is continuous}$$

for every $v \in V$. Sometimes the Hilbert space V is called the *representation space* of π.

[1.0.1] Remark: We cannot and should not attempt to require that $g \to \pi(g)$ be continuous with the *uniform operator topology*

$$|T|_{\text{uniform}} = \sup_{|v|\le 1} |Tv|$$

on operators on V. Simple natural examples (given later) fail to have this (excessive) continuity property.

Nevertheless, the function

$$G \times V \to V \qquad \text{by} \qquad g \times v \to \pi(g)v$$

is continuous:

$$|\pi(g')v' - \pi(g)v| \le |\pi(g')v' - \pi(g')v| + |\pi(g')v - \pi(g)v| \;=\; |v' - v| + |(\pi(g') - \pi(g))(v)|$$

by unitariness of $\pi(g')$. The term $|v' - v|$ certainly can be made small, and the term $|(\pi(g') - \pi(g))(v)|$ is small for g' close to g, by the continuity hypothesis above.

So a representation consists of *both* the Hilbert space V and the group homomorphism π. Thus, in a formal context, a representation is a *pair* (V, π) or (π, V). Nevertheless, for brevity, very often the representation (V, π) will be referred-to simply as 'π' or as 'V'. For a representation referred to simply as 'π', a default notation for the representation space corresponding to it is 'V_π'.

A *G-morphism* or *G-homomorphism* or *G-map* or *G-intertwining operator*

$$\varphi : V \to V'$$

from one G-representation (V, π) to another (V', π') is a continuous linear map $\varphi : V \to V'$ which *commutes with* the action of G in the sense that

$$\varphi \circ \pi(g) \;=\; \pi'(g) \circ \varphi \qquad \text{(for all } g \in G\text{)}$$

We are *not* directly requiring that φ respect the inner products. An intertwining operator is an *isomorphism* if it has a two-sided inverse. The collection of all G-intertwining operators from (V, π) to (V', π') will be denoted by $\mathrm{Hom}_G(\pi, \pi')$ or $\mathrm{Hom}_G(V, V')$ (using the notational abuse in which 'π' refers to the representation more properly denoted by '(π, V)', and so on).

Let (V, π) be a unitary representation of G. A *G-stable subspace* V' of V is a complex subspace V' of V so that, for all $g \in G$ and $v' \in V'$, $\pi(g)v' \in V'$. We will be most interested in *(topologically) closed* G-stable subspaces, but closed-ness is *not* part of the definition.

A *subrepresentation* (V', π') of a representation (V, π) is a (topologically) *closed G-stable subspace* V' of V, and $\pi'(g)$ is just the restriction of $\pi(g)$ to the subspace V'. Since closed subspaces of Hilbert spaces are again Hilbert spaces, and restrictions of unitary operators are again unitary, we see that subrepresentations of unitary representations are again unitary representations.

The *direct sum* representation $(\pi, V) \oplus (\pi', V')$ (or simply $\pi \oplus \pi'$) has representation space the direct sum $V \oplus V'$ of the two Hilbert spaces, with the obvious

$$(\pi \oplus \pi')(g)(v \oplus v') \;=\; \pi(g)v \oplus \pi'(g)v'$$

Note that the direct sum of the two Hilbert spaces has an inner product

$$\langle v_1 \oplus v_1', v_2 \oplus v_2' \rangle \;=\; \langle v_1, v_2 \rangle + \langle v_1', v_2' \rangle$$

It is not hard to check *completeness*, so we really do have a Hilbert space.

A unitary representation (π, V) of G is *irreducible* if there is no (topologically) *closed* G-stable subspace of V other than $\{0\}$ and V itself.

Sometimes for emphasis we would say *topologically irreducible* to emphasize that only *closed* subspaces are considered, and use the phrase *algebraically irreducible* if it is intended *not* to require closedness of the G-stable subspaces. Certainly algebraic irreducibility is a stronger condition in general than topological irreducibility. In finite-dimensional spaces, since every subspace is closed, the distinction is meaningless, but in general in infinite-dimensional spaces there can be proper G-stable subspaces which are not closed. In certain more sophisticated contexts, for particularly nice groups G, it is sometimes possible to prove that topological irreducibility *implies* algebraic irreducibility, but such assertions are far from trivial.

2. *The dual (contragredient) representation*

The *dual space* V^* to a complex vectorspace V is the (complex) vectorspace of continuous linear (complex-valued) functionals on V. Certainly V^* is a complex vectorspace, by

$$(a \cdot \lambda)(v) \;=\; a(\lambda(v)) \qquad \text{(for } \lambda \in V^*,\ a \in \mathbb{C}, \text{ and } v \in V\text{)}$$

On the other hand, when V is a Hilbert space, by the Riesz-Fischer theorem, every $\lambda \in V^*$ is of the form

$$\lambda(v) \;=\; \langle v, v_\lambda \rangle \qquad \text{(for a uniquely-determined } v_\lambda \in V\text{)}$$

A potential confusion arises in the fact that (as can be checked easily)

$$v_{a\lambda} \;=\; \bar{a} \cdot v_\lambda$$

since $\langle,\rangle$ is *conjugate*-linear in its second argument. Thus, it is *not* quite the case that $\lambda \to v_\lambda$ gives a complex-linear isomorphism of V^* to V. Rather, we define the *conjugate* vectorspace $\bar{V}$ to V by redefining scalar multiplication, by

$$a \cdot v \;=\; \bar{a}v$$

(where the multiplication $\bar{a}v$ is the 'old' multiplication in V). Thus, actually

$$V^* \approx \bar{V}$$

Further, we adjust the inner product to be

$$\langle v, w\rangle_{V^*} \;=\; \langle w, v\rangle_V$$

so that it is again complex-linear in the first argument and conjugate-linear in the second argument.

Let (π, V) be a unitary representation of G. The *contragredient* or *dual* representation (π^*, V^*) of G on V^* is defined by

$$\pi^*(g)(\lambda)(v) \;=\; \lambda(\pi(g)^{-1}v)$$

The possibly unexpected inverse in the right-hand side is exactly to assure that

$$\pi^*(gh) \;=\; \pi^*(g)\,\pi^*(h) \qquad\qquad (\text{for } g, h \in G)$$

which follows easily from the definition. Without the inverse there, things would not work right for non-abelian groups.

It is easy to check that (π^*, V^*) is again *unitary*.

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3. *Isotypic components, multiplicities, G-types*

Now we will start to make more use of the notational abuse which allows us to write 'π' for the representation space attached to π, and so on. This makes the notation lighter and curbs the proliferation of parentheses.

Let π be an irreducible unitary representation of G, and let σ be another unitary representation of G. The *algebraic π-isotypic component* or *π-isotype* σ^π_{alg} of π in σ is

$$\sigma^\pi_{\text{alg}} \;=\; \sum_\varphi \varphi(\pi)$$

where the sum is over the space $\text{Hom}_G(\pi, \sigma)$ of all G-intertwining operators $\varphi : \pi \to \sigma$. The *topological π-isotypic component* in σ, denoted σ^π_{top}, is the topological closure of the algebraic π-isotypic component.

The space $\text{Hom}_G(\pi, \sigma)$ of all G-intertwining operators from π to σ is a complex vectorspace. If this dimension is *finite* then this dimension is called the *multiplicity* of π in σ, and π is said to *occur with finite multiplicity* in σ. And, in the case of finite multiplicity, from elementary Hilbert space properties the *algebraic* π-isotypic subspace is already topologically closed, so that the distinction between algebraic and topological π-isotype vanishes.

When a non-zero vector $v \in \sigma$ lies inside $\varphi(\pi)$ for some $\varphi \in \text{Hom}_G(\pi, \sigma)$, then v *has G-type* π (in the strictest sense). If, less stringently, v lies inside the algebraic π-isotypic subspace σ^π_{alg} inside σ, we still say that v has G-type π. Further, even if v only lies inside the topological π-isotype, we still may say that v has G-type π (in the weakest sense). Distinguishing which is meant depends upon the context.

4. Haar measure, measures on quotients, averaging maps

By *topological group* is usually meant a group G which has a locally compact Hausdorff topology such that the group operation and inverse are continuous maps

$$G \times G \to G \text{ by } g \times g' \to gg' \qquad\qquad G \to G \text{ by } g \to g^{-1}$$

To avoid measure-theoretic pathologies, assume that (the topology of) G has a *countable basis*.

Note that a topological group is not simply a group with a topology, nor even such with the requirement that the group operations be continuous. That is, the local compactness and Hausdorff-ness are always implicit, as may be the countability.

Let $C_c^o(G)$ be the set of compactly supported complex-valued functions on G. A positive regular Borel measure (finite on compacts) μ on a topological group G is *right invariant* if

$$\int_G f(gh)\, d\mu(g) \;=\; \int_G f(g)\, d\mu(g) \qquad\qquad (\text{for every } f \in C_c^o(G) \text{ and every } h \in G)$$

Replacing g by gh^{-1} and then h by h^{-1} suggests the abbreviation

$$d\mu(gh) \;=\; d\mu(g)$$

A right invariant positive regular Borel measure is also called a *right Haar measure* on G. The corresponding left invariance condition defines *left Haar measure*.

The notion of *regular Borel measure* is essentially equivalent to the notion of linear functional λ on $C_c^o(G)$ with the continuity property that for every compact subset C of G there is a constant C_K such that

$$|\lambda(f)| \;\le\; C_K \cdot \sup_{g\in K} |f(g)| \qquad\qquad (\text{for every } f \in C_c^o(G) \text{ with support inside } K)$$

This property is an unrolled version of the requirement of continuity with respect to the natural colimit-of-Banach-spaces topology on $C_c^o(G)$, but this technicality is not of immediate use. We will not reprove the following basic theorem.

[4.0.1] Theorem: Up to scalar multiples, there is a unique right Haar measure. ///

[4.0.2] Definition: A topological group G is *unimodular* when a right Haar measure is a left Haar measure.

[4.0.3] Definition: Let μ be a right Haar measure on G. The *modular function* $\Delta = \Delta_G$ of G is defined by

$$d\mu(hg) \;=\; \Delta(h) \cdot d\mu(g)$$

By definition, G is unimodular if and only if $\Delta = 1$.

[4.0.4] Proposition: The modular function Δ is a continuous group homomorphism from G to the positive real numbers (with multiplication). The function Δ is identically 1 on the center of G, on any compact subgroup of G, and on any commutator $ghg^{-1}h^{-1}$ in G.

Proof: That Δ is a group homomorphism is by changing variables. Let μ be a right Haar measure on G. Given $f \in C_c^o(G)$, f is uniformly continuous since it has compact support. Thus, given $\varepsilon > 0$ there is a sufficiently small open neighborhood U of the identity 1 in G such that

$$|f(hg) - f(g)| < \varepsilon \qquad\qquad (\text{for } h \in U \text{ and for all } g \in G)$$

Thus, by the continuity property of the integral

$$\left| \int_G f(hg)\, d\mu(g) - \int_G f(g)\, d\mu(g) \right| \;<\; \varepsilon \cdot \mu(\mathrm{spt}(f)) \qquad\qquad (\text{for } h \in U)$$

That is,

$$|\Delta(h^{-1}) - 1| \cdot \left| \int_G f(g)\, d\mu(g) \right| < \varepsilon \cdot \mu(\mathrm{spt}(f))$$

This proves continuity of Δ at 1, from which easily follows the general case.

Since $d\mu(zg) = d\mu(gz) = d\mu(g)$ certainly Δ is trivial on the center of G. Since the positive reals with multiplication have no non-trivial compact subgroup, and since the continuous image of a compact group is compact, Δ must be continuous on any compact subgroup of G. Finally, since $\mathbb{R}^\times$ is abelian, Δ must be trivial on commutators. ///

Let H be a closed (not necessarily normal) subgroup of a topological group G. Let $H\backslash G$ be the set of cosets Hg, with the quotient topology. Note that G acts on the right on $H\backslash G$ by continuous maps. Define an *averaging map*

$$\alpha : C^o_c(G) \to C^o_c(H\backslash G) \qquad\qquad \text{by} \qquad\qquad \alpha(f)(g) \;=\; \int_H f(hg)\, dh$$

for a right Haar measure dh on H. More generally, for a continuous group homomorphism

$$\omega : H \;\longrightarrow\; \mathbb{C}^\times$$

define

$$C^o_c(H\backslash G, \omega) \;=\; \{f \in C^o(G) : f(hg) \;=\; \omega(h) \cdot f(g), \text{ for all } h \in H,$$
$$g \in G, \;\; \text{and } f \text{ is compactly-supported left mod } H\}$$

The corresponding averaging map

$$\alpha_\omega \;:\; C^o_c(G) \to C^o_c(H\backslash G, \omega)$$

is

$$\alpha_\omega(f)(g) \;=\; \int_H \omega(h)^{-1}\, f(hg)\, dh$$

The following innocent lemma is essential.

[4.0.5] Lemma: The averaging maps (just above) are *surjections*.

Proof: Let q be the quotient map $q : G \to H\backslash G$. First, we show that, given a compact subset C of $H\backslash G$ there is a compact subset C' of G such that $q(C') \;=\; C$. By the *local compactness* of G, we can take an open neighborhood U of the identity in G such that U has compact closure $\overline{U}$. Since a quotient map is *open*, $q(U)$ is open in $H\backslash G$, as are all the translates $q(U) \cdot g$ for $g \in G$. Since

$$C \subset \bigcup_{g : q(g) \in C} q(U) \cdot g$$

and C is compact there is a finite subcover

$$C \subset q(U) g_1 \cup \ldots \cup q(U) g_n$$

The set

$$\overline{U} g_1 \cup \ldots \cup \overline{U} g_n$$

is compact in G, and q^{-1} is at least *closed* in G, so since G is Hausdorff

$$q^{-1}(C) \cap \left(\overline{U}g_1 \cup \ldots \cup \overline{U}g_n\right)$$

is the desired compact set in G.

First consider ω trivial. Given $f \in C_c^o(H\backslash G)$, let C' be a compact subset of G such that $q(C') = \text{spt}(f)$. Via Urysohn's lemma, let φ be in $C_c^o(G)$ such that φ is identically 1 on a neighborhood of C'. Let

$$F(g) \;=\; \varphi(g) \cdot f(g) \in C_c^o(G)$$

Since f is already left H-invariant

$$\alpha(F) \;=\; \alpha(\varphi) \cdot f$$

Thus, noting that $\alpha(\varphi)$ is identically 1 on an open containing the support of f,

$$\alpha(F/\alpha(\varphi)) \;=\; \alpha(\varphi) \cdot f/\alpha(\varphi) \;=\; f$$

Since $\alpha(\varphi)$ is identically 1 on a neighborhood of the support of F, the quotient $F/\alpha(\varphi)$ is continuous. For general ω, a similar trick works, and

$$\alpha_\omega(F/\alpha_\omega(\varphi)) \;=\; \alpha_\omega(\varphi) \cdot f/\alpha_\omega(\varphi) \;=\; f$$

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[4.0.6] Theorem: Let H be a closed (not necessarily normal) subgroup of a topological group G, with the obvious action of G on the right. The quotient $H\backslash G$ has a right G-invariant (positive regular Borel) measure if and only if

$$\Delta_G|_H \;=\; \Delta_H$$

If such a measure exists it is unique up to scalar multiples, and can be uniquely normalized as follows. For given right Haar measure dh on H and for given right Haar measure dg on G there is a unique invariant measure $d\dot{g}$ on $H\backslash G$ such that we have the integration formula

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \left(\int_H f(h\dot{g})\,dh\right) d\dot{g} \qquad\qquad (\text{for } f \in C_c^o(G))$$

Proof: First, prove the *necessity* of the condition on the modular functions. Suppose that there is a right G-invariant measure on $H\backslash G$. Let α be the averaging map as in the previous lemma. The map

$$f \to \int_{H\backslash G} \alpha(f)(\dot{g})\,d\dot{g} \qquad\qquad (\text{for } f \in C_c^o(G))$$

is a right G-invariant functional with the continuity property as above, so must be a constant multiple of the Haar integral

$$f \to \int_G f(g)\,dg$$

Note that the averaging map behaves in a straightforward manner under left translation $L_h f(g) = f(h^{-1}g)$ for $h \in H$:

$$\alpha(L_h f)(g) \;=\; \int_H f(h^{-1}xg)\,dx \;=\; \Delta_H(h) \int_H f(xg)\,dx \qquad\qquad (\text{for } f \in C_c^o(G) \text{ and } h \in H)$$

by replacing x by hx. Then

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \alpha(f)(g)\,d\dot{g} \;=\; \Delta(h)^{-1} \int_{H\backslash G} \alpha(L_h f)(g)\,d\dot{g} \;=\; \Delta(h)^{-1} \int_G f(h^{-1}g)\,dg$$

from the first remark of this proof comparing the iterated integral to the single integral. Replacing g by hg in the integral gives

$$\int_G f(g)\, dg \;=\; \Delta(h)^{-1}\Delta_G(h) \int_G f(g)\, dg$$

Choosing f such that the integral is not 0 implies the condition on the modular functions stated in the theorem.

We prove the sufficiency starting from the existence of Haar measures on G and on H. And first suppose that both these groups are *unimodular*. As expected, we attempt to define an integral on $C_c^o(H\backslash G)$ by

$$\int_{H\backslash G} \alpha f(g)\, d\dot{g} \;=\; \int_G f(g)\, dg$$

invoking the fact that the averaging map α from $C_c^o(G)$ to $C_c^o(H\backslash G)$ is surjective. The potential problem with this is *well-definedness*. It suffices to prove that, if $\alpha f = 0$, then $\int_G f(g)\, dg = 0$. To see this, suppose $\alpha f = 0$. Then, for all $F \in C_c^o(G)$, the integral of F against αf is certainly 0, and we rearrange

$$0 \;=\; \int_G F(g)\, \alpha f(g)\, dg \;=\; \int_G \int_H F(g)\, f(hg)\, dh\, dg \;=\; \int_H \int_G F(h^{-1}g)\, f(g)\, dg\, dh$$

by replacing g by $h^{-1}g$. Then replace h by h^{-1}, so

$$0 \;=\; \int_G \alpha F(g)\, f(g)\, dg$$

The surjectivity of α shows that we can choose F such that αF is identically 1 on the support of f. Then we have shown that the integral of f is 0, as claimed, proving the well-definedness for unimodular H and G.

For not-necessarily-unimodular H and G, in the previous argument the left translation by h^{-1} produces a factor of $\Delta_G(h^{-1})$. Then replacing h by h^{-1} converts right Haar measure to left Haar measure, so produces a factor of $\Delta_H(h)^{-1}$, and the other factor becomes $\Delta_G(h)$. If $\Delta_G(h) \cdot \Delta_H(h)^{-1} = 1$, then the product of these two factors is 1, and the same argument goes through, proving well-definedness. ///

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5. *Regular and biregular representations*

Let

$$\int_G f(g)\, dg \qquad\qquad (\text{for } f \in C_c^o(\mathcal{G}))$$

denote integration with respect to *right* Haar measure. Since we will only be integrating functions, rather than discussing the measure, we will not need to allocate a symbol for the Haar measure.

Let $L^2(G)$ denote the square-integrable complex-valued functions on G (using *right* Haar measure). The *right regular representation* R of G on $L^2(G)$ is defined by

$$R(g)f(h) \;=\; R_g f(h) \;=\; f(hg) \qquad\qquad (\text{for } g, h \in G \text{ and } f \in L^2(G))$$

The *left regular representation* L is similarly defined, by left translation on functions square-integrable with respect to *left* Haar measure on G, by

$$L(g)f(h) \;=\; L_g f(h) \;=\; f(g^{-1}h)$$

The inverse in the formula is necessary to have $L(gg') = L(g)L(g')$ for non-abelian groups, as in the definition of contragredient representation.

For G *unimodular*, that is, when *right* Haar measure is also a *left* Haar measure, the two notions of $L^2(G)$ coincide, and there are *simultaneous* right and left representations on $L^2(G)$. For unimodular G, the *biregular representation* of $G \times G$ on $L^2(G)$ is

$$\pi_{\rm bi}(g \times g')f(h) \;=\; f(g^{-1}hg')$$

In other words,

$$\pi_{\rm bi}(g \times g') \;=\; L(g)\,R(g') \;=\; R(g')\,L(g)$$

The choice of which of the g, g' acts on the left versus right is purely a matter of convention, and can in general be determined only from the context.

[5.0.1] Proposition: The right regular representation of G on $L^2(G)$ is *unitary*. When G is *unimodular*, both left regular and biregular representations are unitary. (The proof is a special case of a more general assertion proven just below).

The construction of $L^2(G)$ can be refined in a manner which turns out to be very important in applications. Let Z be the *center* of G. Since the group operation in G is continuous, and since

$$Z \;=\; \bigcap_{g\in G}\{z \in G : gz = zg\}$$

presents Z as an intersection of closed sets, Z is a closed subgroup of G. Let $\omega : Z \to \mathbb{C}^\times$ be a continuous *unitary character* on Z, meaning that it is a continuous group homomorphism and $|\omega(z)| = 1$ for all $z \in Z$.

Say a function $f \in C^o(G)$ is *compactly supported modulo* Z when there is a compact set C in G so that

$$\mathrm{spt} f \subset Z \cdot C \;=\; C \cdot Z$$

Let

$$C^o_c(Z\backslash G, \omega) \;=\; \{f \in C^o(G) : f(zg) \;=\; \omega(z) \cdot f(g), \quad \text{for all } z \in Z,\ g \in G,\ \text{ and } f \text{ is compactly-supported mod } Z\}$$

This is the collection of continuous functions compactly supported modulo Z with *central character* ω.

Note that for $f_1, f_2 \in C^o_c(Z\backslash G, \omega)$ the product $f_1\bar{f}_2$ lies in $C^o_c(Z\backslash G)$, the space of compactly supported continuous functions on the topological group $Z\backslash G$. (The Hausdorff-ness of this quotient depends upon the fact that Z is *closed*.) Thus, we have an *inner product* on $C^o_c(Z\backslash G, \omega)$ given by

$$\langle f_1, f_2\rangle \;=\; \int_{Z\backslash G} f_1(g)\,\bar{f}_2(g)\,dg$$

where we still write dg for the right Haar measure on the quotient group, rather than using a special notation for the measure on the quotient.

Note that $Z\backslash G$ certainly has a right G-invariant measure, since the condition

$$\Delta_Z \;=\; 1 \;=\; \Delta_G|_Z$$

for existence of such measure is met, where Δ_H is the *modular function* on a topological group H. That the condition really holds follows from the fact that Z is abelian, assuring that $\Delta_Z = 1$, and from the fact that Z is a closed subgroup of the center of G, assuring that $\Delta_G|_Z = 1$.

Then let $L^2(Z\backslash G, \omega)$ be the Hilbert space obtained by completing $C^o_c(Z\backslash G, \omega)$ with respect to the metric associated to this inner product. We define the *right regular representation* of G on $L^2(Z\backslash G, \omega)$ just as above: for $g \in G$ and $f \in L^2(Z\backslash G, \omega)$ define $R_g f$ by

$$R_g f(x) \;=\; f(xg)$$

By the fact that the measure is right G-invariant and $|\omega(z)| = 1$, this representation is *unitary:*

$$\langle R_g f_1, R_g f_2\rangle = \int_{Z\backslash G} f_1(xg)\, f_2(xg)\, dx = \int_{Z\backslash G} f_1(x)\, f_2(x)\, dx \qquad \text{(for } f_1, f_2 \text{ in } L^2(Z\backslash G, \omega),\ g \in G)$$

by replacing x by xg^{-1} in the integral.

When G is *unimodular* (meaning that a right Haar measure is also a left Haar measure), we can define the *left regular representation* $g \to L_g$ of G on $L^2(Z\backslash G, \omega)$ (using the fact that Z is a closed subgroup of the center). And then we also have the *biregular representation* of $G \times G$ on $L^2(Z\backslash G, \omega)$.

In the above construction Z could be any *closed subgroup of the center of* G. Often we might take Z to be the whole center, but this is not necessary. In any such situation the analogous space $L^2(Z\backslash G, \omega)$ may be constructed, with corresponding regular representation of G upon it. The ambiguity of the terminology 'right regular representation' is resolved only by context.

We should check the *continuity* of these representations. We do so for the right regular representation on $L^2(Z\backslash G, \omega)$. Essential use is made of the fact that the Haar measure is a Borel measure, and that all the functions in the Hilbert space can be approximated (in an L^2 sense) by continuous functions with compact support modulo Z, by Urysohn's lemma. Given f in $L^2(Z\backslash G, \omega)$, choose φ in $C_c^o(Z\backslash G, \omega)$ so that in L^2 norm $|\varphi - f| < \varepsilon$. Thus, for $g \in G$,

$$|R_g f - f| \le |R_g f - R_g\varphi| + |R_g\varphi - \varphi| + |\varphi - f| < 2\varepsilon + |R_g\varphi - \varphi|$$

by the unitariness of R_g.

We claim that φ is *uniformly* continuous: if $\operatorname{spt}\varphi \subset Z \cdot C$ with C compact, then certainly φ is uniformly continuous on C. Given $\varepsilon' > 0$, fix a compact neighborhood U_o of 1 in G and take a compact neighborhood $U \subset U_o$ of 1 so that

$$|\varphi(g\theta) - \varphi(g)| < \varepsilon'$$

for $g \in C$, $\theta \in U$. Note that $g\theta$ lies in the compact set $C \cdot U_o$. For g in the support of φ, write $g = zg'$ with $z \in Z$ and $g' \in C$. Then, for $\theta \in U$,

$$|\varphi(g\theta) - \varphi(g)| = |\varphi(zg'\theta) - \varphi(zg')| = |\omega(z)\varphi(g'\theta) - \omega(z)\varphi(g')| = |\varphi(g'\theta) - \varphi(g')| < \varepsilon'$$

since $|\omega(z)| = 1$. That is, elements of $C_c^o(Z\backslash G, \omega)$ really are uniformly continuous.

To prove continuity of the right regular representation, it remains to estimate $|R_g\varphi - \varphi|$. With neighborhoods U, U_o of 1 as in the previous paragraph,

$$|R_g\varphi - \varphi|^2 = \int_{Z\backslash G} |\varphi(xg) - \varphi(x)|^2\, dx \le \int_{Z\backslash \operatorname{spt}\varphi \cdot U_o} (\varepsilon')^2\, dx = (\varepsilon')^2 \times \operatorname{meas}(Z\backslash \operatorname{spt}(\varphi) \cdot U)$$

The latter measure is finite, since the set is compact, so for sufficiently small ε' we can make this less than the original ε.

This proves continuity of all these regular representations. The corresponding proof of continuity for left and biregular representations is identical.

[5.0.2] Remark: At this point, it is easy to illustrate the unreasonableness of trying to require that $\pi : G \to \operatorname{Hom}(V, V)$ be continuous in *operator* norm. Let G be any *non-discrete* group, such as $\mathbb{R}$ or the circle. Fix a compact neighborhood U of 1, and take $g \in U$ with $g \neq 1$. Let $\varphi \in C_c^o(G)$ be a *positive, real-valued* function so that $\int_G |\varphi|^2 = 1$ and so that

$$\operatorname{spt}(\varphi) \cdot g \cap \operatorname{spt}(\varphi) = \phi$$

Then

$$|R_g\varphi - \varphi|^2 \;=\; \langle R_g\varphi - \varphi, R_g\varphi - \varphi\rangle \;=\; |R_g\varphi|^2 - 2\int_G \varphi(xg)\,\varphi(x)\,dx + |\varphi|^2 \;=\; 1 - 0 + 1$$

since the supports of φ and $R_g\varphi$ are disjoint. Thus, we have an estimate on the operator norm $|\;|_{\mathrm{op}}$:

$$|R_g - 1|_{\mathrm{op}} \geq \sqrt{2} \qquad\qquad (\text{for } \textit{any } g \neq 1)$$

That is, in the right regular representation of a non-discrete group, the operator norm $|R_g - 1|_{\mathrm{op}}$ does *not* go to 0 as g goes to 1. Of course, this does *not* contradict the continuity of each operator R_g.

[5.0.3] Remark: The terminology of *right regular* and *left regular* representations is ambiguous, sometimes referring to representations on square-integrable functions, but sometimes to other spaces of functions.

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6. *Hilbert-space valued Gelfand-Pettis integrals*

We need a simple case of vector-valued integrals, namely, integrals of continuous compactly-supported Hilbert space valued functions (with respect to regular Borel measures). The *Gelfand-Pettis*, or *weak* integral considered in the following theorem is sufficient for our present purposes. The argument below takes advantage of the special features of Hilbert spaces.

One important feature of Gelfand-Pettis integrals is the fact (in the corollary below) that they commute with continuous linear maps.

[6.0.1] Theorem: Let X be a locally compact Hausdorff topological space with a countable basis, with a positive regular Borel measure on X giving finite measure to compact subsets. Let V be a Hilbert space, and $f : X \to V$ a continuous compactly-supported function. Then there is a unique vector we'll denote as

$$\int_X f(x)\,dx$$

in V such that for all continuous linear functionals λ on V

$$\lambda\left(\int_X f(x)\,dx\right) \;=\; \int_X \lambda(f(x))\,dx$$

where the latter integral is the usual Lebesgue integral of scalar-valued functions.

[6.0.2] Remark: The assumption that X is locally compact, Hausdorff, and countably based promises that integrals as functionals on $C_c^o(X)$ and regular Borel measures are in bijection, by the Kakutani-Markov-Riesz theorem.

Proof: Since continuous linear functionals on V separate points, there is at most one such integral. The issue is existence. Let $\{e_\alpha : \alpha \in A\}$ be an orthonormal basis for V. Each function $f_\alpha(x) = \langle f(x), e_\alpha\rangle$ is compactly supported continuous $\mathbb{C}$-valued, so has a standard Lebesgue integral

$$I_\alpha \;=\; \int_X f_\alpha(x)\;dx \;=\; \int_X \langle f(x), e_\alpha\rangle\,dx$$

The obvious guess is $I = \sum_\alpha I_\alpha \cdot e_\alpha$. To show convergence, observe that, for a finite $F \subset A$, by Cauchy-Schwarz-Bunyakowsky,

$$\sum_{\alpha\in F} |I_\alpha|^2 \;=\; \sum_{\alpha\in F} \Big|\int \langle f(x), e_\alpha\rangle\,dx\Big|^2 \;=\; \sum_{\alpha\in F} \Big|\int \langle f(x), e_\alpha\rangle\cdot 1\,dx\Big|^2 \leq \sum_{\alpha\in F} \int |\langle f(x), e_\alpha\rangle|^2\;dx \cdot \int_{\mathrm{spt} f} 1\;dx$$

The latter constant is finite because f is compactly supported. This is dominated by

$$\sum_{\alpha\in A}\int |\langle f(x),e_\alpha\rangle|^2\,dx \;=\; \int\sum_{\alpha\in A}|\langle f(x),e_\alpha\rangle|^2\,dx \;=\; \int \|f(x)\|^2\,dx \;<\; \infty$$

since f is continuous and compactly supported, using Fubini-Tonelli and Plancherel. Thus, the tails of the series $\sum_{\alpha\in A}|I_\alpha|^2$ go to zero, and the series converges.

Now verify the weak integral property. For finite $F\subset A$, let $f_F(x)=\sum_{\alpha\in F} f_\alpha(x)\cdot e_\alpha$ and $I_F=\sum_{\alpha\in F} I_\alpha\cdot e_\alpha$. For arbitrary v in the Hilbert space,

$$\langle I_F,v\rangle \;=\; \sum_{\alpha\in F}\int\langle f(x),e_\alpha\rangle\,dx\cdot\langle e_\alpha,v\rangle \;=\; \int\sum_{\alpha\in F}\langle f(x),e_\alpha\rangle\cdot\langle e_\alpha,v\rangle\,dx$$

$$=\; \int\Big\langle \sum_{\alpha\in F}\Big(\langle f(x),e_\alpha\rangle\cdot e_\alpha\Big),\,v\Big\rangle\,dx \;=\; \int\langle f_F(x),v\rangle\,dx$$

Since $|\langle f_F,v\rangle|\le\|f(x)\|\cdot\|v\|$, by the dominated convergence theorem

$$\int\langle f_F(x),v\rangle\,dx \;\longrightarrow\; \int\langle f(x),v\rangle\,dx$$

We showed that $\lim_F I_F = I$, so we have the assertion of the theorem. ///

The following property of Gelfand-Pettis integrals is widely useful.

[6.0.3] Corollary: Let $T:V\to W$ be a continuous linear map of Hilbert spaces. Let $f:X\to V$ be a continuous compactly supported V-valued function on a topological space X with a positive regular Borel measure giving compact subsets finite measure. Then

$$T\left(\int_X f(x)\,dx\right) \;=\; \int_X Tf(x)\,dx$$

Proof: For a continuous linear functional μ on W, by the defining property of the integral,

$$\mu\left(\int_X Tf(x)\,dx\right) \;=\; \int_X \mu\circ Tf(x)\,dx$$

But $\mu\circ T$ is a continuous linear functional on V, so

$$\int_X \mu\circ Tf(x)\,dx \;=\; \mu\circ T\left(\int_X f(x)\,dx\right)$$

and by associativity

$$\mu\circ T\left(\int_X f(x)\,dx\right) \;=\; \mu\left(T\left(\int_X f(x)\,dx\right)\right)$$

which proves as asserted that the image under T of the integral of f is the integral of $T\circ f$. ///

[6.0.4] Remark: The assertions of the theorem and corollary are valid in much greater generality, namely in *quasi-complete locally convex topological vector spaces*, but the argument is more complicated than the above.

7. *Convolution, approximate identities*

Let (π, V) be a unitary representation of G. The map $G \to V$ defined by

$$F(g) \;=\; \varphi(g)\,\pi(g)v \qquad\qquad (\text{fixed } v \in V, \text{ for } \varphi \in C_c^o(G))$$

is continuous and compactly supported. Therefore, by the basic theory of integration of compactly supported continuous Hilbert-space valued functions, there exists an *integral* of F, meaning an element I of the Hilbert space V with the Gelfand-Pettis/weak-integral property

$$\langle I, w\rangle \;=\; \int_G \varphi(g)\,\langle \pi(g)v, w\rangle\; dg \qquad\qquad (\text{for all } w \in V)$$

This integral is denoted

$$\pi(\varphi)v \;=\; \pi(\varphi)(v) \;=\; \int_G \varphi(g)\,\pi(g)v\; dg$$

Using the notation $\pi(\varphi)$ for this *averaged* value of $\pi(g)$ is ambiguous, but this ambiguity is almost always immediately dispelled by context.

[7.0.1] Proposition: For $\varphi \in C_c^o(G)$, for π unitary, $\pi(\varphi)$ is a continuous linear operator with operator norm satisfying the inequality

$$|\pi(\varphi)|_{\text{op}} \le \int_G |\varphi(x)|\, dx \;=\; |\varphi|_{L^1(G)}$$

where the L^1-norm is with respect to *right* Haar measure.

Proof: Compute directly:

$$|\pi(\varphi)|_{\text{op}} \;=\; \sup_{|v|\le 1,\, |w|\le 1} |\langle \pi(\varphi)v, w\rangle| \le \sup_{v,w} \int |\varphi(x)| \cdot |\langle \pi(x)v, w\rangle|\; dx$$

$$\le \sup_{v,w} \int |\varphi(x)| \cdot |\pi(x)v| \cdot |w|\, dx \;=\; \int |\varphi(x)|\, dx$$

as claimed, using unitariness and the Cauchy-Schwarz-Bunyakowsky inequality. ///

[7.0.2] Corollary: The map $\varphi \to \pi(\varphi)$ extends (by continuity) from $C_c^o(G)$ to $L^1(G)$. ///

The collection $C_c^o(G)$ of compactly-supported continuous (complex-valued) functions on G has a uniquely-determined *convolution* product $\varphi \times \longrightarrow \varphi * \psi$ with the property

$$\pi(\varphi * \psi) \;=\; \pi(\varphi) \circ \pi(\psi) \qquad\qquad (\text{for every } \pi)$$

An integral formula for the convolution is obtained directly from this defining property: for fixed $v \in V$,

$$(\pi(\varphi)\circ\pi(\psi))v \;=\; \pi(\varphi)\,(\pi(\psi)v) \;=\; \int_G \varphi(x)\,\pi(x)\,\Big(\int_G \psi(g)\,\pi(g)v\, dg\Big)\; dx$$

$$= \int_G \int_G \varphi(x)\,\psi(g)\,\pi(xg)v\, dx\, dg \;=\; \int_G \int_G \varphi(xg^{-1})\,\psi(g)\,\pi(x)v\, dx\, dg$$

by reversing the order of integration and replacing x by xg^{-1}. Reversing the order of integration again, this is

$$\int_G \Big(\int_G \varphi(xg^{-1})\,\psi(g)\; dg\Big)\,\pi(x)v\; dx$$

This shows that the inner integral in the latter expression should be the definition of convolution (acting on v), that is,

$$\varphi * \psi)(x) \;=\; \int_G \varphi(xg^{-1})\,\psi(g)\,dg$$

giving the desired property

$$\pi(\varphi)\circ\pi(\psi) \;=\; \pi(\varphi*\psi) \;=\; \pi(\varphi)\circ\pi(\psi) \qquad\qquad \text{(for every } \pi)$$

The specific configuration of left/right and the 'inverse', together with use of *right* Haar measure, make convolution *associative*. The computation to verify the associativity illustrates fundamental points, so we give it: let f, φ, ψ be in $C_c^o(G)$. Then

$$(f*(\varphi*\psi))(g) \;=\; \int_G f(gx^{-1})(\varphi*\psi)(x)\,dx \;=\; \int_G\int_G f(gx^{-1})\,\varphi(xy^{-1})\,\psi(y)\,dy\,dx$$

$$= \int_G\int_G f(gy^{-1}x^{-1})\,\varphi(x)\,\psi(y)\,dx\,dy$$

by changing the order of integration and then replacing x by xy. Then this is

$$\int_G (f*\varphi)(gy^{-1})\,\psi(y)\,dy \;=\; ((f*\varphi)*\psi)(g)$$

as asserted.

[7.0.3] Proposition: In the situation as above, the adjoint operator to $\pi(\varphi)$ is $\pi(\varphi^*)$, where we define

$$\varphi^*(g) \;=\; \overline{\varphi(g^{-1})}/\Delta(g)$$

where Δ is the modular function on G, in particular so that with dg denoting right Haar measure

$$dg/\Delta(g) \;=\; \text{left Haar measure}$$

Proof: Computing directly, and without loss of generality taking φ to be in $C_c^o(G)$:

$$\langle\pi(\varphi)v,w\rangle \;=\; \int_G \varphi(x)\,\langle\pi(x)v,w\rangle\,dx \;=\; \int_G \varphi(x)\,\langle v,\pi(x)^*w\rangle\,dx \;=\; \int_G \varphi(x)\,\langle v,\pi(x^{-1})w\rangle\,dx$$

by unitariness of π. Replacing x by x^{-1}, this is

$$\int_G \varphi(x^{-1})\,\langle v,\pi(x)w\rangle\,d(x^{-1}) \;=\; \int_G \varphi(x^{-1})\,\langle v,\pi(x)w\rangle\,d(x)/\Delta(x)$$

since

$$d(x^{-1}) \;=\; d(x)/\Delta(x)$$

By the characterizing property of the Gelfand-Pettis integral, this is

$$\Big\langle v, \int_G \bar\varphi(x^{-1})\,\pi(x)w\,d(x)/\Delta(x)\Big\rangle \;=\; \langle v,\pi(\varphi^*)w\rangle$$

as claimed, where the complex conjugate appears because the inner product is conjugate-linear in its second argument. ///

[7.0.4] Definition: A sequence $\{\varphi_i\}$ of functions in $C_c^o(G)$ is an *approximate identity* (in a strong sense) when
- $\varphi_i(x) \ge 0$ for all $x \in G$ and for all indices i
- $\int_G \varphi_i(x)\,dx = 1$, for all indices i
- For every neighborhood U of 1 in G there is a large-enough index i_U so that for $i \ge i_U$ we have $\mathrm{spt}\varphi_i \subset U$.

[7.0.5] Claim: For locally compact, Hausdorff G, approximate identities exist.

Proof: Given an arbitrarily small neighborhood U of $1 \in G$, using the continuity of multiplication, there is a neighborhood V of 1 such that $V \cdot V \subset U$. The closure $\overline{V}$ of V is contained in $V \cdot V \subset U$. Then invoke Urysohn's lemma to make a continuous function with values between 0 and 1, identically 1 on $\overline{V}$, identically 0 outside U. ///

[7.0.6] Proposition: Let $\{\varphi_i\}$ be an approximate identity, and fix v in the representation space V for the unitary representation π. Then $\pi(\varphi_i)v \to v$ in V.

Proof: Given $\varepsilon > 0$, take a small enough neighborhood U of 1 so that for all $x \in U$ we have $|\pi(x)v - v| \le \varepsilon$, invoking the continuity of the representation. Take $\phi = \phi_i$ with i large enough so that the support of φ is inside U. Since $\int_G \varphi = 1$,

$$|\pi(\varphi)v - v| \;=\; \Big|\int_G \varphi(x)\pi(x)(v)\,dx - v\Big| \le \int_G \varphi(x)\cdot|\pi(x)v - v|\,dx < \int_G \varphi(x)\cdot\varepsilon\,dx \;=\; \varepsilon$$

giving the asserted convergence. ///

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8. *Group representations versus algebra representations*

[8.0.1] Proposition: Let (π, V) and (π', V') be unitary representations of G.
- The collection of *closed* G-stable subspaces of V is identical to the collection of *closed* $C_c^o(G)$-stable subspaces of V.
- A continuous linear map $T : V \to V'$ is a G-homomorphism if and only if it is an $C_c^o(G)$-homomorphism, that is, if and only if

$$\pi'(\varphi)\circ T \;=\; T\circ\pi(\varphi) \qquad\qquad (\text{for all } \varphi \in C_c^o(G))$$

- The representation (π, V) is G-irreducible if and only if it is $C_c^o(G)$-irreducible, that is, if and only if V has no proper closed $C_c^o(G)$-stable subspace.

Proof: The third assertion of the proposition is a special case of the first. A closed G-subspace W of V is a Hilbert space, and the Gelfand-Pettis integral theory applies. The integrals

$$\pi(f)v \;=\; \int_G f(g)\,\pi(g)v\,dg \qquad\qquad (\text{for } f \in C_c^o(G))$$

have values again in W. That is, a closed G-subspace is an $C_c^o(G)$-subspace. On the other hand, for any $C_c^o(G)$-stable subspace W and for $g \in G$, for $w \in W$, with an approximate identity φ_i,

$$\pi(g)w \;=\; \pi(g)\,\lim \pi(\varphi_i)w \;=\; \pi(g)\,\lim \int_G \varphi_i(h)\,\pi(h)w\,dh \;=\; \lim \int_G \varphi_i(h)\,\pi(gh)w\,dh$$

by the continuity of $\pi(g)$ and by the basic properties of the Gelfand-Pettis integral. Replacing h by $g^{-1}h$, this becomes

$$\Delta(g^{-1})\lim \int_G \varphi_i(g^{-1}h)\,\pi(h)w\,dh \;=\; \Delta(g^{-1})\lim \pi(L_g\varphi_i)w$$

where $\Delta(g)$ is the modular function on G, with

$$d(g^{-1}h) \;=\; \Delta(g^{-1})\,dh$$

By the assumption that $C_c^o(G)$ is stable under left translations, the functions $h \to \varphi_i(g^{-1}h)$ are again in $C_c^o(G)$. Thus, W is stable under G, as well.

To prove that a G-homorphism gives rise to an $C_c^o(G)$-homomorphism, repeatedly use properties of Gelfand-Pettis integrals. For $f \in C_c^o(G)$ and $v \in V$,

$$(T \circ \pi(f))v \;=\; T\Big(\int_G f(x)\,\pi(x)v\,dx\Big) \;=\; \int_G T(f(x)\,\pi(x)v)\,dx$$

Further, since T is linear, the latter is

$$\int_G f(x)\,T(\pi(x)v)\,dx \;=\; \int_G f(x)\,\pi'(x)Tv\,dx$$

where finally we use the fact that $T(\pi(x)v) = \pi'(x)Tv$. The latter expression is none other than $\pi'(f)(Tv)$, as asserted.

On the other hand, $\pi(\varphi_i)v \longrightarrow v$ for an approximate identity φ_i, Let $g \in G$. As in the comparison of G-subspaces and $C_c^o(G)$-subspaces,

$$\pi(g)\pi(\varphi_i) \;=\; \Delta(g)^{-1} \cdot \pi(L_g\varphi_i)$$

Using the continuity of $T : V \to V'$

$$T(\pi(g)v) \;=\; \lim T(\pi(g)\pi(\varphi_i)v) \;=\; \Delta(g)^{-1} \lim T(\pi(L_g\varphi_i)v) \;=\; \Delta(g)^{-1} \lim \pi'(L_g\varphi_i)T(v)$$

since T is an $C_c^o(G)$-map. Going backward now, this is

$$\lim \pi'(g)\pi'(\varphi_i)T(v) \;=\; \pi'(g)T(v)$$

by properties of approximate identities. Thus, T is a G-map. ///

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9. *Schur's Lemma for bounded operators*

This result is a corollary of basic spectral theory for continuous linear operators on Hilbert spaces. Let S be a $\mathbb{C}$-algebra with an involution $s \to s^*$, with $\mathbb{C}$-linear ring homomorphisms

$$\pi : S \to \mathrm{End}_{\mathbb{C}}(V) \qquad\qquad \pi' : S \to \mathrm{End}_{\mathbb{C}}(V')$$

for two Hilbert spaces V and V', converting the involution in S to adjoint in the operators:

$$\pi(s^*) \;=\; \pi(s)^* \qquad\qquad \pi'(s^*) \;=\; \pi'(s)^*$$

Let $T : V \to V'$ be a continuous linear map *commuting with* S in the sense that

$$\pi'(s) \circ T \;=\; T \circ \pi(s) \qquad\qquad (\text{for all } s \in S)$$

[9.0.1] Theorem: Suppose that V is S-irreducible, in the sense that there is no proper closed subspace of V stable under $\pi(S)$. Then T is a scalar multiple of an *isometry* of V to a *closed* subspace of V'.

Proof: Consider $\varphi = T^* \circ T$, where $T^* : V' \to V$ is the adjoint of T. The relation

$$\pi'(s) \circ T \;=\; T \circ \pi(s)$$

gives

$$T^* \circ \pi'(s)^* \;=\; \pi(s)^* \circ T^*$$

and since $\pi(S)$ and $\pi'(S)$ are closed under adjoints we conclude that φ commutes with $\pi(S)$, and $\varphi^* = \varphi$. By general spectral theory, the spectrum $\sigma(\varphi)$ of φ is not empty. The self-adjointness implies that $\sigma(\varphi) \subset \mathbb{R}$. If $\sigma(\varphi)$ were not merely a single point, there would be two continuous functions f, g on $\sigma(\varphi)$ such that neither is identically 0, but $fg = 0$ on $\sigma(\varphi)$, by Urysohn's lemma. By the von Neumann and Gelfand spectral theory, the map

$$\mathbb{C}[x] \to \mathbb{C}[\varphi]$$

sending $x \to \varphi$ factors through the restriction $\mathbb{C}[x]|_{\sigma(\varphi)}$ of functions in the polynomial ring to the set $\sigma(\varphi)$. And from there is extends to an *isometry* of algebras (by continuity)

$$C^o_c(\sigma(\varphi))_{\text{sup norm}} \longrightarrow \text{ operator norm closure of } \mathbb{C}[\varphi] \qquad \text{(isomorphism)}$$

Therefore, $f(\varphi) \neq 0$, $g(\varphi) \neq 0$, but $fg(\varphi) = 0$, and all these commute with $\pi(S)$.

Then $\ker f(\varphi)$ is closed, is $\pi(S)$-stable, and is not all of V since $f(\varphi) \neq 0$. Also, $\ker f(\varphi) \neq 0$ since

$$g(\varphi)\,(\ker f(\varphi)) \;=\; 0$$

and $g(\varphi) \neq 0$. Then $\ker f(\varphi)$ would be a proper closed $\pi(S)$ subspace of V, contradiction.

Thus, $\sigma(\varphi)$ is a single point λ. Again by spectral theory for self-adjoint operators, this implies that φ is multiplication by the scalar λ. Then

$$\langle Tx, Ty\rangle_{V'} \;=\; \langle T^*Tx, y\rangle_V \;=\; \lambda \cdot \langle x, y\rangle_V$$

proving the theorem. ///

[9.0.2] Corollary: If π, V is an irreducible unitary representation of G, and if $T \in \mathrm{End}_{\mathbb{C}}(V)$ commutes with the action of G, then T is a scalar. ///

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10. *Schur's lemma for unbounded operators*

This technical strengthening of Schur's lemma is necessary later in discussion of matrix coefficient functions.

Let S be a $\mathbb{C}$-algebra with an involution $s \to s^*$, with $\mathbb{C}$-linear ring homomorphisms

$$\pi : S \to \mathrm{End}_{\mathbb{C}}(V) \qquad \pi' : S \to \mathrm{End}_{\mathbb{C}}(V')$$

for two Hilbert spaces V and V', converting the involution in S to adjoint in the operators:

$$\pi(s^*) = \pi(s)^* \qquad \pi'(s^*) \;=\; \pi'(s)^*$$

[10.0.1] Theorem: Let T be a (not necessarily continuous, but) *closed* linear operator $D_T \to H'$ for a dense subset D_T of V, with D_T stable by $\pi(S)$. Suppose that T commutes with the action of S in the sense that for all $s \in S$

$$\pi'(s) \circ T \;=\; T \circ \pi(s) \qquad \text{(on } D_T\text{)}$$

If H has no proper $\pi(S)$-stable closed subspaces, then T is a scalar multiple of an isometry extending a mapping of V to a closed subspace of V'.

Proof: That T is closed means that, by definition, the graph

$$\Gamma_T \;=\; \{(x, Tx) : x \in D_T\} \subset V \oplus V'$$

is closed. By definition, the graph of $-T^*$ is

$$\Gamma_{-T^*} \;=\; \{(-T^*y, y) : y \in D_{T^*}\} \;=\; \Gamma_T^\perp$$

The denseness of D_T assures the uniqueness of the adjoint T^*. Thus, we have an orthogonal decomposition

$$V \oplus V' \;=\; \Gamma_T \oplus \Gamma_{-T^*}$$

Thus, given $v \in V$, for some $x \in D_T$ and $y \in D_{T^*}$

$$(v, 0) \;=\; (x, Tx) + (-T^*y, y)$$

Write $x = Av$ and $y = Bv$. By orthogonality

$$|v|^2_V \;=\; |(v,0)|^2_{V\oplus V'} \;=\; |(x,Tx)|^2 + |(-T^*y,y)|^2 \;=\; |x|^2_V + |Tx|^2_{V'} + |T^*y|^2_V + |y|^2_{V'} \geq |x|^2_V + |y|^2_{V'}$$

This implies that A and B are bounded (hence, continuous) operators $A : V \to V$ and $B : V \to V'$. Thus we can write

$$v \;=\; Av + T^*Bv \qquad\qquad 0 \;=\; TAv - Bv$$

from which we obtain

$$1_V \;=\; A + T^*B \qquad\qquad 0_{V'} \;=\; TA - B$$

Then $B = TA$. Substitute to obtain

$$1_V \;=\; A + T^*TA \;=\; (1_V + T^*T) \circ A$$

Thus, evidently

$$A \;=\; (1_V + T^*T)^{-1}$$

This shows that $(1_V + T^*T)^{-1}$ extends to a bounded operator. It is immediate that also D_{T^*} is $\pi'(S)$-stable, so by the *bounded* operator version of Schur's lemma $(1_V + T^*T)^{-1}$ is a scalar λ. Then solve to obtain (at first only on D_T)

$$T^*T \;=\; (\frac{1}{\lambda} - 1) \cdot 1_V$$

This extends to a map defined on all of V. Note that λ cannot be 0 since $1_V = (1_V + T^*T)A$. Then as in the bounded case,

$$\langle Tx, Ty\rangle_{V'} \;=\; \langle T^*Tx, y\rangle_V \;=\; (\frac{1}{\lambda} - 1)\langle x, y\rangle_V$$

for $x, y \in D_T$. This shows that T is *bounded*, hence continuous (and extends to all of V). Then apply the bounded case of Schur's lemma to conclude that T is a scalar multiple of an isometry. ///

11. *Central characters of irreducible unitary representations*

Again, the center Z_G of a topological group G is a *closed* subgroup, since

$$Z_G \;=\; \bigcap_{g\in G} \{z\in G : zg = gz\} \;=\; \bigcap_{g\in G} \text{(closed subsets)}$$

and since the group operation is continuous.

[11.0.1] **Corollary:** *(of Schur's lemma)* Let π, V be an irreducible unitary representation of G. Then π sends the center Z_G of G to *scalar* operators on V, with $|\pi(z)| = 1$ for $z\in Z$. ///

[11.0.2] **Remark:** The restriction of π to the center is the *central character* of π, often denoted ω_π.

12. *Matrix coefficient functions, discrete series*

For π, V a unitary representation of G, the *matrix coefficient function* attached to $u, v\in V$ is

$$c_{u,v}(g) \;=\; \langle \pi(g)u, v\rangle$$

Since the map $g\to\pi(g)v$ is continuous, each coefficient function is a continuous $\mathbb{C}$-valued function on G. And by the unitariness, for $g, x, y\in G$, the biregular representation's behavior is

$$c_{u,v}(y^{-1}gx) \;=\; \langle \pi(y^{-1}gx)u, v\rangle \;=\; \langle \pi(g)\cdot\pi(x)u, \pi(y^{-1})^* v\rangle \;=\; \langle \pi(g)\cdot\pi(x)u, \pi(y)v\rangle \;=\; c_{\pi(x)u,\pi(y)v}(g)$$

That is, letting L be the left regular and R be the right regular representation of G on functions on G,

$$L(y)R(x)c_{u,v} \;=\; c_{\pi(x)u,\pi(y)v}$$

Let Z be a closed subgroup of G contained in the center Z_G of G. One might take $Z = \{1\}$ for simplicity, though for G with non-compact center this renders the following result (and others) vacuous. The finesse is to take Z large enough such that Z_G/Z is *compact*. This is not used directly in the following proof, though is necessary for non-vacuity.

[12.0.1] **Theorem:** Take G unimodular. Let π, V be irreducible unitary with central character ω (from Schur's lemma). The following are equivalent.
- π is a subrepresentation of $L^2(Z\backslash G, \omega)$.
- There exist non-zero $u, v\in V$ such that $c_{u,v}\in L^2(Z\backslash G,\omega)$.
- For all $u, v\in V$ we have $c_{u,v}\in L^2(Z\backslash G,\omega)$.

[12.0.2] **Definition:** If these conditions are met, π is a *discrete series* representation of G, or is said to be *square integrable* (modulo Z).

[12.0.3] **Remark:** If the center Z_G is non-compact then, for example, $L^2(G)$ has *no* non-zero unitary subrepresentations, for the following reason. Suppose, to the contrary, that there were such π, V. By Schur's lemma it has a central character ω, and necessarily $|\omega| = 1$, that is, ω is unitary. Take u, v in V. Then

$$\langle u, v\rangle_V \;=\; \int_G u(g)\,\overline{v(g)}\,dg \;=\; \int_{Z_G\backslash G} u(g)\,\overline{v(g)}\left(\int_{Z_G} 1\,dz\right)dg$$

The inner integral is $+\infty$, which means that the integral for the inner product diverges, contradiction.

[12.0.4] Definition: The *discrete spectrum* $L^2_d(Z\backslash G,\omega)$ (with central character ω) is the completion in $L^2(Z\backslash G,\omega)$ of the sum of all irreducible subrepresentations of $L^2(Z\backslash G,\omega)$, that is, of all discrete series representations with central character ω.

Proof: *(of theorem)* First, we prove that the fact that π is a subrepresentation of $L^2(Z\backslash G,\omega)$ implies that *some* matrix coefficient function is square integrable modulo Z. Take u,v in the space of an irreducible π in $L^2(Z\backslash G,\omega)$, with $\langle u,v\rangle\neq 0$. Since $C^o_c(Z\backslash G,\omega)$ is dense, there exists $\varphi\in C^o_c(Z\backslash G,\omega)$ whose orthogonal projection $\mathrm{pr}\,\varphi$ to π is arbitrarily close to v, so

$$c_{u,\mathrm{pr}\,\varphi}(1)=\langle u,\mathrm{pr}\,\varphi\rangle\neq 0$$

Thus, $c_{u,\mathrm{pr}\,\varphi}$ is not the 0 function. And we know that the *averaging map*

$$\alpha:\varphi_o\to\int_Z \omega(z)^{-1}\,\varphi_o(zg)\,dz$$

of $C^o_c(G)$ to $C^o_c(Z\backslash G,\omega)$ is a *surjection* so let φ_o be in $C^o_c(G)$ such that $\alpha\varphi_o=\varphi$. Then

$$c_{u,\mathrm{pr}\,\varphi}(g)\;=\;\int_{Z\backslash G} u(hg)\overline{\varphi(h)}\,dh\;=\;\int_{Z\backslash G} u(hg)\int_Z \omega(z)\,\overline{\varphi_o}(zh)\,dz\,dh\;=\;\int_{Z\backslash G}\int_Z u(zhg)\,\overline{\varphi_o}(zh)\,dz\,dh$$

$$=\;\int_G u(hg)\,\overline{\varphi_o}(h)\,dh\;=\;\int_G u(h^{-1}g)\,\overline{\varphi_o}(h^{-1})\,dh\;=\;L(\varphi_o^*)u(g)$$

where $\varphi^*_o(x)=\overline{\varphi_o}(x^{-1})$ and L is the left regular representation. We need unimodularity of G to know that a right Haar measure is a left Haar measure, and, equivalently, that the left regular representation L of G on $L^2(Z\backslash G,\omega)$ formed with *right* Haar measure is unitary. Then $L(\varphi^*_o)u\in V$, so $c_{u,\mathrm{pr}\,\varphi}$ is in V. Thus, there is a (non-zero) matrix coefficient function which is square integrable modulo Z.

Next, we prove that existence of *some* non-zero matrix coefficient function which is square-integrable modulo Z implies both that *all* coefficient functions are square integrable, and that π imbeds into $L^2(Z\backslash G,\omega)$. Let $u,v\in V$ such that $c_{u,v}\in L^2(Z\backslash G,\omega)$. Let

$$W\;=\;\{x\in V: c_{x,v}\in L^2(Z\backslash G,\omega)\}$$

and on W define

$$Tx\;=\;c_{x,v}$$

Thus, T is a linear map

$$T:W\to L^2(Z\backslash G,\omega)$$

[12.0.5] Lemma: The operator T is a *closed* (possibly unbounded) operator.

Proof: *(of claim)* Let w_n be a sequence of points in W, and suppose that the sequence of points (w_n,Tw_n) has a limit (u,f) in $V\oplus L^2(Z\backslash G,\omega)$. We must show that $f\in T(W)$. *Pointwise*, for $g\in G$, as $w_n\to u$ in V, for any $v\in V$,

$$(Tw_n)(g)\;=\;c_{w_n,v}(g)\;=\;\langle\pi(g)w_n,v\rangle\;=\;\langle w_n,\pi(g)^{-1}v\rangle\to\langle u,\pi(g)^{-1}v\rangle$$

so the limit exists pointwise. The limit is *uniform*:

$$|c_{w_n,v}(g)-c_{u,v}(g)|\;=\;|\langle\pi(g)(w_n-u),v\rangle|\leq\|w_n-u\|\,\|v\|$$

by Cauchy-Schwarz-Bunyakowsky and unitariness. Now we will show that $f=c_{u,v}$ in an L^2 sense. Let $\|\;\|_\omega$ be the norm on $L^2(Z\backslash G,\omega)$, and put

$$X_n\;=\;\{x\in Z\backslash G:|f(x)|\geq\frac{1}{n}\}$$

Necessarily the measure of X_n is finite. Let $\| \|_n$ be the L^2 norm on $L^2(Z\backslash ZX_n, \omega)$. Then

$$\|c_{u,v} - f\|_\omega \le \sum_n \|c_{u,v} - f\|_n \le \sum_n \big(\|c_{u,v} - c_{w_{i_n},v}\|_n + \|c_{w_{i_n},v} - f\|_n\big)$$

where for each index n the index i_n is *any* index. Since the measure of X_n is finite, we can choose each i_n large enough such that both

$$\|c_{u,v} - c_{w_{i_n},v}\|_n < \varepsilon/2^n$$

from the *uniform* convergence, and

$$\|c_{w_{i_n},v} - f\|_n \le \|c_{w_{i_n},v} - f\| < \varepsilon$$

via convergence in $L^2(Z\backslash G, \omega)$. Therefore, $c_{u,v}$ is in $L^2(Z\backslash G, \omega)$ and $c_{u,v} = f$ in $L^2(Z\backslash G, \omega)$. This proves that T is closed. ///

Now $\pi(G)$ certainly stabilizes $D_T = W$, since

$$T(\pi(g)x) \;=\; c_{\pi(g)x,v} \;=\; R_g\, c_{x,v}$$

Thus, as D_T is non-empty and G-stable, it must have closure all of V (by irreducibility), so is dense. Likewise, T commutes with the action of G (on $D_T = W$), so we can invoke the unbounded version of Schur's lemma: T is a scalar multiple of an isometry of V to a closed subspace of $L^2(Z\backslash G, \omega)$.

In particular, $c_{x,v} \in L^2(Z\backslash G, \omega)$ for all $x \in V$, and π is imbedded by T as a subrepresentation of $L^2(Z\backslash G, \omega)$, from the assumption that at least one non-zero coefficient function is square-integrable modulo Z.

The same discussion applied to the map $y \to c_{u,y}$ and left regular representation (using unimodularity to know that the biregular representation is unitary) shows that the square integrability of a single non-zero matrix coefficient functions suffices to prove the square integrability of all of them. ///

[12.0.6] **Corollary:** *(of proof)* (Not necessarily assuming unimodularity) if an irreducible unitary representation has one non-zero square-integrable matrix coefficient c_{u_o,v_o}, then c_{u,v_o} is square integrable for all u in V, and

$$T : x \to c_{x,v_o}$$

is a scalar multiple of an isometry. ///

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13. *Schur orthogonality, inner product relations, formal degree*

Let Z be a closed subgroup of the center Z_G of G such that Z_G/Z is compact.

[13.0.1] **Theorem:** Let π, V and π', V' be non-isomorphic discrete series representations of unimodular G. Suppose that both $\pi(z)$ and $\pi'(z)$ act by a scalar $\omega(z)$ for a unitary character ω of Z. Then, for all $u, v \in V$ and $x, y \in V'$

$$\int_{Z\backslash G} c_{u,v}(g)\, \overline{c_{x,y}}(g)\, dg \;=\; 0$$

where the first matrix coefficient function is attached to π and the second to π'.

Proof: Let $v \in V$ be fixed, and $T : V \to L^2(Z\backslash G, \omega)$ by $Tu = c_{u,v}$. Similarly, let $y \in V'$ be fixed and define $T' : V' \to L^2(Z\backslash G, \omega)$ by $T'x = c_{x,y}$. Letting $\langle, \rangle$ be the inner product on $L^2(Z\backslash G, \omega)$,

$$\langle c_{u,v}, c_{x,y}\rangle \;=\; \langle Tu, T'x\rangle \;=\; \langle u, T^*T'x\rangle$$

Since $T^*T' : V' \to V$ and $\pi \not\approx \pi'$ and T^*T' commutes with the action of G, it must be that $T^*T' = 0$. Thus, we have the theorem. ///

[13.0.2] Theorem: Let π, V be a discrete series representation of unimodular G, in $L^2(Z\backslash G, \omega)$. Then there is $0 < d_\pi < \infty$, the *formal degree* of π (corresponding to choice of Haar measure on G) such that

$$\int_{Z\backslash G} c_{u,v}(g)\, \overline{c_{x,y}}(g)\, dg \;=\; \frac{1}{d_\pi}\, \langle u, x\rangle\, \overline{\langle v, y\rangle} \qquad \text{(for all } u, v, x, y \text{ in } V)$$

Proof: Let T and T' be as in the proof of the previous theorem. Then, as in that proof, $T^*T' = \lambda \cdot 1_V$ for some scalar $\lambda = \lambda_{v,y}$. That is,

$$\langle c_{u,v}, c_{x,y}\rangle \;=\; \lambda_{v,y} \cdot \langle u, x\rangle$$

But, similarly, complex conjugating and replacing g by g^{-1},

$$\overline{\langle c_{u,v}, c_{x,y}\rangle} \;=\; \int_{Z\backslash G} \overline{c_{u,v}(g)}\, c_{x,y}(g)\, dg \;=\; \int_{Z\backslash G} c_{v,u}(g)\, \overline{c_{y,x}(g)}\, dg \;=\; \lambda_{u,x} \cdot \langle v, y\rangle$$

Thus,

$$\int_{Z\backslash G} c_{u,v}(g)\, \overline{c_{x,y}}(g)\, dg \;=\; c\, \langle u, x\rangle\, \overline{\langle v, y\rangle}$$

for some constant c not depending on u, v, x, y. Define $d_\pi = 1/c$. For this to be reasonable, we need $c \neq 0$. And, indeed, for $v \neq 0$ we have $c_{v,v}(1) = \langle v, v\rangle > 0$, and since these coefficient functions are continuous,

$$0 < \langle c_{v,v}, c_{v,v}\rangle \;=\; c \cdot \langle v, v\rangle$$

showing that $c \neq 0$. ///

[13.0.3] Remark: Multiplying the Haar measure by $\alpha > 0$ multiplies d_π by $1/\alpha$, so the invariant is $d_\pi\, dg$.

[13.0.4] Remark: Not surprisingly, for compact $Z\backslash G$, the formal degree (suitably normalized) is simply the *dimension* of the representation space of π.

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14. *Hilbert-Schmidt operators*

We recall some standard facts, with proof, concerning an important concrete class of *compact operators* on Hilbert spaces, namely the *Hilbert-Schmidt* operators, which arise naturally in many applications.

Recall that a *Hilbert-Schmidt* operator $T : V \to W$ between two Hilbert spaces is a continuous linear operator such that for an orthonormal basis $\{e_n\}$ of V

$$\sum_n |Te_n|_W^2 < +\infty$$

The *Hilbert-Schmidt* norm $|T|_{HS} = |T|_2$ on such operators is the square root of the sum, that is,

$$|T|_{HS}^2 \;=\; |T|_2^2 \;=\; \sum_n |Te_n|_W^2$$

The following standard fact is basic to this discussion.

[14.0.1] Lemma: The Hilbert-Schmidt norm is independent of the choice of orthonormal basis. The Hilbert-Schmidt norm of an operator T and its Hilbert-space adjoint T^* coincide.

Proof: Let $\{e_i\}$ and $\{f_j\}$ be two orthonormal bases. Then using Plancherel-Parseval twice

$$\sum_i |Tf_i|^2 \;=\; \sum_i \sum_j |\langle Tf_i, e_j\rangle|^2 \;=\; \sum_{i,j} |\langle f_i, T^*e_j\rangle|^2 \;=\; \sum_j \sum_j |\langle f_i, T^*e_j\rangle|^2 \;=\; \sum_j |T^*e_j|^2$$

The latter expression certainly does not depend upon the f_i, and incidentally shows that the Hilbert-Schmidt norm of the adjoint T^* is the same as that of T. ///

We recall a standard result:

[14.0.2] Proposition: Hilbert-Schmidt operators are compact operators. The space of all Hilbert-Schmidt operators is the completion of the space of finite-rank operators under the Hilbert-Schmidt norm (which dominates the uniform operator norm).

Proof: We claim that the Hilbert-Schmidt norm dominates the uniform operator norm $\| \|_{\rm op}$. Indeed, granting that the norm does not depend upon the choice of orthonormal basis, we may suppose without loss of generality that a given vector x is of length 1 and is the first vector e_1 in an orthonormal basis. Then

$$\|T\|^2_{\rm op} \;=\; \sup_{|x|=1} |Tx|^2_W \;\le\; \sum_n \|Te_n\|^2_W \;=\; \|T\|^2_2$$

Given a Hilbert-Schmidt operator and an orthonormal basis $\{e_i\}$, let T_n be the composition ${\rm pr}_n \circ T$ where ${\rm pr}_n$ is the orthogonal projection to the finite-dimensional space

$$\mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$$

Thus, T_n is a finite-rank operator. Then Hilbert-Schmidt norm of $T - T_n$ is

$$\|T - T_n\|^2_{\rm op} \;=\; \sum_{i \ge n} \|Te_i\|^2_W$$

which goes to 0 as $n \to \infty$, since T has finite Hilbert-Schmidt norm. Since the Hilbert-Schmidt norm dominates the operator norm, $T_n \to T$ in operator norm. Since operator-norm limits of finite-rank operators are compact, T is compact. ///

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15. *Tensor products of representations*

It is an elementary exercise to see that the direct sum $V \oplus W$ of two Hilbert spaces is a Hilbert space, with

$$\langle v \oplus w, v' \oplus w'\rangle^2 \;=\; \langle v, v'\rangle^2 + \langle w, w'\rangle^2$$

and that the obvious linear maps of V and W to the sum are continuous. By contrast, unless one of V, W is finite-dimensional, the usual (algebraic) tensor product $V \otimes W$ is not a Hilbert space, although it certainly has the inner product obtained from

$$\langle v \otimes w, v' \otimes w'\rangle \;=\; \langle v, v'\rangle \cdot \langle w, w'\rangle$$

by extending hermitian-bilinearly. We can *define* a Hilbert-space tensor product

$$V \hat{\otimes} W \;=\; \text{completion of } V \otimes W$$

with respect to the norm coming from the inner product.

[15.0.1] Remark: Unless one of the two Hilbert spaces is finite-dimensional, the tensor product we have denoted with a hatted tensor symbol *is not* a categorical tensor product, insofar as it does not satisfy all the properties a true tensor product would possess. However, the above notational style is common in practice.

Given unitary π, V and π', V' of G and G', we define the (*external*) *tensor product* representation $\pi \otimes \pi'$ of $G \times G'$ by taking the representation space to be $V \hat{\otimes} V'$, taking the obvious definition

$$(\pi \otimes \pi')(g \times g')(v \otimes v') \;=\; \pi(g)(v) \otimes \pi'(g')(v')$$

and extending by hermitian-bilinearity and continuity. Observe that $\pi \otimes \pi'$ is unitary. In general, $V \otimes V'$ is a $(G \times G')$-stable but not topologically closed subspace of $V \hat{\otimes} V'$.

[15.0.2] Lemma: $V \otimes V'$ is a topologically closed subspace of $V \hat{\otimes} V'$ if and only if at least one of V or V' is finite-dimensional, in which case

$$V \otimes V' \;=\; V \hat{\otimes} V'$$

Proof: If V is finite-dimensional, then choose an orthonormal basis $e_1, \ldots, e_n$ for V, and we can express any vector in the tensor product in the form

$$\sum_{i=1}^{n} e_i \otimes w_i$$

for suitable w_i in W. It is easy to see that a sequence of such vectors is a Cauchy sequence if and only if the corresponding vectors w_i form a Cauchy sequence. Thus, for V finite-dimensional the algebraic tensor product is itself already complete.

If, on the other hand, neither V nor W is finite dimensional, let e_i and f_i be orthonormal bases (countable, for simplicity). The vector

$$\sum_{i=1}^{\infty} \frac{1}{n}\, e_i \otimes f_i$$

lies in the completion, but not in the algebraic tensor product. ///

Then the (*internal*) *tensor product* of unitary representations π, V and π', V' of G is defined as above, but restricting the group action to the diagonal copy $\delta(G)$ of G inside $G \times G$. That is,

$$(\pi \otimes \pi')(g)(v \otimes v') \;=\; \pi(g)(v) \otimes \pi'(g)(v')$$

and extend.

[15.0.3] Remark: It is seldom possible to distinguish by purely notation devices between external and internal tensor products. Context is necessary. Some writers use a *square* tensor symbol for external tensor products, but this is not universal.

[15.0.4] Proposition: For two Hilbert spaces V and W, the (completed) tensor product $V^* \hat{\otimes} W$ (with V^* the complex conjugate) is naturally isomorphic to the Hilbert space of *Hilbert-Schmidt operators* $V \to W$ with the Hilbert-Schmidt norm. The (non-completed) $V^* \otimes W$ is naturally isomorphic to the space of finite-rank operators $V \to W$ via

$$(\lambda \otimes w)(v) \;=\; \lambda(v) \cdot w$$

for $\lambda \in V^*$ and $w \in W$, extending linearly.

Proof: The map $\lambda \otimes w$ just defined has image $\mathbb{C} \cdot w$, so is rank 1. Thus, $V^* \otimes W$ consists of finite-rank operators. Conversely, for a finite-rank operator T, let $e_1, \ldots, e_n$ be an orthonormal basis for the image of T. Then take

$$\lambda_i(v) \;=\; \langle Tv, e_i \rangle$$

Then

$$T = \sum_i \lambda_i \otimes v$$

Thus, we really do get all finite-rank operators in this manner. It is straightforward to verify that the tensor product norm and the Hilbert-Schmidt norm agree. ///

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16. *Irreducibility of external tensor products of irreducibles*

[16.0.1] Theorem: Let π, V and π', V' be unitary irreducibles of G and G', respectively. Then $\pi \otimes \pi', V \hat{\otimes} V'$ is an irreducible unitary of $G \times G'$.

[16.0.2] Remark: The converse is not universally true, that is, there are irreducible unitaries of certain groups $G \times G'$ which are not isomorphic to tensor products of irreducibles of G and G'.

Proof: We already constructed the Hilbert space $V \hat{\otimes} V'$ on which $G \times G'$ acts unitarily. In a Hilbert space the orthogonal complement $W^\perp$ of a $G \times G'$-stable subspace W is again stable. Thus, both the orthogonal projection to W and the orthogonal projection to $W^\perp$ are $G \times G'$ maps. They are not scalars. Thus, we have a converse to Schur's lemma, namely that an *reducible* unitary representation of $G \times G'$ has non-scalar $G \times G'$ endomorphisms.

We prove that a $G \times G'$-endomorphism T of $\pi \otimes \pi'$ is scalar. For $V' \in V'$ and $\lambda' \in V'^*$ the map $\varphi_{v',\lambda'} : V \to V$ defined by

$$v \to v \otimes v' \to T(v \otimes v') \stackrel{1_V \otimes \lambda'}{\longrightarrow} V \qquad \text{with} \qquad (1_V \otimes \lambda')(x \otimes y) \;=\; \lambda'(y) \cdot x$$

is a G-map. Thus, by Schur's lemma, since V is irreducible,

$$\varphi_{v',\lambda'}(v) \;=\; \theta_{v',\lambda'} \cdot v \qquad\qquad (\text{for some } \theta_{v',\lambda'})$$

The map $V'^* \to V'*$ by $\lambda' \to (v' \to \theta_{v',\lambda'})$ is a G'-morphism. Since V' is irreducible, V'^* is irreducible, and by Schur's lemma there is a constant c such that

$$\theta_{v',\lambda'} \;=\; c \cdot \langle v', \lambda' \rangle$$

(identifying V'^* with V' conjugate-linearly, as usual). Then for any $\lambda \in V^*$ and $\lambda' \in V'^*$

$$\langle T(v \otimes v'), \lambda \otimes \lambda' \rangle \;=\; \lambda(\theta_{v',\lambda'} \cdot v) \;=\; \lambda(v) \cdot c \cdot \lambda'(v') \;=\; c \cdot \langle v \otimes v', \lambda \otimes \lambda' \rangle$$

Thus, T acts by the scalar c, and we conclude that the tensor product is irreducible. ///

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17. *Decomposition of discrete series*

Still assume that G is unimodular. Fix a closed subgroup Z in the center of G. Let ω be a unitary character (one-dimensional representation) of Z. For π an irreducible unitary representation in the discrete series of G modulo (the closed central subgroup) Z, with character ω on Z, let $L^2(Z\backslash G, \omega)^\pi$ denote the π-isotypic component inside $L^2(Z\backslash G, \omega)$, with the right regular representation R. That is, this is the closure of the sum of all isomorphic copies of π in $L^2(Z\backslash G, \omega)$.

[17.0.1] Theorem: For π in the discrete series with central character ω, the isotypic component $L^2(Z\backslash G,\omega)^\pi$ is stable under the *left* regular representation L, and as $G\times G$-representation with the biregular representation,

$$L^2(Z\backslash G,\omega)^\pi \approx \pi\otimes\pi^*$$

Thus, the discrete spectrum with central character ω is

$$L^2_d(Z\backslash G,\omega) \approx \widehat{\bigoplus}_{\pi \text{ discrete}} \pi\otimes\pi^*$$

where the hat on the sum denotes completion.

[17.0.2] Remark: The Hilbert space corresponding to the tensor product $\pi\otimes\pi'$ is the *completion* inside $L^2(Z\backslash G,\omega)$ of the algebraic tensor product of the two representation spaces.

Proof: For π in the discrete series, the Schur orthogonality relations and inner product relations show that the map

$$\pi\otimes\pi^* \to L^2_d(Z\backslash G,\omega)$$

by extending the map

$$u\otimes v \to c_{u,v}$$

is a $G\times G$ map. Also, for non-isomorphic π and π' the images are orthogonal. What needs to be shown is that there is nothing else in the discrete spectrum than this. In particular, we need to show that the π-isotype is no larger than the image of $\pi\otimes\pi^*$.

If f is in a copy of π inside $L^2(Z\backslash G,\omega)$ but is orthogonal to all coefficient functions $c_{u,v}$ coming from $\pi\otimes\pi^*$, then by the stability of matrix coefficient functions under the left regular representation L we find that $L(g)f$ is still orthogonal to all these coefficient functions.

Take $\varphi\in C^o_c(G)$ real-valued such that (with $\varphi^*(g) \;=\; \varphi(g^{-1})$)

$$\pi(\varphi^*)(f) \;\neq\; 0$$

Existence of approximate identities assures the existence of such. Then

$$\pi(\varphi^*)f(g) \;=\; \int_G f(h^{-1}g)\,\varphi^*(h)\,dh \;=\; \int_G f(hg)\,\varphi(h)\,dh \;=\; \int_{Z\backslash G} f(hg)\,\left(\int_Z \varphi(zh)\,\omega(z)\,dz\right)\,dh$$

Let $\varphi_\omega(h)$ denote the complex conjugate of the last inner integral, and let η be the orthogonal projection of φ_ω to the π-isotype. Then

$$\pi(\varphi^*)f(g) \;=\; c_{f,\varphi_\omega}(g) \;=\; c_{f,\eta}(g)$$

and thus

$$\langle\pi(\varphi^*)f, c_{u,v}\rangle \;=\; \langle c_{f,\eta}, c_{u,v}\rangle \;=\; \frac{1}{d_\pi}\cdot\langle f,\eta\rangle\cdot\overline{\langle\eta,v\rangle}$$

In particular, taking $u=f$ and $v=\eta$ gives a contradiction. Thus, there is no function non-zero f in the π-isotype such that f is orthogonal to all the matrix coefficient functions. That is, $\pi\otimes\pi^*$ is all of the π-isotype. ///

18. $C_c^o(G)$ yields Hilbert-Schmidt operators on discrete series

Take G unimodular, and Z a closed subgroup of the center, ω a unitary character on Z. For $\varphi \in C_c^o(G)$ write

$$\varphi_\omega(g) \;=\; \int_Z \omega(z)^{-1}\,\varphi(zg)\,dz \in C_c^o(Z\backslash G,\omega)$$

For π an irreducible unitary representation in the discrete series with central character ω, let pr_π be the orthogonal projection to the π-isotype $L^2(Z\backslash G,\omega)^\pi$ in $L^2(Z\backslash G,\omega)$, which we now know to be isomorphic to $\pi\otimes\pi^*$ under the biregular representation.

[18.0.1] Theorem: For $\varphi\in C_c^o(G)$ and π, V an irreducible subrepresentation of $L^2(Z\backslash G,\omega)$ (that is, in the discrete series for $Z\backslash G$ and ω) with the right regular representation R, the *Hilbert-Schmidt norm* of $\pi(\overline{\varphi})$ is expressible as

$$\|\pi(\overline{\varphi})\|_{HS} \;=\; \frac{1}{d_\pi}\,|\mathrm{pr}_\pi(\varphi_\omega)|_{L^2(Z\backslash G,\omega)}$$

Proof: Let e_i be an orthonormal basis for V. Write $c_{ij} = c_{e_i,e_j}$ for the corresponding matrix coefficient functions. Then

$$\|\pi(\overline{\varphi})\|^2_{HS} \;=\; \sum_i |\pi(\overline{\varphi})(e_i)|^2 \;=\; \sum_{ij} |\langle \pi(\overline{\varphi})(e_i), e_j\rangle|^2$$

by Plancherel. Further, this is

$$\sum_{ij}\Big|\langle \int_G \overline{\varphi}(g)\,\pi(g)(e_i)\,dg, e_j\rangle\Big|^2 \;=\; \sum_{ij}\Big|\int_G \langle\overline{\varphi}(g)\,\pi(g)(e_i), e_j\rangle\,dg\Big|^2 \;=\; \sum_{ij}\Big|\int_G \overline{\varphi}(g)\,\langle\pi(g)(e_i), e_j\rangle\,dg\Big|^2$$

since the main property of Gelfand-Pettis integrals, that they commute with continuous linear functions, allows us to move the integral out of the inner product. (We are only using the continuous compactly supported case, with values in a Hilbert space.) The inner product inside the integral is exactly the coefficient function c_{ij}, so, more succinctly, this is

$$\sum_{ij}\Big|\int_G \overline{\varphi}(g)\,c_{ij}(g)\,dg\Big|^2 \;=\; \sum_{ij}\Big|\int_G \overline{\varphi}(g)\,c_{ij}(g)\,dg\Big|^2 \;=\; \sum_{ij}\Big|\int_{Z\backslash G} c_{ij}(g)\Big(\int_Z \omega(z)\overline{\varphi_\omega}(zg)\,dz\Big)\;dg\Big|^2$$

$$=\; \sum_{ij}\Big|\int_{Z\backslash G} \overline{\varphi_\omega}(g)\,c_{ij}(g)\,dg\Big|^2 \;=\; \sum_{ij}|\langle c_{ij},\varphi_\omega\rangle|^2 \;=\; \sum_{ij}|\langle \varphi_\omega, c_{ij}\rangle|^2$$

using the usual averaging trick.

The Schur inner product relations directly tell us that the normalizations $\sqrt{d_\pi}c_{ij}$ of the matrix coefficient functions c_{ij} are an orthonormal basis for the π-isotype $\pi\otimes\pi^*$ in $L^2(Z\backslash G,\omega)$. Thus, the last sum of integrals

$$\|\pi(\overline{\varphi})\|^2_{HS} \;=\; \frac{1}{d_\pi}\,|\mathrm{pr}_\pi(\varphi_\omega)|^2_{L^2(Z\backslash G,\omega)}$$

19. *Admissibility for discrete series*

Suppose that the unimodular G has a *compact open subgroup* K. This certainly occurs in p-adic groups and other totally disconnected groups. Then the characteristic function (or indicator function) ch_K of K is in $C_c^o(G)$. Normalize by

$$e = e_K = \frac{\mathrm{ch}_K}{\mathrm{meas}\,(K)}$$

[19.0.1] Theorem: For π, V in the discrete series in $L^2(Z\backslash G, \omega)$, the space of K*-fixed vectors*

$$V^K = \{v \in V : \pi(k)v = v, \text{ for all } k \in K\}$$

is *finite-dimensional.* In fact,

$$\sum_\pi d_\pi \cdot \dim_{\mathbb{C}} V_\pi^K = |e_\omega|^2_{L^2(Z\backslash G,\omega)} < \infty$$

where π is summed over isomorphism classes of (irreducible) discrete series representations with central ω, and e_ω is the averaged

$$e_\omega(g) = \int_Z \omega(z)^{-1}\, e_K(zg)\, dz \in C_c^o(Z\backslash G, \omega)$$

Proof: It is immediate that $\pi(e_K)$ is the identity operator on V^K, and the latter is a closed subspace of V, so is a Hilbert space in its own right. Further $\pi(e_K)$ maps all of V to V^K. By the previous result, $\pi(e_K)$ is Hilbert-Schmidt, so compact. For the identity operator on a Hilbert space to be compact requires that the space be finite dimensional. Further, again by the previous result,

$$\sum_\pi d_\pi \cdot \dim V_\pi^K = \sum_\pi d_\pi \cdot \|\pi(e_K)\|^2_{HS} = \sum_\pi \|\mathrm{pr}_\pi(e_\omega)\|^2 = \|e_\omega\|^2_{L^2(Z\backslash G,\omega)}$$

This is the stronger conclusion. ///

20. *Compactness of operators on compact $Z\Gamma\backslash G$*

Still take G unimodular, Z a closed subgroup of the center, and ω a unitary character of Z. Let Γ be a *discrete* subgroup of G such that $Z\Gamma$ is discrete in $Z\backslash G$, and $\omega = 1$ on $Z \cap \Gamma$. Define

$$C_c^o(Z\Gamma\backslash G, \omega) = \Big\{ f \in C^o(G) : f(z\gamma g) = \omega(z) f(g) \text{ for } z \in Z, \gamma \in \Gamma,\ f \text{ compactly supported left mod } Z\Gamma \Big\}$$

Note that Γ and G are both unimodular, so $\Gamma\backslash G$ and $Z\Gamma\backslash G$ have right G-invariant measures. Let $L^2(Z\Gamma\backslash G, \omega)$ be the completion of $C_c^o(Z\Gamma\backslash G, \omega)$ with respect to the metric arising from the norm attached to the inner product

$$\langle f_1, f_2 \rangle = \int_{Z\Gamma\backslash G} f_1(g)\, \overline{f_2(g)}\, dg$$

The group G acts by right translation on $C_c^o(Z\Gamma\backslash G, \omega)$, and since we integrate with respect to a right G-invariant measure on the quotient $Z\Gamma\backslash G$ the inner product just defined is preserved by this action. Thus, by continuity, the right translation action of G extends to the completion. Let R denote the right translation representation.

Our significant hypothesis now is that

$$Z\Gamma\backslash G \text{ is compact}$$

In particular, this compactness implies that

$$C_c^o(Z\Gamma\backslash G,\omega) \;=\; C^o(Z\Gamma\backslash G,\omega)$$

and that, with the sup norm, this is a *Banach* space (as opposed to $C_c^o(X)$ for a non-compact space X, which is in general only a *colimit* of Banach spaces, and definitely not complete metric).

[20.0.1] Lemma: With $Z\Gamma\backslash G$ compact, for $\varphi \in C_c^o(G)$, the operator $R(\varphi)$ is a compact operator on the Banach space $C_c^o(Z\Gamma\backslash G,\omega)$. Hence, $R(\varphi)$ is a compact operator on the Hilbert space $L^2(Z\Gamma\backslash G,\omega)$.

Proof: The second assertion follows from the first, since the compactness of the quotient implies that (with invariant measure) it has finite total measure, so continuous functions are square-integrable.

Let

$$\psi(g) \;=\; \int_Z \omega(z)\,\varphi(zg)\,dz \qquad\qquad \Psi(g,h) \;=\; \sum_{\gamma\in\Gamma} \psi(g^{-1}\gamma h)$$

[20.0.2] Sublemma: The function Ψ is *uniformly* continuous, and is in

$$C_c^o\big(Z\Gamma\backslash G \times Z\Gamma\backslash G,\; \omega\otimes\overline{\omega}\big)$$

Proof: *(of sublemma)* The point of the assertion of the sublemma is that for g and h both in a fixed compact subset C of $Z\backslash G$, the set

$$\Gamma_C \;=\; \{\gamma\in\Gamma : \psi(g^{-1}\gamma h)\neq 0 \text{ for } \textit{some } g,h\in C\}$$

is *finite* modulo Z, that is, $Z\backslash Z\cdot\Gamma_C$ is finite. That this is so is easy to see:

$$g^{-1}\gamma h \in \mathrm{spt}\psi \quad\text{implies}\quad \gamma \in C\cdot \mathrm{spt}(\psi)\cdot C^{-1}\cap Z\Gamma$$

The set $C\mathrm{spt}(\psi)C^{-1}$ is compact mod Z, and $Z\Gamma$ is discrete mod Z, yielding the result. Therefore, for $g,h\in C$, the sum defining Ψ is a uniformly finite sum of continuous functions, so is continuous. The equivariance under the center is formal. Finally, since $|\omega|=1$ and $Z\Gamma\backslash G$ is *compact* we have *uniform* continuity. ///

Returning to the proof of the lemma, we need to show that for given φ there is a constant c such that

$$\|R(\varphi)f\|_{\sup} \le c\cdot |f|_{L^2(Z\Gamma\backslash G,\omega)} \qquad\qquad (\text{for } f\in C_c^o(Z\Gamma\backslash G,\omega))$$

To this end, note that

$$(R(\varphi)f)(g) \;=\; \int_G \varphi(h)\,f(gh)\,dh \;=\; \int_G \varphi(g^{-1}h)\,f(h)\,dh \;=\; \int_{Z\backslash G} f(h)\,\Big(\int_Z \omega(z)\varphi(g^{-1}zh)\,dz\Big)\,dh$$

$$=\; \int_{Z\backslash G} f(h)\,\psi(g^{-1}h)\,dh \;=\; \int_{Z\Gamma\backslash G} f(h)\,\Bigg(\sum_{\gamma\in Z\backslash Z\Gamma}\psi(g^{-1}\gamma h)\Bigg)\,dh \;=\; \int_{Z\Gamma\backslash G} f(h)\,\Psi(g,h)\,dh$$

Then by Cauchy-Schwarz-Bunyakowsky

$$|R(\varphi)f(g)|^2 \le \int_{Z\Gamma\backslash G}|f|^2\cdot\int_{Z\Gamma\backslash G}|\Psi(g,\cdot)|^2 \le \|f\|_{L^2}^2\cdot\sup|\Psi|^2\cdot\mathrm{meas}\,(Z\Gamma\backslash G)$$

as desired. From this it follows that $R(\varphi)$ maps $L^2(Z\Gamma\backslash G,\omega)$ to $C_c^o(Z\Gamma\backslash G,\omega)$, since L^2 limits are necessarily mapped to C_c^o limits.

Thus, if we can show that $R(\varphi)$ maps the unit ball in $L^2(Z\Gamma\backslash G,\omega)$ to a *uniformly equicontinuous* set of functions in $C_c^o(Z\Gamma\backslash G,\omega)$, then the Arzela-Ascoli theorem would imply that the image by $R(\varphi)$ of this ball is *totally bounded*, so has compact closure, proving that $R(\varphi)$ is a compact operator. The computation above also yields

$$|R(\varphi)f(g') - R(\varphi)f(g)|^2 \le \|f\|^2_{L^2} \cdot \int_{Z\Gamma\backslash G} |\Psi(g',h) - \Psi(g,h)|^2\, dh$$

The uniform continuity of Ψ and the finite measure of the quotient $Z\Gamma\backslash G$ complete the argument. ///

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21. *Discrete decomposition of compact-quotient* $L^2(Z\Gamma\backslash G,\omega)$

Keep the previous assumptions. In particular, $Z\Gamma\backslash G$ is compact.

[21.0.1] Theorem: Under the right translation action R of G, the Hilbert space $L^2(Z\Gamma\backslash G,\omega)$ is the completed direct sum of π-isotypic subspaces $L^2(Z\Gamma\backslash G,\omega)^\pi$ where π varies over isomorphism classes of irreducible unitary representations. Further, the multiplicity of each such π is *finite*. Thus,

$$L^2(Z\Gamma\backslash G,\omega) \;\approx\; \widehat{\bigoplus_\pi}\, m_\pi\cdot\pi \qquad\qquad (\text{with each } m_\pi \text{ } \textit{finite})$$

Proof: This will be a corollary of the compactness of the operators $R(\varphi)$ proven above for $\varphi\in C_c^o(G)$.

Take $\varphi\in C_c^o(G)$ such that $\varphi^*=\varphi$, where $\varphi^*(g)=\overline{\varphi(g^{-1})}$. Then $R(\varphi)$ is a self-adjoint compact operator on $X=L^2(Z\Gamma\backslash G,\omega)$, so gives a decomposition

$$X \;=\; \widehat{\bigoplus_\lambda}\, X(\lambda)$$

where $X(\lambda)$ is the λ-eigenspace, and is finite-dimensional for $\lambda\neq 0$. We need

[21.0.2] Lemma: Take $v\neq 0$ in $X(\lambda)$ with $\lambda\neq 0$. Let W be the closure of the subspace of X spanned by $R(g)v$ for $g\in G$. Then W is Artinian as a G-space, that is, a descending chain of closed G-stable subspaces must stabilize after finitely-many steps. In particular, W contains a non-zero irreducible closed G-space.

Proof: *(of lemma)* For closed G-subspaces $W_1\subset W_2\subset W$ with $W_1\neq W_2$ we will show that

$$W_1(\lambda) \not= W_2(\lambda)$$

Let p_i be the orthogonal projection to W_i. These are G-maps, so commute with $T=R(\varphi)$. If $W_1(\lambda)=W_2(\lambda)$, then necessarily $p_1(v)=p_2(v)$, so

$$R(G)\cdot p_i(v) \;=\; p_i(R(G)\cdot v) \;=\; p_i(\text{dense subspace of } W) \;=\; \text{dense subspace of } W_i$$

Since the W_i are closed, they are equal. Since the λ-eigenspaces are finite-dimensional, in effect we have shown that the dimension of the λ-eigenspaces may be used to unambiguously index closed G-subspaces of W. Thus we obtain the Artinian-ness. In particular, we can take X_o to be a minimal non-zero closed subspace of W, thus an irreducible G-space. ///

Returning to the proof of the theorem, let $\{X_\alpha\}$ be a maximal set of irreducible subrepresentations such that $\sum_\alpha X_\alpha = \bigoplus_\alpha X_\alpha$. (Invoke Zorn's lemma.) Suppose that the closure of this sum is *not* the whole space X. Let X' be the orthogonal complement of this closure. Existence of approximate identities assures that there is $\varphi\in C_c^o(G)$ such that $R(\varphi\neq 0$ on X'. Then either $R(\varphi+\varphi^*)$ or $R(\varphi-\varphi^*)/i$ is non-zero, and yields a non-zero self-adjoint compact operator on X'. Then the lemma implies that X' has a non-zero irreducible subspace, contradicting maximality. That is, $X = L^2(Z\Gamma\backslash G, \omega)$ is the completion of a direct sum of irreducibles.

To see that a π-isotype is of finite multiplicity, again use compactness of the operators from $C_c^o(G)$. With $\varphi = \varphi^*$, $R(\varphi)$ acts by scalar λ on $X^\pi(\lambda)$. Non-zero scalar operators are compact only for finite-dimensional spaces, so $X^\pi(\lambda)$ is finite-dimensional for $\lambda \neq 0$. Given π, take $\varphi = \varphi^*$ such that $R(\varphi)$ is non-zero on the representation space V_π of π, invoking the existence of approximate identities. Then $V_\pi(\lambda) \neq 0$ for some non-zero λ. The previous finite-dimensionality result gives

$$\text{multiplicity of } \pi \;=\; \frac{\dim_{\mathbb{C}} X^\pi(\lambda)}{\dim_{\mathbb{C}} V_\pi(\lambda)} < \infty$$

This finishes the proof of the theorem. ///

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22. *Peter-Weyl theorem for compact quotients*

Keep the assumptions from above, in particular that $Z\Gamma\backslash G$ is compact. Note that this includes the case that G is compact and Z and Γ are trivial, thus incidentally applying to compact G. But the case that Γ is non-trivial is more interesting.

[22.0.1] Theorem: Given $f \in C_c^o(Z\Gamma\backslash G, \omega) = C^o(Z\Gamma\backslash G, \omega)$, and given $\varepsilon > 0$, there is a finite set F of irreducibles π occurring in $X = L^2(Z\Gamma\backslash G, \omega)$ and *continuous* f_π in the π-isotype X^π of X, such that

$$\sup_{g \in G} \Big| f(g) - \sum_{\pi \in F} f_\pi \Big| \;<\; \varepsilon$$

Proof: First, we need

[22.0.2] Lemma: For f *uniformly* continuous on G, and for an approximate identity φ_i in $C_c^o(G)$ (in the strong sense, namely that the supports shrink to $\{1\}$), the functions

$$f_i(g) \;=\; R(\varphi_i)f(g) \;=\; \int_G f(gh)\,\varphi_i(h)\,dh$$

tend to f in sup norm.

Proof: (of lemma) Given $\varepsilon > 0$, take a neighborhood U of 1 such that $|f(gu) - f(g)| < \varepsilon$ for all $g \in G$ and $u \in U$. Take φ_i with support inside U. Then for all $g \in G$, since the integral of φ_i is 1,

$$|f_i(g) - f(g)| \;=\; \Big|\int_G \varphi_i(h)\,(f(gh) - f(g))\;dh\Big| \;<\; \int_G \varphi_i \cdot \varepsilon \;=\; \varepsilon$$

This proves the lemma. ///

Thus, given $f \in C_c^o(Z\Gamma\backslash G, \omega)$, f is necessarily uniformaly continuous since $|\omega| = 1$ and $Z\Gamma\backslash G$ is compact, from the lemma take $\varphi \in C_c^o(G)$ such that

$$\sup |f - R(\varphi)f| < \varepsilon$$

On the other hand, f has an L^2 expansion

$$f \;=\; \sum_{\text{all } \pi} f_\pi \quad (f_\pi \in X^\pi)$$

From above, there is a constant c such that for all $\psi \in X$ the image $R(\varphi)\psi$ is in $C_c^o(Z\Gamma\backslash G, \omega)$ and

$$\|R(\varphi)\psi\|_{\sup} \leq C \cdot \|\psi\|_{L^2}$$

Take a finite set F of π's such that

$$\|f - \sum_{\pi \in F} f_\pi\| < \varepsilon/C$$

Then

$$\|R(\varphi)f - \sum_{\pi \in F} R(\varphi)f_\pi\|_{\sup} < \varepsilon$$

By remarks above, each $R(\varphi)f_\pi$ is continuous, and since $R(\varphi)$ stabilizes isotypes is still in X^π. ///

[22.0.3] Corollary: In the above situation, for $g \in G$, g not in $Z\Gamma$, for any unitary ω there exists π occurring in $L^2(Z\Gamma\backslash G, \omega)$ such that $\pi(g) \neq 1$.

Proof: Since g is not in $Z\Gamma$, there is $f \in C_c^o(Z\Gamma\backslash G, \omega)$ such that $f(g) = 1$ and $f(1_G) = 0$. Take a finite sum $\sum_{\pi \in F} f_\pi$ with f_π in the π-isotype in $L^2(Z\Gamma\backslash G, \omega)$ such that

$$\sup \left| f - \sum_{\pi \in F} f_\pi \right| \; < \; \frac{1}{2}$$

Then some f_π must have different values at 1_G and g. Thus,

$$R(g)f_\pi(1) \; = \; f_\pi(g) \; \neq \; f_\pi(1)$$

so $R(g)f_\pi \neq f_\pi$ pointwise. Both f_π and $R(g)f_\pi$ are continuous, so $R(g)f_\pi \neq f_\pi$ in an L^2 sense. ///

23. *Admissibility for compact quotients*

Suppose that G has a compact open subgroup K, that $Z\Gamma\backslash G$ is compact, and ω is a unitary character of Z.

[23.0.1] Corollary: For π irreducible unitary occuring in $L^2(Z\Gamma\backslash G, \omega)$ with $Z\Gamma\backslash G$ compact, for a compact open subgroup K of G, the space

$$V_\pi^K \; = \; \{v \in V_\pi : \pi(k)v = v \text{ for all } k \in K\}$$

of K-invariant vectors in the representation space V_π of π is *finite-dimensional*:

$$\dim_{\mathbb{C}} V_\pi^K < \infty$$

Proof: As earlier, the trick is that we can let

$$\varphi \; = \; \text{ characteristic function of } K/\text{meas}\,(K)$$

so the right-translation action $R(\varphi)$ is the identity on V_π^K. From above, $R(\varphi)$ is compact. A non-zero scalar operator is compact only on finite-dimensional spaces, so we have the finiteness conclusion. ///

24. *Unitarizability of representations of compact groups*

We specialize our considerations to compact G.

[24.0.1] Theorem: Let $V, \langle,\rangle$ be a Hilbert space, G a compact group, and $\pi : G \times V \to V$ a continuous, not necessarily unitary, representation. Then there is another inner product $\langle,\rangle_{\text{new}}$ on V such that the topology given by the new inner product on V is the same as the old topology. That is, there are $0 < c < c' < \infty$ such that for all $v \in V$

$$c \cdot \bar{v}, v\rangle \le \langle v, v\rangle_{\text{new}} \le c' \cdot \langle v, v\rangle$$

and π is *unitary* with respect to the new inner product.

Proof: By continuity, for all v the set $\{\pi(g)v : g \in G\}$ is compact (in V), so bounded. Thus, by Banach-Steinhaus (uniform boundedness), the operators $\pi(g)$ are *uniformly equicontinuous.* Thus, given $\varepsilon > 0$, the exists $\delta > 0$ such that for all $v \in V$ and for all $g \in G$

$$|v| < \delta \quad \text{implies} \quad |\pi(g)v| < \varepsilon$$

Take $\varepsilon = 1$. With the corresponding $\delta > 0$

$$|v| \le 1 \quad \text{implies} \quad |\pi(g)v| \le \frac{1}{\delta}$$

Thus, the operator norm of $\pi(g)$ is at most $1/\delta$ for all g. That is,

$$|\pi(g)v| \le \frac{1}{\delta}\,|v|$$

Replacing g by g^{-1} and v by $\pi(g)v$, we also have

$$|v| \le \frac{1}{\delta}\,|\pi(g)v|$$

which yields

$$\delta\,|v| \le |\pi(g)v|$$

Therefore,

$$\delta^2\,|v|^2 \cdot \text{meas}\,(G) \le \int_G |\pi(g)v|^2\,dg \le \frac{1}{\delta^2}\,|v|^2 \cdot \text{meas}\,(G)$$

Let

$$|v|^2_{\text{new}} \;=\; \int_G |\pi(g)v|^2\,dg$$

This is clearly G-invariant, and the inequality shows that the topologies are the same. Of course, not every norm arises from an inequality. We must show that

$$|u+v|^2_{\text{new}} - |u-v|^2_{\text{new}} \;=\; 2\,|u|^2_{\text{new}} + 2|v|^2_{\text{new}}$$

But this just follows from integrating the corresponding identity for the old norm. ///

Therefore, all our prior discussion of unitary representations applies to arbitrary (continuous) Hilbert-space representations of compact groups.

[24.0.2] Corollary: A finite-dimensional continuous representation of compact G is a direct sum of irreducibles. Any finite-dimensional representation of compact G is unitarizable.

Proof: From basic functional analysis, a finite-dimensional topological vector space has a unique Hausdorff topology such that vector addition and scalar multiplication are continuous. Thus, without loss of generality we may suppose that the representation space is $\mathbb{C}^n$ with the usual inner product.

Certainly finite-dimensional spaces are Artinian, so an easy induction on dimension proves that finite-dimensional G-spaces decompose as orthogonal direct sums of irreducibles. ///

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25. *Compact G and compact $Z\backslash G$*

The previous results are decisive for G and $Z\backslash G$ compact, with Z a closed subgroup of the center of G. One can think of the simple case in which $Z = \{1\}$ if one wants, but we will not treat this separately. The following assertions are special cases or nearly immediate corollaries of prior results. These results are often proven in their own right, but here it is economical to obtain them as corollaries.

[25.0.1] Corollary: Let $Z\backslash G$ be compact. (Necessarily G is unimodular.) Then
(i) Every irreudible unitary π of $Z\backslash G$ is finite-dimensional, and is in the discrete series $L^2(Z\backslash G, \omega)$ for the unitary character ω obtained by restricting π to the closed central Z.
(ii) For a unitary character ω of Z, the biregular representation of $G \times G$ on $L^2(Z\backslash G, \omega)$ decomposes the latter Hilbert space as

$$L^2(Z\backslash G, \omega) \approx \widehat{\bigoplus}_{\pi|_Z=\omega} \pi \otimes \pi^*$$

and the right regular representation decomposes as

$$L^2(Z\backslash G, \omega) \approx \widehat{\bigoplus}_{\pi|_Z=\omega} \dim\pi \cdot \pi$$

(iii) *(Peter-Weyl)* The subspace $\pi \otimes \pi^*$ of $L^2(Z\backslash G, \omega)$ consists entirely of *continuous* functions. Given $f \in C_c^o(Z\backslash G, \omega)$, and given $\varepsilon > 0$, there is a *finite* set F of irreducibles π and $f_\pi \in \pi \otimes \pi^*$ such that

$$\sup_{g\in G} |f(g) - \sum_{\pi\in F} f_\pi| < \varepsilon$$

(iv) For an irreducible unitary π of G,

$$\text{formal degree of } \pi \;=\; \frac{\dim_{\mathbb{C}} \pi}{\text{meas}\,(Z\backslash G)}$$

Proof: The fundamental point here is that the compactness of $Z\backslash G$ implies that *every* matrix coefficient function c_{uv} is square integrable modulo the center. Thus, all irreducibles are discrete series representations.

And with $\Gamma = \{1\}$, the quotient $Z\Gamma\backslash G = Z\backslash G$ is compact, so every irreducible occurs with finite multiplicity. Then the decomposition of the biregular representation

$$L^2(Z\backslash G, \omega) \approx \widehat{\bigoplus}_\pi \pi \otimes \pi^*$$

shows that the multiplicity of π is $\dim\pi^*$. Thus, the dimension of π^*, hence of π, is finite.

Since by the existence of approximate identitities there is π such that $R(\varphi)\pi \otimes \pi^*$ is dense in $\pi \otimes \pi^*$, and since $\pi \otimes \pi^*$ is finite-dimensional, the image is the whole $\pi \otimes \pi^*$. And each $R(\pi)f$ is continuous. Then the remainder of the Peter-Weyl assertion follows from the general case.

Last, let e_i be an orthonormal basis for a (finite-dimensional, from above) irreducible unitary π, and write $c_{ij} = c_{e_ie_j}$. Then by Plancherel-Parseval and unitariness

$$\sum_{ij} |c_{ij}|^2 \;=\; \sum_i \sum_j |\langle \pi(g)e_i, e_j\rangle|^2 \;=\; \sum_i |\pi(g)e_i|^2 \;=\; \sum_i |e_i|^2 \;=\; \sum_i 1 \;=\; \dim\pi$$

Then

$$\int_{Z\backslash G} \sum_{ij} |c_{ij}(g)|^2 \, dg \;=\; \dim\pi \cdot \text{meas}\,(Z\backslash G)$$

On the other hand, by the Schur inner-product relations,

$$\int_{Z\backslash G} \sum_{ij} |c_{ij}(g)|^2 \, dg \;=\; \sum_{ij} \frac{1}{d_\pi} \;=\; \frac{(\dim\pi)^2}{d_\pi}$$

Comparing these two equalities gives the assertion. ///

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26. *Induced representations with* $\Delta_H = \Delta_G|_H$

Our point is simply to give the definition in this case, and see that it gives a means of constructing unitary representations of G from unitary representations of H.

Let H be a closed subgroup of G with the modular function condition $\Delta_H = \Delta_G|_H$ met, so that there is a right G-invariant measure on $H\backslash G$. Let σ, V_σ be a unitary (not necessarily irreducible) representation of H, and let

$$C_c^o(H\backslash G,\sigma) \;=\; \begin{Bmatrix}\text{continuous } V_\sigma\text{-valued functions } f \text{ on } G, \text{ compactly supported left} \\ \text{modulo } H, \text{ such that } f(hg)=\sigma(h)f(g) \text{ for all } h\in H \text{ and } g\in G\end{Bmatrix}$$

We let G act by the right translation action R. For f_1 and f_2 in $C_c^o(H\backslash G,\sigma$, define an inner product by

$$\langle f_1, f_2\rangle \;=\; \int_{H\backslash G} \langle f_1(g), f_2(g)\rangle_\sigma \, dg$$

The unitariness of σ assure that the integrand depends only upon the coset Hg. And it is clear that the right translation action of G preserves this inner product. Let

$$\text{Ind}_H^G\,\sigma \;=\; \text{Ind}_H^G\,\sigma \;=\; L^2(H\backslash G,\sigma)$$

be the completion of $C_c^o(H\backslash G,\sigma)$ with respect to the metric associated to the norm associated to this inner product. This is a unitary representation of G.

For example, with $\sigma = 1$ on $V_\sigma = \mathbb{C}$, this simplifies to

$$\text{Ind}_H^G\, 1 \approx L^2(H\backslash G)$$

[26.0.1] Remark: There are many other genres of induced representations, hence an inevitable need for clarification from context.

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27. *Principal series* $\text{Ind}_P^G\,\sigma\Delta^{1/2}$

Now drop the modular function condition for a right-invariant measure on $P\backslash G$ for a closed subgroup P of G. Instead, suppose that G is unimodular, and that there is a compact subgroup K of G such that we have an *Iwasawa decomposition*

$$G \;=\; P\cdot K \;=\; \{pk : p\in P, k\in K\}$$

Let Δ denote the modular function of P. For f_1 and f_2 in

$$C_c^o(P\backslash G,\Delta^{1/2}\sigma) \;=\; \begin{Bmatrix}\text{continuous } V_\sigma\text{-valued functions } f \text{ on } G, \text{ compactly supported left} \\ \text{modulo } P, \text{ such that } f(pg)=\Delta^{1/2}\sigma(p)f(g) \text{ for } p\in P \text{ and } g\in G\end{Bmatrix}$$

define

$$\langle f_1, f_2\rangle \;=\; \int_K \langle f_1(g), f_2(g)\rangle_\sigma \, dk$$

Note that the integration is over the compact subgroup K, not over the whole group G. While it is clear that G acts by right translations R on $C^o_c(P\backslash G, \Delta^{1/2}\sigma)$, it is not at all clear that G preserves the inner product defined just above. In fact, it might not be entirely clear that this pairing is positive-definite.

[27.0.1] Theorem: The inner product defined above is G-invariant and positive-definite, and the completion is a unitary representation of G.

Proof: We need

[27.0.2] Lemma: The functional on $C^o_c(G)$ defined by

$$f \to \int_P \int_K f(pk)\, \frac{dp}{\Delta(p)}\, dk$$

with right Haar measure dp and dk on P and K is a right Haar integral. That is, it is invariant under right translation of f by elements of G.

Proof: *(of lemma)* The group $P \times K$ acts transitively on $G = PK$ by $(p, k)(g) = pgk^{-1}$, and the isotropy group of 1_G is

$$\Theta \;=\; \{(\theta, \theta) : \theta \in P \cap K\} \subset P \times K$$

which is compact since P is closed and K is compact. Then

$$\Delta_\Theta \;=\; 1 \;=\; \Delta_{P\times K}|\Theta$$

since Θ is compact, so $G \approx (P \times K)/\Theta$ has a unique right $P \times K$-invariant measure μ (up to constants). Let q be the quotient map

$$P \times K \to (P \times K)/\Theta \approx G$$

The lemma just below shows that this map is a homeomorphism. Compute

$$\int_G f(g)\, d\mu(g) \;=\; \int_P \int_K f(q(p, k))\, \frac{dp}{\Delta(p)}\, dk$$

since $\frac{dp}{\Delta(p)}$ is a *left* Haar measure on P. This is

$$\int_P \int_K f(pk^{-1})\, \frac{dp}{\Delta(p)}\, dk \;=\; \int_P \int_K f(pk)\, \frac{dp}{\Delta(p)}\, dk$$

since K is unimodular (being compact). On the other hand, the *usual* Haar integral on G also is left P-invariant and right K-invariant. The uniqueness result proves that the double integral in terms of P and K must be a multiple of the usual Haar integral. ///

Returning to the proof of the theorem, the function

$$\Psi(g) \;=\; \langle f_1(g), f_2(g)\rangle$$

is in $C^o_c(P\backslash G, \Delta)$, so by the general result on surjectivity of averaging maps there is ψ in $C^o_c(G)$ such that

$$\Psi(g) \;=\; \int_P \Delta(p)^{-1}\, \psi(pg)\, dp$$

Then

$$\langle R(g)f_1, R(g)f_2\rangle \;=\; \int_K R(g)\Psi(k)\,dk \;=\; \int_K \Psi(kg)\,dk \;=\; \int_K\int_P \psi(pkg)\,\frac{dp}{\Delta(p)}\,dk \;=\; \int_G \psi(hg)\,dh \;=\; \int_G \psi(h)\,dh$$

by the lemma, and then replacing h by hg^{-1}. Thus, the inner product is G-invariant. The positive-definiteness follows similarly, and the unitariness on the completion is then clear. ///

Finally, we prove the following standard lemma on comparison of topologies.

[27.0.3] Lemma: Let G be a locally compact, Hausdorff topological group, with a countable basis. Let X be a Hausdorff topological space with a continuous transitive action of G upon it. (The map $G \times X \to X$ is continuous.) Let x_0 be a fixed element of X, and let

$$G_{x_0} \;=\; \{g \in G : gx_0 = x_0\}$$

be the *isotropy group* of x_0 in G. Then the natural map

$$G/G_{x_0} \to X$$

by $gG_{x_0} \to gx_0$ is a homeomorphism.

Proof: The map $gG_{x_0} \to gx_0$ is a continuous bijection, by assumption. We need to show that it is *open*. Let U be an open subset of G, and take a compact neighborhood V of $1 \in G$ so that $V^{-1} = V$ and $gV^2 \subset U$ for fixed $g \in U$. Since G has a countable basis, there is a countable list $g_1, g_2, \ldots$ of elements of G so that $G = \bigcup_i g_i V$. Let $W_n = g_n V x_o$. By the transitivity, $X = \bigcup_i W_i$. Now W_n is compact, being a continuous image of a compact set, so is closed since X is Hausdorff.

Since X is locally compact and Hausdorff, by Urysohn's Lemma it is *regular*. In particular, if no W_n contained an open set, then there would be a sequence of non-empty open sets U_n with compact closure in X so that

$$U_{n-1} - W_{n-1} \supset \bar{U}_n$$

and

$$\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \ldots$$

Then $\bigcap \bar{U}_i \neq \emptyset$, yet this intersection fails to meet any W_n, contradiction.

Therefore, some $W_m = g_m V x_0$ contains an open set S of X. For $h \in V$ so that $hx_0 \in S$,

$$gx_0 \;=\; gh^{-1}hx_0 \in gh^{-1}S \subset gh^{-1}Vx_0 \subset gV^{-1} \cdot Vx_0 \subset Ux_0$$

Therefore, gx_0 is an interior point of Ux_0, for all $g \in U$. ///

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28. *Frobenius reciprocity for discrete series*

We introduce only a special case of Frobenius reciprocity provable in this general setting, and applicable to unitary representations.

Let Z be a closed central subgroup of unimodular G, and $Z \subset H \subset G$ for another closed subgroup H with $Z\backslash H$ compact. Fix a unitary character ω of Z. Let π be an irreducible subrepresentation of $L^2(Z\backslash G, \omega)$, that is, in the *discrete series* of G with central character ω. Let σ be an irreducible unitary of H (hence, from above, finite-dimensional, since $Z\backslash H$ is compact).

[28.0.1] Theorem:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\,\sigma) \approx_{\mathbb{C}} \mathrm{Hom}_H(\pi|_H, \sigma)$$

Proof: We need

[28.0.2] Lemma: Letting H act on $L^2(Z\backslash G, \omega)$ by left translation L, we have a natural isomorphism

$$\mathrm{Ind}_H^G\, \sigma \approx (L^2(Z\backslash G, \omega) \otimes \sigma)^H$$

That is, the induced representation is the H-fixed vectors in the indicated tensor product.

Proof: *(of Lemma)* Since $Z\backslash H$ is compact,

$$L^2(H\backslash G, \sigma) \subset L^2(Z\backslash G, \omega)$$

Map

$$L^2(Z\backslash G, \omega) \otimes \sigma)^H \to L^2(Z\backslash G, \omega)$$

by

$$(f \otimes v)(g) \;=\; f(g) \cdot v$$

Since σ is finite-dimensional the conclusion of the lemma is immediate and formal. ///

For the theorem, using the lemma,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \;=\; \mathrm{Hom}_G(\pi, L^2(H\backslash G, \sigma)) \;=\; \mathrm{Hom}_G(\pi, (L^2(Z\backslash G, \omega) \otimes \sigma)^H)$$

$$=\; \mathrm{Hom}_G(\pi, \pi - \text{isotype in } (L^2(Z\backslash G, \omega) \otimes \sigma)^H) \;=\; \mathrm{Hom}_G(\pi, ((\pi \otimes \pi^*) \otimes \sigma)^H)$$

since the actions of G and H commute, by our decomposition results for discrete series in the biregular representation. Continuing, this is

$$\mathrm{Hom}_G(\pi, \pi \otimes (\pi^* \otimes \sigma)^H) \approx (\pi^* \otimes \sigma)^H \approx \mathrm{Hom}_H(\pi|_H, \sigma)$$

since σ is finite-dimensional. ///

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29. *Traces, characters, central functions for $Z\backslash G$ compact*

Take $Z\backslash G$ compact with Z a closed subgroup of the center of G (so G is unimodular). From above, every irreducible unitary of G is finite dimensional, with some unitary central ω on Z.

[29.0.1] Definition: Let π, V be an irreducible (finite-dimensional) of G. The *character* χ_π of π is the function on G defined by

$$\chi_\pi(g) \;=\; \text{trace } \pi(g) \;=\; \sum_i \langle \pi(g)e_i, e_i \rangle$$

for any orthonormal basis e_i of π.

[29.0.2] Remark: If π were not finite-dimensional, the character χ_π of π could not be defined as a pointwise-evaluatable function, but only as some more general sort of entity. Such a discussion would require more structure on G, as well as hypotheses on π.

[29.0.3] Definition: The *central functions* $L^2_{\text{cen}}(Z\backslash G, \omega)$ in $L^2(Z\backslash G, \omega)$ are the conjugation-invariant functions

$$L^2_{\text{cen}}(Z\backslash G, \omega) \;=\; \{f \in L^2(Z\backslash G, \omega) : f(h^{-1}gh) = f(g) \text{ for } h, g \in G\}$$

(The equalities are in the almost-everywhere sense.) Since the defining conditions are *closed* conditions, this is a closed subspace of $L^2(Z\backslash G, \omega)$, though it is not generally G-stable. Of course it *is* stable under the conjugation action of G

$$\pi_{\text{conj}}(h)f(g) \;=\; f(h^{-1}gh)$$

Also, $L^2(Z\backslash G,\omega)$ is a unitary representation of G with the conjugation action, and so $L^2_{\rm cen}(Z\backslash G,\omega)$ is unitary under this action, as well. The central character under the conjugation action is trivial.

Since trace is invariant under conjugation, it is immediate that each χ_π is a central function.

[29.0.4] **Theorem:** The collection

$$\{\chi_\pi : \pi \text{ irreducible }, \pi|Z=\omega\}$$

is an orthogonal basis for $L^2_{\rm cen}(Z\backslash G,\omega)$, and

$$\langle \chi_\pi, \chi_\pi\rangle \;=\; \text{meas}\,(Z\backslash G)$$

Proof: The orthogonality and inner products' values follow from the expression of χ_π in terms of matrix coefficient functions, from Schur's inner product relations, and from the fact that $d_\pi = \dim(\pi)/\text{meas}\,(Z\backslash G)$ for $Z\backslash G$ compact.

Since by the compactness of $Z\backslash G$

$$L^2(Z\backslash G,\omega) \;=\; \widehat{\bigoplus}_{\pi|_Z=\omega}\, \pi\otimes\pi^*$$

and $\pi\otimes\pi^*$ is conjugation-stable, it suffices to show that the central functions in $\pi\otimes\pi^*$ are exactly the multiples of χ_π. From above the central functions in $\pi\otimes\pi^*$ (using finite-dimensionality of π, V)

$$(V\otimes V^*)^G \approx \text{Hom}_G(V,V)\approx \mathbb{C}$$

since π is irreducible, invoking (an easy case of) Schur's lemma. That is, the space of central functions in $\pi\otimes\pi^*$ is one-dimensional, so must be just $\mathbb{C}\cdot\chi_\pi$. ///

[29.0.5] **Corollary:** With G as above, the characters of mutually non-isomorphic irreducible unitary representations are linearly independent. ///

[29.0.6] **Corollary:** Two irreducible unitary representations of G are isomorphic if and only if their characters are equal. ///

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30. *Complete reducibility for $Z\backslash G$ compact*

Let σ, V_σ be an arbitrary (not necessarily irreducible, nor necessarily finite-dimensional) unitary representation of G, such that $\sigma|_Z$ is scalar ω, where ω is a unitary character of the closed central subgroup Z, and $Z\backslash G$ is compact. The following theorem asserts that σ is *completely reducible*, meaning that it is a (completed) direct sum of irreducibles.

[30.0.1] **Theorem:** With these hypotheses, σ decomposes as a (completed) orthogonal direct sum of its isotypic components, as

$$V_\sigma \approx \widehat{\bigoplus}_\pi\, V_\sigma^\pi$$

where π runs over isomorphism classes of irreducible unitary representations of G with central character ω, and V_σ^π is the π-isotypic component of V_σ. The orthogonal projector to the π-isotypic component is

$$q_\pi(v) \;=\; \int_{Z\backslash G} \overline{\chi_\pi(g)}\,\sigma(g)v\,dg\,/\,\text{meas}\,(Z\backslash G)$$

[30.0.2] Remark: Decomposition as a *sum* rather than a more general *integral* of irreducibles can be guaranteed only under various relatively special hypotheses. This discreteness property fails in very simple circumstances, such as $L^2(\mathbb{R})$ with the translation action of $\mathbb{R}$. The Fourier Inversion formula exactly expresses this Hilbert space as an *integral* of one-dimensional spaces, each spanned by $x \to e^{i\xi x}$, but these exponential functions do not themselves lie inside $L^2(\mathbb{R})$.

Proof: The orthogonal complement of the indicated sum is certainly a G subspace of V_σ. Thus, it suffices to show that if $V_\sigma^\pi = 0$ for all π then $V_\sigma = 0$.

Fix $w \in V_\sigma$. Define $T : V_\sigma \to L^2(Z\backslash G, \omega)$ by

$$Tv(g) \;=\; \langle \sigma(g)v, w\rangle$$

We have continuity:

$$|Tv(g)|^2_{L^2(Z\backslash G,\omega)} \;=\; \int_{Z\backslash G} |\langle\sigma(g)v, w\rangle|^2\, dg \le \int_{Z\backslash G} |\sigma(g)v|^2 \cdot |w|^2\, dg \;=\; \text{meas}\,(Z\backslash G)\cdot |v|^2\cdot|w|^2$$

by Cauchy-Schwarz-Bunyakowsky and unitariness, and using the finiteness of the measure of $Z\backslash G$. Since we have the decomposition

$$L^2(Z\backslash G, \omega) \;=\; \widehat{\bigoplus_\pi}\, \pi\otimes\pi^*$$

(for π running over irreducibles with central character ω), T composed with *some* orthogonal projector q_π to an isotypic subspace $\pi\otimes\pi^*$ must be non-zero. Then $q_\pi \circ T$ gives a non-zero isomorphism from $\ker(q_\pi\circ T)^\perp$ to a π-isotypic subspace, contradiction.

To prove the last assertion about orthogonal projections: the integral for $q\pi$ exists in a strong sense: as $Z\backslash G$ is *compact* there is $\varphi \in C^o_c(G)$ such that

$$\chi_\pi(g) \;=\; \int_Z \omega(z)^{-1}\,\varphi(zg)\, dz$$

by the surjectivity of these averaging maps (proven earlier). Thus, by integration theory,

$$q_\pi \;=\; \sigma(\varphi/\text{meas}\,(Z\backslash G))$$

Therefore, it suffices to take v in a copy of $V_{\pi'}$ inside V_σ where π' is an irreducible, and show that

$$q_\pi(v) \;=\; \begin{cases} 0 & \pi\not\approx\pi' \\ v & \pi\approx\pi'\end{cases}$$

To this end, we compute

$$\text{meas}\,(Z\backslash G)\cdot q_\pi(\sigma(g)v) \;=\; \int_{Z\backslash G} \overline{\chi_\pi}(h)\,\sigma(hg)v\, dh \;=\; \int_{Z\backslash G} \overline{\chi_\pi}(h)\,\sigma(gh)v\, dh$$

by replacing h by ghg^{-1}, using the central-ness of χ_π. This makes visible the fact that the map q_π is a G-hom, so on any irreducible $V_{\pi'}$ is a scalar (by Schur). To evaluate this scalar, we may as well compute inside $L^2(Z\backslash G,\omega)$, for example on the function $\chi_{\pi'}$. The latter is a continuous function, so has unambiguous pointwise values (unlike an L^2 function), so the scalar is 0 for $\pi\not\approx\pi'$, and for $\pi\approx\pi'$ it is

$$\frac{q_\pi(\chi_{\pi'})(1)}{\chi_{\pi'}(1)} \;=\; \frac{\int_{Z\backslash G}\chi_{\pi'}(g)\,\overline{\chi_\pi}(g)\,dg}{\text{meas}\,(Z\backslash G)\cdot\dim\pi'} \;=\; \frac{(\dim\pi)^2}{d_\pi}\cdot\frac{1}{\text{meas}\,(Z\backslash G)\cdot\dim\pi} = 1$$

by Schur's inner product relations. ///

31. *Tensor factorization of irreducibles of $G \times G'$*

We showed earlier that (external) tensor products $\pi \otimes \pi'$ of irreducible unitaries π and π' of G and G' yield irreducible unitaries of $G \times G'$. The converse is not true without some additional hypotheses on G or G'. The easiest first case is the following.

[31.0.1] Theorem: Suppose that $Z\backslash G$ is compact, for a closed subgroup Z of the center of G. Then any irreducible unitary σ of $G \times G'$ is isomorphic to $\pi \otimes \pi'$ for an irreducible (finite-dimensional) unitary π of G and an irreducible unitary π' of G'.

Proof: From Schur's lemma, σ is scalar on the center of $G \times G'$. Thus, from the previous result for some π, V irreducible unitary of G the π-isotype σ^π is non-zero. As V is finite-dimensional (since $Z\backslash G$ is compact),

$$\mathrm{Hom}_G(V, V_\sigma) \approx (V^* \otimes V_\sigma)^G$$

And $V^* \otimes V$ is a Hilbert space (since V^* is finite-dimensional), so its fixed vectors $(V^* \otimes V_\sigma)^G$ under G are a Hilbert space as well, being the intersection of closed linear subspaces. Further, $(V^* \otimes V_\sigma)^G$ is a *unitary* representation π', V' of G'. Define

$$T : V \otimes V' \to V_\sigma$$

by

$$T(v \otimes \varphi) \;=\; \varphi(v)$$

using the fact that V' consists of linear maps from V to V_σ. This map is continuous, is a $G \times G'$-homomorphisms. Since V_σ is irreducible, and the map is not the zero map, T must be a surjection. We must show that it is injective. Indeed, if $\sum_i \varphi_i(v_i) = 0$, with a finite set of linearly independent φ_i, then $\varphi_1(v_1)$ is in the sum of the images of the *other* φ_i, so $v_1 = 0$ by linear independence of the maps φ_i. Thus, T is injective. And, last, if π' had a proper closed G'-subspace then $\pi \otimes \pi'$ would also have a proper closed $G \times G'$-subspace, contradicting the irreducibility. ///

32. *Unitarizability and positive-definite functions*

Sometimes a representation's potential to be *completed* to a unitary Hilbert space representation is not entirely clear.

Let V be a complex vector space (not necessarily with a topology), and

$$\pi : G \to GL_{\mathbb{C}}(V)$$

a group homomorphism from a topological group G to the group of all $\mathbb{C}$-linear automorphisms of V (with no topological requirements).

[32.0.1] Definition: The representation π is *unitarizable* or *pre-unitary* if there is an inner product $\langle , \rangle$ on V (that is, a positive-definite hermitian form) so that
(i) with the metric topology on V from the norm $|v| = \langle v, v\rangle^{1/2}$ for all $v \in V$ the map $g \to \pi(g)v$ is a continuous function $G \to V$, and
(ii) $\langle , \rangle$ is G-invariant, in the sense that

$$\langle \pi(g), v, \pi(g)w\rangle \;=\; \langle v, w\rangle$$

for all $v, w \in V$ and $g \in G$.

[32.0.2] Proposition: In the situation just described, a unitarizable representation π can be extended (continuously) to give a *unitary* representation of G on the Hilbert space $\tilde{V}$ obtained by completing V with respect to the norm.

Proof: The proof proceeds in the natural manner. Define $\tilde{\pi}$ on $\tilde{V}$ in the obvious way: for v_i in V and $v_i \to \tilde{v}$ with $\tilde{v}$ in the completion, try to define

$$\tilde{\pi}(g)\tilde{v} \;=\; \lim_i \pi(g)v_i$$

Since $\pi(g)$ is unitary, $\pi(g)v_i$ is still a Cauchy sequence, so has a limit $\tilde{\pi}(g)\tilde{v}$ in the completion. It is easy to check that this extension is still $\mathbb{C}$-linear and preserves inner products. Then, given $\varepsilon > 0$ and $\tilde{v} \in \tilde{V}$, choose $v \in V$ such that $|\tilde{v}| < \varepsilon$. Choose a small enough neighborhood U of g such that for $h \in U$ we have $|\pi(h)v - \pi(g)v| < \varepsilon$. Then

$$|\tilde{\pi}(g)\tilde{v} - \tilde{\pi}(h)\tilde{v}| \le |\tilde{\pi}(g)\tilde{v} - \pi(g)v| + |\pi(g)v - \pi(h)v| + |\pi(h)v - \tilde{\pi}(h)\tilde{v}|$$

$$= \; |\tilde{v} - v| + |\pi(g)v - \pi(h)v| + |v - \tilde{v}| < \varepsilon + \varepsilon + \varepsilon$$

This proves the required continuity. ///

[32.0.3] Definition: A function $f \in \mathbb{C}^o(G)$ is *positive definite* if, for all finite sets of g_i in G and c_i in $\mathbb{C}$,

$$\sum_{ij} c_i\, \overline{c_j} f(g_j^{-1} g_i) \ge 0$$

[32.0.4] Lemma: If f is a positive definite function, then $f(1) \ge 0$, $f(g^{-1}) = \overline{f(g)}$, and $|f(g)| \le f(1)$.

Proof: Taking a single element g_1 in G and $c_1 = 1$, we obtain $f(1) \ge 0$. The other two assertions will be obtained by taking the set of elements $1, g$ in G and complex numbers $1, z$. The definition of positive-definiteness gives

$$f(1)(1 + z\overline{z}) + zf(g^{-1}) + \overline{z}f(g) \ge 0$$

Therefore, for all complex z, $zf(g^{-1}) + \overline{z}f(g)$ is *real*. Taking $z = 1$ and $z = i$ gives two equations solution of which yields $f(g^{-1} = \overline{f(g)}$. If $f(g) = 0$, the fact that $f(1) \ge 0$ trivially gives $|f(g)| \le f(1)$. Now suppose that $f(g) \ne 0$. Let

$$z \;=\; t \cdot \frac{f(z)}{|f(z)|}$$

with t real. Using $f(g^{-1} = \overline{f(g)}$ we have

$$f(1)(1 + t^2) + 2t\,|f(g)| \ge 0$$

for all $t \in \mathbb{R}$. The minimum of the left-hand side occurs where the derivative with respect to t is 0, namely where

$$2t \cdot f(1) + 2\,|f(g)| \;=\; 0$$

or

$$t \;=\; -|f(g)|/f(1)$$

Substituting this back into the inequality, and multiplying through by an additional factor of $f(1)$, we have

$$f(1)^2\,[1 + \left(\frac{|f(g)|}{f(1)}\right)^2] - 2|f(g)|^2 \ge 0$$

or

$$f(1)^2 + |f(g)|^2 - 2|f(g)|^2 \ge 0$$

from which we obtain $f(1) \geq |f(g)|$. ///

[32.0.5] Theorem: Let f be a positive definite function on G, and V the $\mathbb{C}$-vectorspace of all finite linear combinations of right translates of f by G, with G acting on V by right translations R. Then V is unitarizable, with inner product given by the formula

$$\Big\langle \sum_i c_i R(g_i) f \,,\, \sum_j c_j R(g_j) f \Big\rangle \;=\; \sum_{ij} c_i\, \overline{c_j}\, f(g_j^{-1} g_i) \geq 0$$

Conversely, for any unitary representation π of G on a Hilbert space V, for any $v \in V$ the matrix coefficient function $f(g) = \langle \pi(g)v, v\rangle$ is positive definite.

Proof: First we prove the easy half. Let π be unitary. Then

$$0 \leq |\sum_i c_i \pi(g_i) v|^2 \;=\; \sum_{ij} \langle c_i \overline{c_j} \pi(g_i) v_i, \pi(g_j) v\rangle \;=\; \sum_{ij} \langle c_i \overline{c_j} \pi(g_j)^* \pi(g_i) v_i, v\rangle \;=\; \sum_{ij} \langle c_i \overline{c_j} \pi(g_j^{-1} g_i) v_i, v\rangle$$

by unitariness.

On the other hand, suppose f is positive definite. Let

$$E(g) \;=\; \sum_i c_i\, f(g g_i) \qquad F(g) \;=\; \sum_j d_j\, f(g h_j)$$

Using $f(g^{-1}) = \overline{f}(g)$,

$$\langle E, F\rangle \;=\; \sum_{ij} c_i \overline{d_j} f(h_j^{-1} g_i) \;=\; \sum_j \overline{d_j} E(h_j^{-1}) \;=\; \sum_i c_i \overline{F}(g_i^{-1})$$

Thus, the inner product $\langle E, F\rangle$ is independent of the *expressions* for E and F, but only depends on the *functions*. That is, the inner product is *well-defined*.

That the inner product has the positive semi-definiteness property $\langle F, F\rangle \geq 0$ is immediate from the definition of positive-definiteness. Therefore, a Cauchy-Schwarz-Bunyakowsky in equality holds for it, even without knowing that the inner product is actually positive definite. Thus, if $\langle F, F\rangle = 0$, then

$$F(g) \;=\; \sum_i c_i f(g g_i) \;=\; \langle F, R_g f\rangle$$

and then

$$|F(g)|^2 \;=\; |\langle F, R_g f\rangle|^2 \leq \langle F, F\rangle \cdot \langle R_g f, R_g f\rangle$$

from which F is *identically* 0. Thus, $\langle,\rangle$ is positive *definite*, not merely *semi-definite*.

For G-invariance, expanding the definition makes clear that it was designed to ensure this invariance:

$$\langle R_g E, R_g F\rangle \;=\; \sum_{ij} c_i \overline{d}_j\, f((g h_j)^{-1}\,(g g_i)) \;=\; \sum_{ij} c_i \overline{d}_j\, f(h_j^{-1}\, g_i) \;=\; \langle E, F\rangle$$

The continuity is not difficult. Using unitariness and G-invariance

$$|R_g F - R_h F|^2 \;=\; \langle R_g F, R_g F\rangle - \langle R_g F, R_h F\rangle - \langle R_h F, R_g F\rangle + \langle R_h F, R_h F\rangle$$

$$=\; \langle F, F\rangle - \langle R_g F, R_h F\rangle - \langle R_h F, R_g F\rangle + \langle F, F\rangle \;=\; \sum_{ij} c_i \overline{c}_j\; \big(2 f(g_j^{1} g_i) - f(g_j^{-1} h^{-1} g g_i) - f(g_j^{-1} g^{-1} h g_i)\big)$$

As g approaches h, the continuity of f assures that both $f(g_j^{-1}h^{-1}gg_i)$ and $f(g_j^{-1}g^{-1}hg_i)$ approach $f(g_j^{-1}g_i)$. Since the sums are finite, this shows that the whole approaches 0. ///

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33. *Type I groups, CCR/liminal groups*

Returning to the question of factoring irreducible unitaries of $G \times G'$ as tensor products of irreducibles of G and G', we can isolate some useful technical properties while explaining definitions. Similar preparations are relevant to a second eventual goal, that of a spectral decomposition of $L^2(Z\backslash G)$ and/or $L^2(Z\Gamma\backslash G)$ *without* the assumption of compactness of $Z\backslash G$ or $Z\Gamma\backslash G$.

Let S be a set of continuous linear operators on a Hilbert space V. The *commutator* S' of S is defined to be

$$S' \;=\; \{T \text{ continuous linear operators on } V : T\Phi = \Phi T \text{ for all } \Phi \in S\}$$

[33.0.1] Definition: A *factor representation* π of a group G on a Hilbert space V is a (usually unitary) representation π such that

$$\pi(G) \cap \pi(G)' \;=\; \text{scalar operators}$$

[33.0.2] Remark: Schur's lemma implies that *irreducible* unitary representations π, V have this property, since in that case already the commutator is small:

$$\pi(G)' \;=\; \text{scalar operators}$$

[33.0.3] Definition: A unitary Hilbert space representation σ, V_σ of G is (strongly) *isotypic* if it is a (completed orthogonal) direct sum

$$V_\sigma \approx \widehat{\bigoplus}_{\alpha\in A} V_\alpha$$

of irreducible unitaries π_α, V_α all isomorphic to a *single* irreducible unitary π, V. To be more specific, we could say that σ is π-isotypic.

[33.0.4] Definition: A topological group G is *type I* if every factor representation is (strongly) isotypic.

[33.0.5] Definition: A topological group G is *CCR* (Kaplansky's 'completely continuous representations') or (Dixmier's) *liminal* or *liminaire* if for every irreducible unitary representation π, V of G the image in $\mathrm{End}(V)$ of $C_c^o(G)$ consists entirely of *compact* operators.

The following result is not too hard, but we will not prove it here just now.

[33.0.6] Proposition: If G is a Type I group and if π is an irreducible unitary Hilbert space representation of $G \times H$, then π is of the form $\pi_1 \otimes \pi_2$ where π_1 is an irreducible unitary representation of G and π_2 is an irreducible unitary representation of H.

The first general theorem we would want to prove is

[33.0.7] Theorem: Liminal groups are Type I.

The second theorem we eventually would want to prove, for applications to representations of adele groups, especially automorphic representations, is

[33.0.8] Theorem: A real, complex, or p-adic reductive group is *liminal*, hence of Type I.

[33.0.9] Remark: In fact, the proof of the liminality proceeds by showing something stronger, namely that each irreducible unitary π, V of these reductive groups G is *admissible* in the sense that for a maximal compact subgroup K for each irreducible ρ of K the ρ-isotype V^σ is of *finite multiplicity* (equivalently, of finite dimension). The proof of admissibility is non-trivial. Then one notes that the K-conjugation-invariant elements of $C_c^o(G)$ stabilize the K-isotypic components, so are finite rank operators. Thus, the closure (in operator norm) would consist entirely of compact operators. These conjugation-invariant functions are dense in $C_c^o(G)$ in the L^1 norm, so all of $C_c^o(G)$ consists of compact operators.

[33.0.10] Remark: The following bibliography is meant to give a rough idea of the chronology of some aspects of abstract harmonic analysis. The *papers* (as opposed to monographs or texts) mentioned below are typically one of the earliest in a series of several papers on related matters by the same author(s). One thing we have neglected in our discussion is the general finer structure theory of topological groups, though this was a topic of interest in the earlier works in abstract harmonic analysis.

[Borel 1972] A. Borel, *Representations de Groups Localement Compacts*, Lect Notes in Math. **276**, Springer-Verlag, 1972.

[Dixmier 1964] J. Dixmier, *Les C^*-algebras et leurs Representations*, Gauthier-Villars, Paris, 1964.

[Gaal 1973] S. Gaal, *Linear Analysis and Representation Theory*, Springer-Verlag, 1973.

[Godement 1947] R. Godement, *Analyse harmonique dans les groupes centraux, I. Fonctions centrale et caracter'eres*, C.R. Acad. Sci. Paris **225** (1947), pp. 19-21.

[Grosser, Moskowitz 1967] S. Grosser, M. Moskowitz, *Representation theory of central topological groups*, Trans. A.M.S., vol. 129, no. 3, December, 1967.

[Loomis 1953] L. Loomis, *Abstract Harmonic Analysis*, Van Nostrand, N.J., 1953.

[Mackey 1952] G. Mackey, *Induced representations of locally compact groups, I*, Ann. of Math. (2) **55** (1952), pp. 101-139.

[Mautner 1953] F. Mautner, *Infinite-dimensional representations of certain groups*, Proc. A.M.S. **1**, (1949), pp. 582-84.

[Pontrjagin 1939] L. Pontrjagin, *Topological Groups*, Princeton Univ. Press, 1939.

[Robert 1983] A. Robert *Introduction to the Representation Theory of Compact and Locally Compact Groups*, London Math. Soc. Lecture Notes **80**, Cambridge Univ. Press, 1983.

[Segal 1947] I. Segal, *The group algebra of a locally compact group*, Trans. A.M.S. **61** (1947), pp. 69-105.

[Weil 1953] A. Weil, *L'integration dan les groups topologiques, et ses applications*, Hermann, Paris, 1953.
