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Stone - von Neumann theorem

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The bibliography indicates the origins of these issues in proving uniqueness, up to isomorphism, of the model for the commutation rules of quantum mechanics.

As Mackey and Weil observed, some form of the argument succeeds in quite general circumstances. The argument here is an amalgam and adaptation of those in sources in the bibliography. The proof here applies nearly verbatim, with suitable adjustment of the notion of *Schwartz space*, to Heisenberg groups over other local fields in place of $\mathbb{R}$.

Fix a finite-dimensional $\mathbb{R}$-vectorspace V with non-degenerate alternating form $\langle , \rangle$. The corresponding *Heisenberg group* has Lie algebra $\mathfrak{h} = V \oplus \mathbb{R}$, with Lie bracket

$$[v+z, v'+z'] \;=\; \langle v, v' \rangle \;\in\; \mathbb{R} \;\approx\; \{0\} \oplus \mathbb{R} \;\subset\; V \oplus \mathbb{R}$$

Exponentiating to the Lie group H, the group operation is

$$\exp(v+z) \cdot \exp(v'+z') \;=\; \exp\Big(v + v' + z + z' + \frac{\langle v, v' \rangle}{2}\Big)$$

This is explained by literal matrix exponentiation: for example, with $v = (x, y) \in \mathbb{R}^2$,

$$\exp(v+z) \;=\; \exp\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad (\text{with } v = (x, y))$$

We use Lie algebra coordinates on H, often suppressing the exponential on abelian subgroups, but reinstating an explicit exponential notation when necessary to avoid confusion.

[0.0.1] Theorem: *(Stone-vonNeumann)* For fixed non-trivial unitary central character, up to isomorphism there is a unique irreducible unitary representation of the Heisenberg group with that central character. Further, *any* unitary representation with that central character is a *multiple* of that irreducible.

Proof: We construct the irreducible for given non-trivial central character ω as an L^2 induced representation, prove its irreducibility, and then prove uniqueness.

Given a unitary representation π of the Heisenberg group H, the associated *Weyl transform* [1] $\pi\varphi$ of a Schwartz function φ on the alternating space V is the action of π integrated over $V \subset H$:

$$\pi\varphi \;=\; \int_V \varphi(v) \cdot \pi(v)\, dv \qquad (\varphi \in \mathscr{S}V, \text{ unitary } \pi)$$

[1] Unitaries π with central character ω are acted-on by functions φ on H that are compactly supported modulo the center, and transform by the complex conjugate character $\overline{\omega}$ under the center, by

$$\varphi \cdot x \;=\; \int_{H/Z} f(h)\, \pi(h)(x)\; dx \qquad (\text{for } x \in \pi)$$

The convolution product on such functions is

$$(\varphi * \psi)(g) \;=\; \int_{H/Z} \varphi(gh^{-1})\, \psi(h)\; dh$$

Over $\mathbb{R}$, as opposed to $\mathbb{Q}_p$, one can also consider the *reduced* Heisenberg group $H/\ker\omega$, which has the feature that $Z/\ker\omega$ has become *compact*. This device is unavailable for $\mathbb{Q}_p$.

Given a Lagrangian subspace W of V, $W+Z$ is a maximal abelian subgroup of H. Extend a non-trivial central character ω *trivially* to $W+Z$ by

$$\omega(w+z) \;=\; \omega(z)$$

[0.0.2] Claim: For the L^2 induced representation $\sigma = \mathrm{Ind}_{W+Z}^H\omega$ the map $\varphi \to \sigma\varphi$ on $\mathscr{S}V$ extends to an *isometric* bijection from $L^2(V)$ to Hilbert-Schmidt operators on σ.

Proof: We determine the kernel functions for operators $\sigma\varphi$. Let $V = W \oplus W'$ with complementary Lagrangian subspace W' to W. On non-abelian subgroups of H we may revert to explicit denotation of the exponential map. The relation $e^{x+y} = e^x \cdot e^y \cdot e^{-\langle x,y\rangle/2}$ for $x,y\in V$ gives an operator equality

$$\sigma\varphi \;=\; \int_{W'}\int_W \varphi(x+y)\,\sigma(e^{x+y})\,dx\,dy \;=\; \int_{W'}\int_W \varphi(x+y)\,\sigma(e^x\,e^y)\,\omega\Big(\frac{-\langle x,y\rangle}{2}\Big)\,dx\,dy$$

The action of $e^x\in H$ on f in the usual model of $\mathrm{Ind}_{W+Z}^H\omega$ is by right translation $(e^xf)(e^{w'}) = f(e^{w'}\cdot e^x)$. The relation $e^{w'}e^x = e^x\,e^{w'}\,e^{-\langle x,w'\rangle}$ gives

$$(e^xf)(e^{w'}) \;=\; f(e^{w'}e^x) \;=\; f(e^x\,e^{w'}\,e^{-\langle x,w'\rangle}) \;=\; \omega(-\langle x,w'\rangle)\cdot f(e^{w'}) \qquad\qquad (\text{for } w'\in W' \text{ and } x\in W)$$

Identify the representation space of σ with $L^2(W')$ via $\exp(W+Z)\backslash H \approx \exp(W') \approx W'$. For $f\in L^2(W')$ and $w'\in W'$,

$$(\sigma\varphi)f(w') \;=\; \int\limits_{W'}\int\limits_W \omega\Big(\frac{-\langle x,y\rangle}{2}\Big)\,\varphi(x+y)\,(e^xe^yf)(w')\,dx\,dy$$

From

$$(e^xe^yf)(w') \;=\; \omega(-\langle x,w'\rangle)\cdot(e^yf)(w') \;=\; \omega(-\langle x,w'\rangle)\cdot f(y+w')$$

rearrange to see the kernel function:

$$(\sigma\varphi)f(w') \;=\; \int\limits_{W'}\int\limits_W \omega\Big(\frac{-\langle x,y\rangle}{2}\Big)\omega(-\langle x,w'\rangle)\,\varphi(x+y)\,f(y+w')\,dx\,dy$$

$$=\int\limits_{W'}\int\limits_W \omega\Big(\frac{-\langle x,y+2w'\rangle}{2}\Big)\,\varphi(x+y)\,f(y+w')\,dx\,dy \;=\; \int\limits_{W'}\int\limits_W \omega\Big(\frac{-\langle x,y+w'\rangle}{2}\Big)\,\varphi(x+y-w')\,f(y)\,dx\,dy$$

$$=\int_{W'} K_\varphi(\cdot,y)\,f(y)\,dy \qquad\quad \Big(\text{with kernel } K_\varphi(w',y) = \int_W \omega\Big(\frac{-\langle x,y+w'\rangle}{2}\Big)\,\varphi(x+y-w')\,dx\Big)$$

The integral in $x\in W$ is a Fourier transform F_W along $W\subset V$, with respect to the character $x\to\omega(\langle x,y\rangle/2)$, making the Fourier transform a Schwartz function on W'. Indeed, this (partial) Fourier transform is a homeomorphism $\mathscr{S}V\to\mathscr{S}V$, so

$$K_\varphi(w',y) \;=\; \big(F_W\varphi\big)\Big(\frac{y+w'}{2},\,y-w'\Big) \;\in\; \mathscr{S}(W'\times W')$$

Further, the partial Fourier transform is an isometry in the corresponding L^2 metrics, so extends by continuity from $\mathscr{S}V\to\mathscr{S}V$ to an isometry $L^2(W\times W')\to L^2(W'\times W')$. Thus, $\varphi\in L^2(W\times W')$ gives a Hilbert-Schmidt operator on $L^2(W')$. Conversely, every Hilbert-Schmidt operator is given by such a kernel, which arises from some $\varphi\in L^2(W\times W')$. ///

[0.0.3] Corollary: Finite-rank operators are in the image of $L^2(V)$ under σ. ///

[0.0.4] Corollary: The only continuous linear operators on σ commuting with σh for all $h\in H$ are *scalars*.

Proof: Such T commutes with all integral operators $\sigma\varphi$ for $\varphi \in \mathscr{S}V$, therefore with all operators arising from $\varphi \in L^2(V)$, therefore with all Hilbert-Schmidt operators, including all finite-rank operators. Thus, for any vector $e \in V$, the rank-one orthogonal projector P to $\mathbb{C} \cdot e$ commutes with T, and $P \circ T = T \circ P$ implies that T stabilizes the line $\mathbb{C} \cdot e$. Necessarily T acts by a scalar λ on any line it stabilizes. Likewise, for another vector e', T acts by a scalar λ' on $\mathbb{C} \cdot e'$, and by a scalar on the line $\mathbb{C} \cdot (e + e')$, as well. Thus, $\lambda = \lambda'$, and T is a scalar. ///

[0.0.5] Corollary: *The representation σ is irreducible.*

Proof: For a closed stable subspace π, the orthogonal projector P to π commutes with H. By the previous corollary, P is scalar, so is 0 or 1. ///

Let $*$ be convolution of $(Z, \overline{\omega})$-equivariant smooth compactly-supported functions on H:

$$(f * g)(x) \;=\; \int_{H/Z} f(xy^{-1})\, g(y)\; dy$$

Transport this convolution to $C_c^\infty(V)$, via the identification of $\varphi \in C_c^\infty(V)$ with $(Z, \overline{\omega})$-equivariant functions $\widetilde{\varphi}$ on H by

$$\widetilde{\varphi}(e^v \cdot z) \;=\; \overline{\omega}(z) \cdot \varphi(v) \qquad\qquad (\text{for } v \in V \text{ and } z \in Z)$$

Thus, for $f, g \in C_c^\infty(V)$,

$$(f * g)(v) \;=\; \int_V f(e^v \cdot e^{-y})\, g(e^y)\; dy \;=\; \int_{H/Z} f(e^{v-y} \cdot e^{\frac{-\langle v,y\rangle}{2}})\, g(e^y)\; dy \;=\; \int_V \omega\Big(\frac{\langle v, y\rangle}{2}\Big)\, f(v-y)\, g(y)\; dy$$

Unsurprisingly, this convolution is not the convolution on the additive group V. This convolution extends to Schwartz functions $\mathscr{S}V$ on V.

Under convolution, the Schwartz functions φ on V give an algebra A of operators $\sigma\varphi$ on the irreducible σ, compatible with composition of operators: $\sigma\varphi \circ \sigma\varphi' = \sigma(\varphi * \varphi')$. For any unitary π of H with central character ω, $\varphi \to \pi\varphi$ maps $\mathscr{S}V$ to an algebra A_π of operators on π, and respects Hilbert space *adjoints*.

For $f \in \mathscr{S}W'$ with $|f|_{L^2(W')} = 1$, the orthogonal projector P to the line $\mathbb{C} \cdot f$ is Hilbert-Schmidt, so is in the image of the Weyl transform. That is, $P = \sigma\varphi$ for $\sigma = \mathrm{Ind}_{W+Z}^V \omega$ and some $\varphi \in L^2(V)$, with

$$\varphi * \varphi = \varphi \qquad \varphi^\vee = \varphi \qquad \sigma\varphi \circ \sigma h \circ \sigma\varphi = \langle \sigma h f, f\rangle \cdot \sigma\varphi \qquad\qquad (\text{for } h \in H)$$

In fact, $\varphi \in \mathscr{S}V$, because $f \in \mathscr{S}V$, so the kernel function of P is $K(u,v) = f(u)\,\overline{f}(v)$, and we have the relation

$$f(w')\,\overline{f}(y) \;=\; K(w', y) \;=\; K_\varphi(w', y) \;=\; \big(F_W \varphi\big)\Big(\frac{y + w'}{2},\, y - w'\Big)$$

The (partial) Fourier transform maps Schwartz functions to Schwartz functions. Thus, for any unitary π with central character ω, the integral operator $\pi\varphi$ makes sense.

[0.0.6] Claim: With fixed φ as above, the images $(\pi h \circ \pi\varphi)(x)$ for $h \in H$ and $x \in \pi$ are *dense* in the Hilbert space π.

Proof: Let y be orthogonal to all the indicated images. For $h = e^u$ in H, suppressing π,

$$0 = \langle h \cdot \varphi \cdot h^{-1} x, y\rangle = \int_V \varphi(v)\, \langle e^u \cdot e^v \cdot e^{-u} x, y\rangle\, dv = \int_V \varphi(v)\, \langle e^{v + \langle u,v\rangle} x, y\rangle\, dv = \int_V \varphi(v)\, \omega\langle u, v\rangle\, \langle e^v \cdot x, y\rangle\, dv$$

The function $v \to \varphi(v) \cdot \langle e^v \cdot x, y\rangle$ is continuous and in $L^2(V)$, because φ is Schwartz, and because

$$|\langle e^v \cdot x, y\rangle| \;=\; |\langle \sigma(e^v)(x),\, y\rangle| \;\le\; |\sigma(e^v)x| \cdot |y| \;=\; |x| \cdot |y| \qquad\qquad (\text{because } \sigma \text{ is unitary})$$

The vanishing

$$0 \;=\; \int_V \varphi(v)\cdot\omega\langle u,v\rangle\cdot\langle e^v\cdot x,y\rangle\,dv$$

is vanishing of the Fourier transform of $v\to\varphi(v)\cdot\langle e^v\cdot x,y\rangle$. Since $\varphi\in\mathscr{S}V$, its Fourier transform is also in $\mathscr{S}V$, so is 0 *pointwise*, giving the claimed density. ///

From $\varphi*\varphi=\varphi$ and $\varphi^\vee=\varphi$, the image $\pi\varphi$ of φ under any unitary representation π is a projector to the subspace $X'=(\pi\varphi)(Y)$ where Y is the representation space of π. As X' is the kernel of $1_Y-\pi\varphi$, it is *closed*.

With X the representation space of σ, we want to define an H-isomorphism $j:X\otimes X'\to Y$ by

$$j\big((\sigma h)f\otimes x'\big) \;=\; \pi h(x') \qquad\qquad (\text{for } x'\in X'=(\pi\varphi)Y,\text{ and } h\in H))$$

The inner product on an algebraic tensor product $C=A\otimes B$ of inner product spaces is given by

$$\langle a\otimes b, a'\otimes b'\rangle_C \;=\langle a,a'\rangle_A\cdot\langle b,b'\rangle_B$$

[0.0.7] Claim: For $h_1,h_2\in H$ and $x_1,x_2\in X'$,

$$\Big\langle(\pi h_1\circ\pi\varphi)(x_1),\,(\pi h_2\circ\pi\varphi)(x_2)\Big\rangle_Y \;=\; \Big\langle(\sigma h_1)(f),\,(\sigma h_2)(f)\Big\rangle_X\cdot\Big\langle(\pi h_1)(x_1),(\pi h_2)(x_2)\Big\rangle_{X'}$$

Proof: Suppress σ and π. Moving all the operators to one side, using the unitariness of π, and $\varphi^\vee=\varphi$,

$$\langle h_1\cdot\varphi\cdot x_1, h_2\cdot\varphi\cdot x_2\rangle_Y \;=\; \langle\varphi\cdot h_2^{-1}h_1\cdot\varphi\cdot x_1,x_2\rangle_Y$$

Using $\varphi\cdot h\cdot\varphi=\langle h\cdot f,f\rangle_X\cdot\varphi$,

$$\langle\varphi\cdot h_2^{-1}h_1\cdot\varphi\cdot x_1,x_2\rangle_Y \;=\; \langle h\cdot f,f\rangle_X\cdot\langle\varphi\cdot x_1,x_2\rangle_Y$$

Since $\varphi*\varphi=\varphi$, and since π is unitary,

$$\langle\varphi\cdot x_1,x_2\rangle_Y \;=\; \langle\varphi\cdot\varphi\cdot x_1,x_2\rangle_Y \;=\; \langle\varphi\cdot x_1,\varphi\cdot x_2\rangle_Y$$

giving the asserted equality. ///

Since σ is irreducible, the collection of all finite complex-linear combinations $\sum_i c_i\,\sigma h_i\cdot f$ with $h_i\in H$ is *dense* in the representation space X of σ. Try to define

$$j:X\otimes X'\longrightarrow Y \qquad\qquad \text{by} \qquad\qquad j\Big(\sum_i(\sigma h_i)f\otimes x_i'\Big) \;=\; \sum_i(\pi h_i)(x_i')$$

This is *well-defined*: the inner product formula just proven shows that, for $\sum_i(\sigma h_i)f\otimes x_i'=0$,

$$\Big|\sum_i(\pi h_i)(x_i')\Big|^2 \;=\; \sum_{ij}\Big\langle(\pi h_i)(x_i'),\,(\pi h_j)(x_j')\Big\rangle \;=\; \sum_{ij}\Big\langle(\sigma h_i)f,\,(\sigma h_j)f\Big\rangle_Y\cdot\Big\langle(\pi h_i)(x_i'),\,(\pi h_j)(x_j')\Big\rangle_{X'}$$

$$=\;\Big|\sum_i(\sigma h_i)f\otimes x_i'\Big|^2_{X\otimes X'} \;=\; 0$$

Thus, the map is an isometry on a dense subspace of a Hilbert space, with dense image in another Hilbert space. Extend the map by continuity (isometry). The extension is *surjective*, hence an isometric isomorphism $X\otimes X'\to Y$, and

$$j\circ(\sigma h\otimes 1_{X'})\circ j^{-1} \;=\; \pi h \qquad\qquad (\text{for all } h\in H)$$

so π is a multiple of σ in this sense. In particular, for π irreducible, necessarily $X'\approx\{0\}$, and $\pi\approx\sigma$.
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