Moments for L–functions for $GL_r \times GL_{r-1}$

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Abstract: *We establish a spectral identity for moments of Rankin-Selberg L–functions on $GL_r \times GL_{r-1}$ over arbitrary number fields, generalizing our previous results for $r = 2$.*

1. *Introduction*

Let k be an algebraic number field with adele ring $\mathbb{A}$. Fix an integer $r \geq 2$ and consider the general linear groups $GL_r(k)$, $GL_r(\mathbb{A})$ of $r \times r$ invertible matrices with entries in k, $\mathbb{A}$, respectively. Let Z^+ be the positive real scalar matrices in GL_r. Let π be an irreducible cuspidal automorphic representation in $L^2(Z^+GL_r(k)\backslash GL_r(\mathbb{A}))$. Let π' run over irreducible unitary cuspidal representations in $L^2(Z^+GL_{r-1}(k)\backslash GL_{r-1}(\mathbb{A}))$, where now Z^+ is the positive real scalar matrices in GL_{r-1}. For brevity, denote a sum over such π' by $\sum_{\pi'}$. For complex s, let $L(s, \pi \times \pi')$ denote the Rankin-Selberg convolution L–function. A *second integral moment* over the spectral family GL_{r-1} is described roughly as follows. For each irreducible cuspidal automorphic π' of GL_{r-1}, assign a constant $c(\pi') \geq 0$. Letting π_∞ be the archimedean component of π and π'_∞ the archimedean factor of each π', let $M(s, \pi_\infty, \pi'_\infty)$ be a function of complex s, whose possibilities will be described in more detail later. The corresponding second moment of π is

$$\sum_{\pi'} c(\pi') \int_{\mathrm{Re}\, s = \frac{1}{2}} |L(s, \pi \times \pi')|^2 \cdot M(s, \pi_\infty, \pi'_\infty)\, ds$$

In fact, there are corresponding further correction terms corresponding to non-cuspidal parts of the spectral decomposition of $L^2(Z^+GL_{r-1}(k)\backslash GL_{r-1}(\mathbb{A}))$, but the cuspidal part presumably dominates.

The theory of second integral moments on $GL_2 \times GL_1$ has a long history, although the early papers treated mainly the case that the groundfield k is $\mathbb{Q}$. For example, see [Hardy-Littlewood 1918], [Ingham 1926], [Heath-Brown 1975], [Sarnak 1985], [Good 1983,1986], [Motohashi 1997], [Jutila 1997], [Petridis-Sarnak 2001], [Bruggeman-Motohashi 2001,2003], [CFKRS 2005], [Diaconu-Goldfeld-Hoffstein 2005], [Diaconu-Goldfeld 2006a,2006b], [Diaconu-Garrett 2009]. Second integral moments of level-one holomorphic elliptic modular forms were first treated in [Good 1983,1986], the latter using an idea that is a precursor of part of the present approach. The study of second integral moments for $GL_2 \times GL_1$ with arbitrary level, groundfield, and infinity-type is completely worked out in [Diaconu-Garrett 2010].

The main aim of this paper is to establish an identity relating the second integral moment, described above, to the integral of a certain Poincaré series $\mathfrak{P}$ against the absolute value squared $|f|^2$ of a distinguished cuspform $f \in \pi$. Acknowledging that the spectral decomposition of $L^2(Z_\mathbb{A}GL_r(k)\backslash GL_r(\mathbb{A}))$ also has a non-cuspidal part generated by Eisenstein series and their residues, the identity we obtain takes the form

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$$\int\limits_{Z_{\mathbb{A}} GL_r(k)\backslash GL_r(\mathbb{A})} \mathfrak{P}(g,\varphi_\infty)\cdot |f(g)|^2\, dg \;=$$

$$\sum_{\pi'} |\rho(\pi')|^2 \int_{\mathrm{Re}\, s=\frac{1}{2}} |L(s,\pi\times\pi')|^2 \cdot M(s,\pi_\infty,\pi'_\infty,\varphi_\infty)\, ds \;+\; \text{(non-cuspidal part)}$$

Here $M(s,\pi_\infty,\pi'_\infty,\varphi_\infty)$ is a weighting function depending on the complex parameter s, on the archimedean components π_∞ and π'_∞, and on archimedean data φ_∞ defining the Poincaré series. The global constants $\rho(\pi')$ are analogues of the leading Fourier coefficients of GL_2 cuspforms. The spectral expansion of the Poincaré series $\mathfrak{P}$ relates the second integral moment to automorphic spectral data. Remarkably, the cuspidal data appearing in the spectral expansion of $\mathfrak{P}$ comes only from GL_2.

These new identities have some similarities to the Kuznetsov trace formula [Bruggeman 1978], [Wallach 1992], [Ye 2000], [Goldfeld 2006], in that they are derived via the spectral resolution of a Poincaré series, but they are clearly of a different nature. We have in mind application not only to cuspforms, but also to truncated Eisenstein series or wave packets of Eisenstein series, thus applying harmonic analysis on GL_r to L–functions attached to GL_1, touching upon higher integral moments of the zeta function $\zeta_k(s)$ of the ground field k.

In connection to the present work, we mention the recent mean-value result of [Young 2009],

$$\int_{-T^{1-\varepsilon}}^{T^{1-\varepsilon}} \sum_{T<t_j\le 2T} |L(\tfrac{1}{2}+it,u_j\times\varphi)|^2\, dt \;\ll\; T^{3+\varepsilon} \qquad\qquad (\text{for } \varepsilon>0)$$

where φ is on GL_3, and where u_j on GL_2 has spectral data t_j, as usual. From this the t-aspect convexity bound can be recovered. Also, [Li 2009] obtains a t-aspect subconvexity bound for standard L–functions for $GL_3(\mathbb{Q})$ for Gelbart-Jacquet lifts.

For context, we review the [Diaconu-Goldfeld 2005] treatment of spherical waveforms f for $GL_2(\mathbb{Q})$. In that case, the sum of moments is a single term

$$\int_{Z_{\mathbb{A}} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})} \mathfrak{P}(g,z,w)\, |f(g)|^2\, dg = \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} L(z+s,f)\cdot \overline{L}(s,f)\cdot \Gamma(s,z,w,f_\infty)\, ds$$

where $\Gamma(s,z,w,f_\infty)$ is a ratios of products of gammas, with arguments depending upon the archimedean data of f. Here the Poincaré series $\mathfrak{P}(g)=\mathfrak{P}(g,z,w)$ is specified completely by complex parameters z,w, and has a *spectral expansion*

$$\mathfrak{P}(g,z,w) \;=\; \frac{\pi^{\frac{1-w}{2}}\Gamma(\frac{w-1}{2})}{\pi^{-\frac{w}{2}}\Gamma(\frac{w}{2})}\cdot E_{1+z}(g) \;+\; \tfrac{1}{2}\sum_{F \text{ on } GL_2} \rho_{\overline{F}}\cdot L(\tfrac{1}{2}+z,\overline{F})\cdot \mathcal{G}(\tfrac{1}{2}-it_F,z,w)\cdot F(g)$$

$$+\frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\zeta(z+s)\,\zeta(z+1-s)}{\xi(2-2s)}\,\mathcal{G}(1-s,z,w)\cdot E_s(g)\, ds \qquad\qquad (\text{for } \mathrm{Re}\,(z)\gg\tfrac{1}{2},\ \mathrm{Re}\,(w)\gg 1)$$

where $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\mathcal{G}$ is essentially a product of gamma function values

$$\mathcal{G}(s,z,w) = \pi^{-(z+\frac{w}{2})}\frac{\Gamma(\frac{z+1-s}{2})\,\Gamma(\frac{z+s}{2})\,\Gamma(\frac{z-s+w}{2})\,\Gamma(\frac{z+s-1+w}{2})}{\Gamma(z+\frac{w}{2})}$$

and F is summed over (an orthogonal basis for) spherical (at finite primes) cuspforms on GL_2 with Laplacian eigenvalues $\frac{1}{4}+t_F^2$, and E_s is the usual spherical Eisenstein series. The continuous part, the *integral* of Eisenstein series E_s, cancels the pole at $z=1$ of the leading term, and when evaluated at $z=0$ is

$$\mathfrak{P}(g,0,w) \;=\; (\text{holomorphic at } z{=}0) \;+\; \tfrac{1}{2} \sum_{F \text{ on } GL_2} \rho_{\overline{F}} \cdot L(\tfrac{1}{2}, \overline{F}) \cdot \mathcal{G}(\tfrac{1}{2} - it_F, 0, w) \cdot F(g)$$

$$+\frac{1}{4\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\zeta(s)\,\zeta(1-s)}{\xi(2-2s)}\, \mathcal{G}(1-s,0,w) \cdot E_s(g)\, ds$$

In this spectral expansion, the coefficient in front of a cuspform F includes $\mathcal{G}$ evaluated at $z = 0$ and $s = \frac{1}{2} \pm it_F$, namely

$$\mathcal{G}(\tfrac{1}{2} - it_F, 0, w) = \pi^{-\frac{w}{2}} \frac{\Gamma(\frac{\frac{1}{2}-it_F}{2})\,\Gamma(\frac{\frac{1}{2}+it_F}{2})\,\Gamma(\frac{w-\frac{1}{2}-it_F}{2})\,\Gamma(\frac{w-\frac{1}{2}+it_F}{2})}{\Gamma(\frac{w}{2})}$$

The gamma function has poles at $0, -1, -2, \ldots$, so this coefficient has poles at $w = \frac{1}{2} \pm it_F$, $-\frac{3}{2} \pm it_F$, ... Over $\mathbb{Q}$, among spherical cuspforms (or for any fixed level) these values have no accumulation point. The continuous part of the spectral side at $z = 0$ is

$$\frac{1}{4\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\xi(s)\,\xi(1-s)}{\xi(2-2s)}\, \frac{\Gamma(\frac{w-s}{2})\,\Gamma(\frac{w-1+s}{2})}{\Gamma(\frac{w}{2})} \cdot E_s\, ds$$

with gamma factors grouped with corresponding zeta functions, to form the completed L–functions ξ. Thus, the evident pole of the leading term at $w = 1$ can be exploited, using the continuation to $\mathrm{Re}\,(w) > 1/2$. A contour-shifting argument shows that the continuous part of this spectral decomposition has a meromorphic continuation to $\mathbb{C}$ with poles at $\rho/2$ for zeros ρ of ζ, in addition to the poles from the gamma functions.

Already for GL_2, over general ground fields k, infinitely many Hecke characters enter both the spectral decomposition of the Poincaré series and the moment expression. This naturally complicates isolation of literal moments, and complicates analysis of poles via the spectral expansion. Suppressing constants, the moment expansion is a sum of twists by Hecke characters χ, of the form

$$\int_{Z_{\mathbb{A}} GL_2(k)\backslash GL_2(\mathbb{A})} \mathfrak{P}(s,z,w,\varphi_\infty) \cdot |f(g)|^2 = \sum_\chi \int_{\mathrm{Re}\,(s)=\frac{1}{2}} L(z+s, f\otimes\chi) \cdot L(1-s, \overline{f}\otimes\overline{\chi}) \cdot M(s,z,\chi_\infty,\varphi_\infty)\, ds$$

where $M(s,z,\chi_\infty,\varphi_\infty)$ depends upon complex parameters s, z and archimedean components χ_∞, f_∞, and upon auxiliary archimedean data φ_∞ defining the Poincaré series. Again suppressing constants, the spectral expansion is

$$\mathfrak{P}(g,z,\varphi_\infty) \;=\; (\infty - \text{part}) \cdot E_{1+z}(g) \;+\; \sum_{F \text{ on } GL_2} (\infty - \text{part}) \cdot \rho_{\overline{F}} \cdot L(\tfrac{1}{2}+z, \overline{F}) \cdot F(g)$$

$$+\quad \sum_\chi \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{L(z+s,\overline{\chi})\, L(z+1-s,\chi)}{L(2-2s,\overline{\chi}^2)}\, \mathcal{G}(s,\chi_\infty) \cdot E_{s,\chi}(g)\, ds$$

where the factor denoted ∞-part depends only upon the archimedean data, as does $\mathcal{G}(s,\chi_\infty)$.

In the simplest case beyond GL_2, take f a spherical cuspform for $GL_3(\mathbb{Q})$ generating an irreducible cuspidal automorphic representation $\pi = \pi_f$. We can construct a weight function $\Gamma(s,z,w,\pi_\infty,\pi'_\infty)$ with explicit asymptotic behavior, depending upon complex parameters s, z, and w, and upon the *archimedean* components π_∞ for π and for π' irreducible cuspidal automorphic on GL_2, such that the *moment expansion* has the form

$$\int_{Z_{\mathbb{A}} GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \mathfrak{P}(g,z,w) \cdot |f(g)|^2\, dg = \sum_{\pi' \text{ on } GL_2} |\rho(\pi')|^2 \frac{1}{2\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} |L(s,\pi\times\pi')|^2 \cdot \Gamma(s,0,w,\pi_\infty,\pi'_\infty)\, ds$$

$$+\frac{1}{4\pi i}\frac{1}{2\pi i} \sum_{k\in\mathbb{Z}} \int_{\mathrm{Re}\,(s_1)=\frac{1}{2}} \int_{\mathrm{Re}\,(s_2)=\frac{1}{2}} \frac{|L(s_1, \pi\times\pi_{E^{(k)}_{1-s_2}})|^2}{|\xi(1-2it_2)|^2} \cdot \Gamma(s_1,0,w,\pi_\infty,E^{(k)}_{1-s_2,\infty})\, ds_1\, ds_2$$

where π' runs over (an orthogonal basis for) all level-one cuspforms on GL_2, with *no* restriction on the right K_∞-types, $E_s^{(k)}$ is the usual level-one Eisenstein series of K_∞-type k, and the notation $E^{(k)}_{1-s_2,\infty}$ means that the dependence is only upon the archimedean component. Here and throughout, for $\mathrm{Re}(s) = 1/2$, use $1-s$ in place of $\overline{s}$, to maintain holomorphy in complex-conjugated parameters.

More generally, for an irreducible cuspidal automorphic representation π on GL_r with $r \geq 3$, whether over $\mathbb{Q}$ or over a numberfield, the *moment expansion* includes an infinite sum of terms $|L(s, \pi\times\pi')|^2$ for π' ranging over irreducible cuspidal automorphic representations on GL_{r-1}, as well as *integrals* of products of L–functions $L(s,\pi\times\pi'_1)\ldots L(s,\pi\times\pi'_\ell)$ for $\pi'_1,\ldots,\pi'_\ell$ ranging over ℓ-tuples of cuspforms on $GL_{r_1}\times\ldots\times GL_{r_\ell}$ for all partitions $(r_1,\ldots,r_\ell)$ of r.

Generally, the spectral expansion of the Poincaré series for GL_r is an induced-up version of that for GL_2. Suppressing constants, using groundfield $\mathbb{Q}$ to skirt Hecke characters, the spectral expansion has the form

$$\mathfrak{P} \;=\; (\infty-\mathrm{part})\cdot E^{r-1,1}_{z+1} \;+\; \sum_{F \text{ on } GL_2} (\infty-\mathrm{part})\cdot \rho_{\overline{F}} \cdot L(\tfrac{rz+r-2}{2}+\tfrac{1}{2}, \overline{F})\cdot E^{r-2,2}_{\frac{z+1}{2},F}$$

$$+\; \int_{\mathrm{Re}(s)=\frac{1}{2}} (\infty-\mathrm{part})\cdot \frac{\zeta(\frac{rz+r-2}{2}+\frac{1}{2}-s)\cdot\zeta(\frac{rz+r-2}{2}+\frac{1}{2}+s)}{\zeta(2-2s)} \;\cdot\; E^{r-2,1,1}_{z+1,\,s-\frac{z+1}{2},\,-s-\frac{z+1}{2}}\;ds$$

where F is summed over an orthonormal basis for spherical cuspforms on GL_2, and where the Eisenstein series are naively normalized spherical, with $E^{r-1,1}_s$ a degenerate Eisenstein series attached to the parabolic corresponding to the partition $r-1,1$, and $E^{r-2,1,1}_{s_1,s_2,s_3,\chi}$ a degenerate Eisenstein series attached to the parabolic corresponding to the partition $r-2,1,1$.

Again over $\mathbb{Q}$, the *most-continuous* part of the moment expansion for GL_r is of the form

$$\int_{\mathrm{Re}(s)=\frac{1}{2}}\int_{t\in\Lambda} |L(s,\pi\times\pi_{E^{\min}_{\frac{1}{2}+it}})|^2\, M_t(s)\;ds\,dt = \int\int_\Lambda \left|\frac{\Pi_{1\leq\ell\leq r-1}\, L(s+it_\ell,\pi)}{\Pi_{1\leq j<\ell<n}\,\zeta(1-it_j+it_\ell)}\right|^2 M_t(s)\;ds\,dt$$

where

$$\Lambda = \{t\in\mathbb{R}^{r-1} \;:\; t_1+\ldots+t_{r-1}=0\}$$

and where M_t is a weight function depending upon π. More generally, let $r-1 = m\cdot b$. For π' irreducible cuspidal automorphic on GL_m, let

$${\pi'}^{\Delta} \;=\; \pi'\otimes\ldots\otimes\pi'$$

on $GL_m\times\ldots\times GL_m$. Inside the moment expansion we have (recall Langlands-Shahidi)

$$\int_{\mathrm{Re}(s)=\frac{1}{2}}\int_\Lambda |L(s,\pi\times\pi_{E_{\pi'^\Delta,\;\frac{1}{2}+it}})|^2\, M_{\pi',t}(s)\;ds\,dt = \int\int \left|\frac{\Pi_{1\leq\ell\leq b}\, L(s+it_\ell,\pi\times\pi')}{\Pi_{1\leq j<\ell\leq b}\, L(1-it_j+it_\ell,\pi'\times{\pi'}^\vee)}\right|^2 M\;ds\,dt$$

Replacing the cuspidal representation π on $GL_r(\mathbb{Q})$ by a (truncated) minimal-parabolic Eisenstein series E_α with $\alpha\in\mathbb{C}^{n-1}$, the most-continuous part of the moment expansion contains a term

$$\int\int_\Lambda \left|\frac{\Pi_{1\leq\mu\leq n,\;1\leq\ell\leq r-1}\,\zeta(\alpha_\mu+s+it_\ell)}{\Pi_{1\leq j<\ell<r}\,|\zeta(1-it_j+it_\ell)}\right|^2 ds\,dt$$

Taking $\alpha = 0\in\mathbb{C}^{r-1}$ gives

$$\int\int_\Lambda \left|\frac{\Pi_{1\leq\ell\leq r-1}\;\zeta(s+it_\ell)^r}{\Pi_{1\leq j<\ell<r}\,\zeta(1-it_j+it_\ell)}\right|^2 M\,ds\,dt$$

For example, for GL_3, where $\Lambda = \{(t,-t)\}\approx\mathbb{R}$,

$$\int\int_{\mathbb{R}} \left|\frac{\zeta(s+it)^3\cdot\zeta(s-it)^3}{\zeta(1-2it)}\right|^2 M\;ds\,dt$$

and for GL_4

$$\int_{(s)} \int_{\Lambda} \left| \frac{\zeta(s+it_1)^4 \cdot \zeta(s+it_2)^4 \cdot \zeta(s+it_3)^4}{\zeta(1-it_1+it_2)\,\zeta(1-it_1+it_3)\,\zeta(1-it_2+it_3)} \right|^2 M\,ds\,dt$$

2. *Background and normalizations*

We recall some facts concerning Whittaker models and Rankin-Selberg integral representations of L–functions, and spectral theory for automorphic forms, on GL_r. To compare zeta local integrals formed from vectors in cuspidal representations to local L–functions attached to the representations, we specify distinguished vectors in irreducible representations of p-adic and archimedean groups. Locally at both p-adic and archimedean places, Whittaker models with spherical vectors have a natural choice of distinguished vector, namely, the spherical vector taking value 1 at the identity element of the group.

Even in general, for the specific purposes here, at finite places the facts are clear. At archimedean places the facts are more complicated, and, further, the situation dictates choices of data, and we are not free to make ideal choices. See [Cogdell 2002], [Cogdell 2003], [Cogdell 2004] for detailed surveys, and references to the literature, mostly papers of Jacquet, Piatetski-Shapiro, and Shalika. The spectral theory is due to [Langlands 1976], [Moeglin-Waldspurger 1995], and proof of conjectures of [Jacquet 1983] in [Moeglin-Waldspurger 1989].

Fix an integer $r \geq 2$ and consider the general linear group $G = GL_r$ over a fixed algebraic number field k. For a positive integer ℓ, in the following we use the notation '$\ell \times \ell$' to denote an ℓ-by-ℓ matrix, and let 1_ℓ denote the $\ell \times \ell$ identity matrix. Then $G = GL_r$ has the following standard subgroups:

$$\begin{aligned}
P &= P^{r-1,1} = \left\{ \begin{pmatrix} (r-1)\times(r-1) & * \\ 0 & 1\times 1 \end{pmatrix} \right\} \\
U &= \left\{ \begin{pmatrix} 1_{r-1} & * \\ 0 & 1 \end{pmatrix} \right\} \quad H = \left\{ \begin{pmatrix} (r-1)\times(r-1) & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
N &= \{\text{upper-triangular unipotent elements in } H\} \\
&= (\text{unipotent radical of standard minimal parabolic in } H) \\
Z &= \text{center of } GL_r
\end{aligned}$$

Let $\mathbb{A} = \mathbb{A}_k$ be the adele ring of k. For a place v of k let k_v be the corresponding completion, with ring of integers $\mathfrak{o}_v$ for finite v. For an algebraic group defined over k, let G_v be the k_v-valued points of G. For $G = GL_r$ over k, let K_v be the standard maximal compact subgroup of G_v: for $v < \infty$, $K_v = GL_r(\mathfrak{o}_v)$ for $v \approx \mathbb{R}$, $K_v = O_r(\mathbb{R})$, and for $v \approx \mathbb{C}$, $K_v = U(r)$.

A standard choice of non-degenerate character on $N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}$ is

$$\psi(n \cdot u) = \psi_0(n_{12} + n_{23} + \ldots + n_{r-2,r-1}) \cdot \psi_0(u_{r-1,r})$$

where ψ_0 is a fixed non-trivial character on $\mathbb{A}/k$. A cuspform f has a Fourier-Whittaker expansion along NU

$$f(g) = \sum_{\xi \in N_k \backslash H_k} W_f(\xi g) \quad \text{where} \quad W_f(g) = \int_{N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}} \overline{\psi}(nu)\, f(nug)\, dn\, du$$

The Whittaker function $W_f(g)$ factors over primes, and a careful normalization of this factorization is set up below. Cuspforms F on H have corresponding Fourier-Whittaker expansions

$$F(h) = \sum_{\xi \in N'_k \backslash H'_k} W_F(\xi h) \quad \text{where} \quad W_F(g) = \int_{N'_k \backslash N'_{\mathbb{A}}} \overline{\psi}(n)\, F(nh)\, dn$$

where $H' \approx GL_{r-2}$ sits inside H as H sits inside G, $N' = N \cap H'$, and ψ is restricted from NU to N. This Whittaker function also factors $W_F = \bigotimes_v W_{F,v}$.

At finite places v, given an irreducible admissible representation π_v of G_v admitting a Whittaker model, [Jacquet-PS-Shalika 1981] shows that there is an essentially unique *effective vector* $W^{\rm eff}_{\pi_v}$, generalizing the characterization of *newform* in [Casselman 1973], as follows. For π_v spherical, $W^{\rm eff}_{\pi_v}$ is the usual unique spherical Whittaker vector taking value 1 at the identity element of the group, as in [Shintani 1976], [Casselman-Shalika 1980]. For non-spherical local representations, define *effective vector* as follows. Let

$$U_v^{\rm opp}(\ell) \;=\; \{\begin{pmatrix} 1_{r-1} & 0 \\ x & 1 \end{pmatrix} \;:\; x = 0 \bmod \mathfrak{p}^\ell\}$$

Let $K_v^H \approx GL_{r-1}(\mathfrak{o}_v)$ be the standard maximal compact of H_v. Define a congruence subgroup of K_v by

$$K_v(\ell) \;=\; K_v^H \cdot (U_v \cap K_v) \cdot U_v^{\rm opp}(\ell)$$

For a non-spherical Whittaker model π_v there is a unique positive integer ℓ_v, the *conductor* of π_v, such that π_v has *no* non-zero vectors fixed by $K_v(\ell')$ for $\ell' < \ell_v$, and has a one-dimensional space of vectors fixed by $K_v(\ell_v)$. The remaining ambiguous constant is completely specified by requiring that local Rankin-Selberg integrals

$$Z_v(s, W^{\rm eff}_{\pi_v} \times W^o_{\pi'_v}) \;=\; \int_{N_v\backslash H_v} |\det Y|^s \, W^{\rm eff}_{\pi_v}\begin{pmatrix} Y & \\ & 1 \end{pmatrix} W^o_{\pi'_v}(Y)\, dY$$

produce the correct local factors $L_v(s, \pi_v \times \pi'_v)$ of $GL_r \times GL_{r-1}$ Rankin-Selberg L–functions for every *spherical* representation π'_v of the local GL_{r-1}, with normalized spherical Whittaker vector $W^o_{\pi'_v}$ in π'_v. That is,

$$Z_v(s, W^{\rm eff}_{\pi_v} \times W^o_{\pi'_v}) \;=\; L_v(s, \pi_v \times \pi'_v)$$

with no additional factor on the right-hand side. See Section 4 of [Jacquet-PS-Shalika 1983], and comments below. Cuspidal automorphic representations $\pi \approx \bigotimes'_v \pi_v$ of $G_{\mathbb{A}}$ admit local Whittaker models at all finite places, so locally at all finite places have a unique effective vector.

Facts concerning archimedean local Rankin-Selberg integrals for $GL_m \times GL_n$ for general m, n are more complicated than the non-archimedean cases. See [Stade 2001], [Stade 2002], [Cogdell-PS 2003], as well as the surveys [Cogdell 2002], [Cogdell 2003], [Cogdell 2004]. The *spherical* case for $GL_r \times GL_{r-1}$ admits fairly explicit treatment, but this is insufficient for our purposes. Fortunately, for us there is no compulsion to attempt to specify the archimedean local data for Rankin-Selberg integrals. Indeed, the local archimedean Rankin-Selberg integrals will be *deformed* into shapes essentially unrelated to the corresponding L-factor, in any case. Thus, in the *moment expansion* in the theorem below we can use *any* systematic specification of distinguished vectors e_{π_v} in irreducible representations π_v of G_v, and $e_{\pi'_v}$ in π'_v of H_v, for v archimedean. For $v|\infty$ and π_v a Whittaker model representation of G_v with a spherical vector, let the distinguished vector e_{π_v} be the spherical vector normalized to take value 1 at the identity element of the group. Similarly, for π'_v a Whittaker model representation of H_v with a spherical vector, let the distinguished vector $e_{\pi'_v}$ be the normalized spherical vector. Anticipating that cuspforms generating spherical representations at archimedean places make up the bulk of the space of automorphic forms, we do not give an explicit choice of distinguished vector in other archimedean representations. Rather, we formulate the normalizations below, and the moment expansion, in a fashion applicable to *any* choice of distinguished vectors in archimedean representations.

Let π be an automorphic representation of $G_{\mathbb{A}}$, factoring over primes as $\pi \approx \bigotimes'_v \pi_v$ admitting a global Whittaker model. Each local representation π_v has a Whittaker model, since π has a global Whittaker model. At each finite place v, let $W^{\rm eff}_{\pi_v}$ be the normalized effective vector, and e_{π_v} the distinguished vector at $v|\infty$. Let $f \in \pi$ be a moderate-growth automorphic form on $G_{\mathbb{A}}$ corresponding to a monomial tensor in π, consisting of the effective vector at all finite primes, and the distinguished vector e_{π_v} at $v|\infty$. Then the global Whittaker function of f is a globally-determined constant multiple of the product of the local functions:

$$W_f \;=\; \rho_f \cdot \bigotimes_{v|\infty} e_{\pi_v} \;\otimes\; \bigotimes_{v<\infty} W^{\rm eff}_{\pi_v}$$

where ρ_f is a general analogue of the leading Fourier coefficient $\rho_f(1)$ in the $GL_2(\mathbb{Q})$ theory.

Let π' be an automorphic representation of $H_{\mathbb{A}}$ *spherical* at all finite primes, admitting a global Whittaker model. Let π' factor as $j : \bigotimes'_v \pi'_v \to \pi'$. Certainly each π'_v admits a Whittaker model. At each finite v, let $W^o_{\pi'_v}$ be the normalized spherical vector in π'_v, and at archimedean v let $e_{\pi'_v}$ be the distinguished vector. For a moderate-growth automorphic form $F \in \pi'$ corresponding to a monomial vector in the factorization of π', at all finite places corresponding to the spherical Whittaker function $W^o_{\pi'_v}$, and to the distinguished vector $e_{\pi'_v}$ at archimedean places, again specify a constant ρ_F by

$$W_F \;=\; \rho_F \cdot \bigotimes_{v|\infty} e_{\pi'_v} \;\otimes\; \bigotimes_{v<\infty} W^o_{\pi'_v}$$

When π' occurs discretely in the space of L^2 automorphic forms on H, each of the local factors of π' is unitarizable, and uniquely so up to a constant, by irreducibility. For an arbitrary vector $\varepsilon = \varepsilon_\infty$ in π'_∞, let F^ε be the automorphic form corresponding to $\bigotimes_{v<\infty} W^o_{\pi_v} \otimes \varepsilon$ by the isomorphism j. Define ρ_{F^ε} by

$$W_{F^\varepsilon} \;=\; \rho_{F^\varepsilon} \cdot \bigotimes_{v<\infty} W^o_{\pi_v} \otimes \varepsilon$$

By Schur's Lemma, the comparison of ρ_F and ρ_{F^ε} depends only upon the comparison of archimedean data, namely,

$$\frac{\rho_{F^\varepsilon}}{\rho_F} \;=\; \frac{|\varepsilon|_{\pi'_\infty}}{|\otimes_{v|\infty} e_{\pi'_v}|_{\pi'_\infty}}$$

with Hilbert space norms on the representation π'_∞ at archimedean places. The ambiguity of these norms by a constant disappears in taking ratios.

Indeed, the global constants ρ_F and ρ_{F^ε} can be compared by a similar device (and induction) for F and F_ε in any irreducible π' occurring in the L^2 automorphic spectral expansion for H. We do not do carry this out explicitly, since this would require setting up normalizations for the full spectral decomposition, while our main interest is in the cuspidal (hence, discrete) part.

With f cuspidal and F moderate growth, corresponding to distinguished vectors, as above, the Rankin-Selberg zeta integral is the finite-prime Rankin-Selberg L–function, with global constants ρ_f and ρ_F, and with archimedean local Rankin-Selberg zeta integrals depending upon the distinguished vectors at archimedean places:

$$\int_{H_k\backslash H_{\mathbb{A}}} |\det Y|^{s-\frac{1}{2}}\, F(Y)\, f\begin{pmatrix} Y & \\ & 1\end{pmatrix} dY \;=\; \rho_f \cdot \rho_F \cdot L(s, \pi\times\pi') \cdot \prod_{v|\infty} Z_v(s, e_{\pi_v} \times e_{\pi'_v})$$

The finite-prime part of the Rankin-Selberg L–function appears regardless of the archimedean local data. The global constants ρ_f and ρ_F do depend partly upon the local archimedean choices, but are global objects.

We need a spectral decomposition of part of $L^2(H_k\backslash H_{\mathbb{A}})$, as follows. Let K^H_{fin} be the standard maximal compact $GL_{r-1}(\widehat{\mathfrak{o}})$ of H_{fin}, where as usual $\widehat{\mathfrak{o}}$ is $\prod_{v<\infty} \mathfrak{o}_v$, with $\mathfrak{o}_v$ the local integers at the finite place v of k. First, there is the Hilbert direct-integral decomposition by characters ω on the *central archimedean split component* Z^+ of H: let

$$i \;:\; y \;\longrightarrow\; (y^{\frac{1}{d}}, \ldots, y^{\frac{1}{d}}, 1, 1, \ldots) \qquad\qquad (\text{for } y>0, \text{ with } d=[k:\mathbb{Q}])$$

be the diagonal imbedding of the positive real numbers in the archimedean factors of the ideles of k. The central archimedean split component is

$$Z^+ \;=\; \{ j(y) = \begin{pmatrix} i(y)^{1/(r-1)} & & \\ & \ddots & \\ & & i(y)^{1/(r-1)} \end{pmatrix} \in H_{\mathbb{A}} \;:\; y>0\}$$

The point of our parametrization is that (with idele norms)

$$|\det j(y)| \;=\; |i(y)| \;=\; y \qquad\qquad (\text{with } y>0)$$

The corresponding spectral decomposition is

$$L^2(H_k\backslash H_{\mathbb{A}}) \;\approx\; \int_{\mathbb{R}}^{\oplus} L^2(Z^+H_k\backslash H_{\mathbb{A}}, \omega_{it})\, dt$$

where $L^2(Z^+H_k\backslash H_{\mathbb{A}}, \omega_{it})$ is the isotypic component of functions Φ with $|\Phi|$ in $L^2(Z^+H_k\backslash H_{\mathbb{A}})$ transforming by

$$\Phi(j(y)\cdot h) \;=\; y^{it}\cdot \Phi(h) \qquad\qquad (\text{for } y>0 \text{ and } h\in H_{\mathbb{A}})$$

under Z^+. The projections and spectral synthesis along Z^+ can be written as

$$F(h) \;=\; \int_{\mathbb{R}} \Big(\int_0^\infty F(j(y)\cdot h)\, y^{-it}\, \frac{dy}{y}\Big)\, dt$$

Each isotypic component $L^2(Z^+H_k\backslash H_{\mathbb{A}}, \omega_{it})$ has a direct integral decomposition as a representation of $H_{\mathbb{A}}$, of the form

$$L^2(Z^+H_k\backslash H_{\mathbb{A}}, \omega_{it}) \;\approx\; \int_{\Xi}^{\oplus} \pi'\otimes|\det|^{it}\, d\pi'$$

where Ξ is the set of irreducibles π' occuring in $L^2(Z^+H_k\backslash H_{\mathbb{A}}, \omega_0)$. That is, the irreducibles for general archimedean split-component character ω_{it} differ merely by a determinant twist from the trivial split-component character case. The measure is not described explicitly here, apart from the observation that the discrete part of the decomposition, including the cuspidal part, has counting measure.

For our applications, we are concerned with the subspaces $L^2(Z^+H_k\backslash H_{\mathbb{A}}/K^H_{\text{fin}}, \omega)$ of right K^H_{fin}-invariant functions. Since each π' factors over primes as a restricted tensor product $\pi'\approx\bigotimes'_v \pi'_v$ of irreducibles π'_v of the local points H_v, the decomposition of $L^2(Z^+H_k\backslash H_{\mathbb{A}}/K^H_{\text{fin}}, \omega)$ only involves the subset Ξ^o consisting of irreducibles $\pi'\in\Xi$ such that for every *finite* place v the local representation π'_v is *spherical*. Let π'_∞ be the archimedean factor of π', and π'_{fin} the finite-place factor, so $\pi'\approx\pi'_\infty\otimes\pi'_{\text{fin}}$. Let π'^o_{fin} be the one-dimensional space of K^H_{fin}-fixed vectors in π'_{fin}. As a representation of the archimedean part H_∞ of $H_{\mathbb{A}}$,

$$L^2(Z^+H_k\backslash H_{\mathbb{A}}/K^H_{\text{fin}}, \omega_{it}) \;\approx\; \int_{\Xi^o}^{\oplus} \big(\pi'_\infty\otimes\pi'^o_{\text{fin}}\big)\otimes|\det|^{it}\, d\pi'$$

An automorphic spectral decomposition for F in $L^2(Z^+H_k\backslash H_{\mathbb{A}}/K^H_{\text{fin}}, \omega_{it})$ can be written in the form

$$F \;=\; \int_{\Xi^o} \sum_j \langle F, \Phi_{\pi' j}\otimes|\det|^{it}\rangle\cdot \Phi_{\pi' j}\otimes|\det|^{it}\, d\pi'$$

where Ξ^o indexes spherical automorphic representations π' with trivial archimedean split-component character entering the spectral expansion, for each of these j indexes an orthonormal basis in the archimedean component π'_∞, and $\Phi_{\pi' j}$ is the corresponding moderate-growth spherical automorphic form in the global π'. The pairing is the natural one, namely,

$$\langle F, \Phi_{\pi' j}\otimes|\det|^{it}\rangle \;=\; \int_{H_k\backslash H_{\mathbb{A}}} F(h)\, \overline{\Phi}_{\pi' j}(h)\, |\det h|^{-it}\, dh$$

3. *Moment expansion*

We define a Poincaré series $\mathfrak{P} = \mathfrak{P}_{z,\varphi_\infty}$ depending on archimedean data φ_∞ and a complex *equivariance* parameter z. With various simplifying choices of archimedean data φ_∞ depending only on a complex parameter w, the Poincaré series $\mathfrak{P} = \mathfrak{P}_{z,w}$ is a function of the two complex parameters z, w. By design, for a cuspform f of conductor ℓ on $G = GL_r$ over a number field k, for *any* choice of data for the Poincaré series sufficient for convergence, the integral

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} |f|^2 \cdot \mathfrak{P}$$

is an *integral moment* of L–functions attached to f, in the sense that it is a sum and integral over a spectral family, namely, a weighted average over spectral components with respect to $L^2(GL_{r-1}(k)\backslash GL_{r-1}(\mathbb{A}))$. Subsequently, we will obtain a spectral expansion of the more-simply parametrized Poincaré series $\mathfrak{P} = \mathfrak{P}_{z,w}$, giving the meromorphic continuation of this integral in the complex parameters z, w.

For $z \in \mathbb{C}$, let

$$\varphi \;=\; \bigotimes_v \varphi_v$$

where $z \in \mathbb{C}$ specifies an equivariance property of φ, as follows. For v finite,

$$\varphi_v(g) \;=\; \begin{cases} \big|(\det A)/d^{r-1}\big|_v^z & \text{(for } g = mk \text{ with } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \text{ in } Z_v H_v \text{ and } k \in K_v) \\ 0 & \text{(otherwise)} \end{cases}$$

For v archimedean require right K_v-invariance and left equivariance

$$\varphi_v(mg) \;=\; \left|\frac{\det A}{d^{r-1}}\right|_v^z \cdot \varphi_v(g) \qquad \text{(for } g \in G_v\text{, for } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_v H_v)$$

Thus, for $v|\infty$, the further data determining φ_v consists of its values on U_v. A simple useful choice of archimedean data parametrized by a single complex parameter w is

$$\varphi_v\begin{pmatrix} 1_{r-1} & x \\ 0 & 1 \end{pmatrix} \;=\; (1 + |x_1|^2 + \ldots + |x_{r-1}|^2)^{-[k_v:\mathbb{R}]w/2} \qquad \text{(where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix}\text{, and } w \in \mathbb{C})$$

The norm $|x_1|^2 + \ldots + |x_{r-1}|^2$ is normalized to be invariant under K_v. Thus, φ is left $Z_{\mathbb{A}} H_k$-invariant. We attach to any such φ a *Poincaré series*

$$\mathfrak{P}(g) \;=\; \mathfrak{P}_\varphi(g) \;=\; \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g)$$

3.1 Remark: There is an essentially unique choice of (parametrized) archimedean data $\varphi_\infty = \varphi_{z,w,\infty}$ such that the associated Poincaré series at $z = 0$ has a functional equation (as in [Diaconu-Garrett-Goldfeld 2008]). For instance, when $G = GL_3$ over $\mathbb{Q}$ this choice is

$$\varphi_\infty\begin{pmatrix} I_2 & u \\ & 1 \end{pmatrix} \;=\; \varphi_{0,w,\infty}\begin{pmatrix} I_2 & u \\ & 1 \end{pmatrix} \;=\; 2^{-w}\sqrt{\pi}\;\frac{\Gamma(\frac{w}{2})\left(1 + ||u||^2\right)^{-\frac{w}{2}} F\left(\frac{w}{2}, \frac{w}{2}; w; \frac{1}{1+||u||^2}\right)}{\Gamma(\frac{w+1}{2})} \qquad \text{(for } z = 0)$$

with F the usual hypergeometric function

$$F(\alpha, \beta; \gamma; x) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \sum_{m=0}^{\infty} \frac{1}{m!}\,\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)}\, x^m \qquad \text{(for } |x| < 1)$$

The functional equation of the Poincaré series $\mathfrak{P}_{0,w}(g)$ attached to this choice of $\varphi = \varphi_w$ when $z = 0$ is: *the function*

$$\sin\left(\frac{\pi w}{2}\right)\mathfrak{P}_{0,w}(g) \;+\; \frac{\pi\,\zeta(w)\,\zeta(2-w)}{2\,(1-w)\,\pi^{\frac{1}{2}-w}\,\Gamma(w-\frac{1}{2})\,\zeta(2w-1)} \cdot E^{1,1,1}\Big(g, \frac{w}{3}, 1 - \frac{2w}{3}\Big)$$

is invariant as $w \to 2 - w$, where $E^{1,1,1}(g, s_1, s_2) = E^{1,1,1}_{s_1,s_2}(g)$ is the minimal parabolic Eisenstein series. After our discussion of the spectral expansion of the Poincaré series, we give a general prescription for archimedean data producing Poincaré series admitting a functional equation: with suitable archimedean data, the functional equation is visible from the spectral expansion.

With subscripts ∞ denoting the archimedean parts of various objects, for $h, m \in H_\infty$, define

$$\mathcal{K}(h,m) \;=\; \mathcal{K}_{\varphi_\infty}(h,m) \;=\; \int_{U_\infty} \varphi_\infty(u)\, \psi_\infty(huh^{-1})\, \overline{\psi}_\infty(mum^{-1})\, du$$

Let $\pi \approx \otimes' \pi_v$ be a cuspidal automorphic representation of G, with finite set S of finite primes such that π_v is spherical for finite $v \notin S$, and π_v has conductor ℓ_v for $v \in S$. We say a cuspform f in π is a *newform* if it is spherical at finite $v \notin S$ and is right $K_v(\ell_v)$-fixed for $v \in S$. As above, the global Whittaker function W_f of f factors as

$$W_f \;=\; \rho_f \cdot \bigotimes_{v<\infty} W^{\mathrm{eff}}_{\pi_v} \otimes \bigotimes_{v|\infty} e_{\pi_v}$$

Let $e_{\pi_\infty} = \otimes_{v|\infty} e_{\pi_v}$. Let π' be an automorphic representation of H admitting a global Whittaker model, with unitarizable archimedean factor π'_∞, with orthonormal basis $\varepsilon_{\pi' j}$ for π'_∞. Recalling that $\mathcal{K}(h,m) = \mathcal{K}_{z,\varphi_\infty}(h,m)$ depends on the parameter z and the data φ_∞, the gamma factors appearing in the moment expansion below are

$$\Gamma(e_{\pi_\infty}, \pi'_\infty, s) \;=\; \Gamma_{z,\varphi_\infty}(e_{\pi_\infty}, \pi'_\infty, s)$$

$$= \sum_j \int_{N_\infty\backslash H_\infty} \int_{N_\infty\backslash H_\infty} \int_{K_\infty} e_{\pi_\infty}(hk)\varepsilon_{\pi' j}(h)|\det h|^{z+s-\frac{1}{2}}\overline{e}_{\pi_\infty}(mk)\overline{\varepsilon}_{\pi' j}(m)|\det m|^{\frac{1}{2}-s}\mathcal{K}(h,m)\, dm\, dh\, dk$$

The sum over the orthonormal basis for π'_∞ is simply an expression for a projection operator, so is necessarily independent of the orthonormal basis indexed by j. Thus, the sum indeed depends only on the archimedean Whittaker model π'_∞.

For each automorphic representation π' of H occurring (continuously or discretely) in the automorphic spectral expansion for H, and admitting a global Whittaker model, and *spherical* at all finite primes, let $F_{\pi'}$ be an automorphic form in π' corresponding to the spherical vector at all finite places and to the *distinguished vector* $e_{\pi'_\infty}$ in the archimedean part.

3.2 Theorem: Let f be a cuspform, as just above. For $\mathrm{Re}(z) \gg 1$ and $\mathrm{Re}(w) \gg 1$, we have the moment expansion

$$\int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} |f|^2 \cdot \mathfrak{P} \;=\; |\rho_f|^2 \int_{\Xi^o} |\rho_{F_{\pi'}}|^2 \int_{\mathbb{R}} L(\tfrac{1}{2}+it+z, \pi\otimes\pi')\, L(\tfrac{1}{2}-it, \overline{\pi}\otimes\overline{\pi}')\, \Gamma(e_{\pi_\infty}, \pi'_\infty, \tfrac{1}{2}+it)\, dt\, d\pi'$$

Proof: The typical first unwinding is

$$\int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2\, dg = \int_{Z_{\mathbb{A}} H_k\backslash G_{\mathbb{A}}} \varphi(g)\, |f(g)|^2\, dg$$

Express f in its Fourier-Whittaker expansion, and unwind further:

$$\int_{Z_{\mathbb{A}} H_k\backslash G_{\mathbb{A}}} \varphi(g) \sum_{\eta\in N_k\backslash H_k} W_f(\eta g)\, \overline{f}(g)\, dg \;=\; \int_{Z_{\mathbb{A}} N_k\backslash G_{\mathbb{A}}} \varphi(g)\, W_f(g)\, \overline{f}(g)\, dg$$

Use an Iwasawa decomposition $G = (HZ)UK$ everywhere locally to rewrite the whole integral as

$$\int_{N_k\backslash H_{\mathbb{A}}\times U_{\mathbb{A}}\times K_{\mathbb{A}}} \varphi(huk)\, W_f(huk)\, \overline{f}(huk)\, dh\, du\, dk$$

At finite primes $v \notin S$, the right integral over K_v can be dropped, since all the functions in the integrand are right K_v-invariant. At finite primes $v \in S$, using the newform assumption on f, the one-dimensionality of the $K_v(\ell_v)$-fixed vectors in π_v implies that the K_v-type in which the effective vector lies is *irreducible*. Thus, by Schur orthogonality and inner product formulas, a diagonal integral of $f(xk_v) \cdot \overline{f}(yk_v)$ over $k_v \in K_v$ is a positive constant multiple of $f(x)\overline{f}(y)$, for all $x, y \in G_{\mathbb{A}}$. Thus, the integrals over K_v for v finite can be dropped entirely, and, up to a positive constant depending only upon the right K_v-type of f at $v \in S$, the whole integral is

$$\int_{N_k\backslash H_{\mathbb{A}} \times U_{\mathbb{A}} \times K_\infty} \varphi(huk)\, W_f(huk)\, \overline{f}(huk)\, dh\, du\, dk$$

Since $\overline{f}$ is left H_k-invariant, it decomposes along $H_k\backslash H_{\mathbb{A}}$. The function $h \to f(huk)$ with $u \in U_{\mathbb{A}}$ and $k \in K_\infty$ is right $K^H_{\rm fin}$-invariant. Thus, $\overline{f}$ decomposes as

$$\overline{f}(huk) \;=\; \int_{\mathbb{R}} \int_{\Xi^o} \sum_j \Phi_{\pi' j}(h)\, |\det h|^{it} \int_{H_k\backslash H_{\mathbb{A}}} \overline{\Phi}_{\pi' j}(m) |\det m|^{-it} \overline{f}(muk)\, dm\; d\pi'\; dt$$

Unwind the Fourier-Whittaker expansion of $\overline{f}$

$$\overline{f}(huk) \;=\; \int_{\Xi^o} \sum_j \Phi_{\pi' j}(h) |\det h|^{it} \int_{H_k\backslash H_{\mathbb{A}}} \overline{\Phi}_{\pi' j}(m)\, |\det m|^{-it} \sum_{\eta \in N_k\backslash H_k} \overline{W}_f(\eta muk)\, dm\, dk\, d\pi'$$

$$= \int_{\Xi^o} \Phi_{\pi' j}(h)\, |\det h|^{it} \int_{N_k\backslash H_{\mathbb{A}}} \overline{\Phi}_{\pi' j}(m)\, |\det m|^{-it}\, \overline{W}_f(muk)\, dm\, dk\, d\pi'$$

Then the whole integral is

$$\int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2\, dg$$

$$= \int_{\mathbb{R}} \int_{\Xi^o} \sum_j \int_{N_k\backslash H_{\mathbb{A}}} \int_{U_{\mathbb{A}}} \int_{K_\infty} \varphi(huk) \Phi_{\pi' j}(h) |\det h|^{it} W_f(huk) \int_{N_k\backslash H_{\mathbb{A}}} \overline{W}_f(muk) \overline{\Phi}_{\pi' j}(m) |\det m|^{-it} dm\, dh\, du\, dk\, d\pi'\, dt$$

The part of the integrand that depends upon $u \in U$ is

$$\int_{U_{\mathbb{A}}} \varphi(huk)\, W_f(huk)\, \overline{W}_f(muk)\, du \;=\; \varphi(h)\, W_f(hk)\, \overline{W}_f(mk) \cdot \int_{U_{\mathbb{A}}} \varphi(u)\, \psi(huh^{-1})\, \overline{\psi}(mum^{-1})\, du$$

The latter integrand and integral visibly factor over primes. We need the following:

3.3 Lemma: Let v be a finite prime. For $h, m \in H_v$ such that $W^{\rm eff}_{\pi_v}(h) \neq 0$ and $W^{\rm eff}_{\pi_v}(m) \neq 0$,

$$\int_{U_v} \varphi_v(h)\, \psi_v(huh^{-1})\, \overline{\psi}_v(mum^{-1})\, du \;=\; \int_{U_v \cap K_v} 1\; du$$

Proof: At a finite place v, $\varphi_v(u) \neq 0$ if and only if $u \in U_v \cap K_v$, and for such u

$$\psi_v(huh^{-1}) \cdot W_{\pi_v}(h) = W^{\rm eff}_{\pi_v}(huh^{-1} \cdot h) = W^{\rm eff}_{\pi_v}(hu) = W^{\rm eff}_{\pi_v}(h) \cdot 1$$

by the right $U_v \cap K_v$-invariance, since f is a *newform*, in our present sense. Thus, for $W^{\rm eff}_{\pi_v}(h) \neq 0$, $\psi_v(huh^{-1}) = 1$, and similarly for $\psi_v(mum^{-1})$. Thus, the finite-prime part of the integral over U_v is just the integral of 1 over $U_v \cap K_v$, as indicated. ///

Returning to the proof of the theorem, the archimedean part of the integral does not behave as the previous lemma indicates the finite-prime components do, because of its non-trivial deformation by φ_∞. Thus, with subscripts ∞ denoting the infinite-adele part of various objects, for $h, m \in H_\infty$, as above, let

$$\mathcal{K}(h, m) \;=\; \int_{U_\infty} \varphi_\infty(u)\, \psi_\infty(huh^{-1})\, \overline{\psi}_\infty(mum^{-1})\, du$$

The whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2 \, dg$$

$$= \int_{\mathbb{R}} \int_{\Xi^o} \sum_j \int_{K_\infty} \int_{N_k\backslash H_{\mathbb{A}}} \int_{N_k\backslash H_{\mathbb{A}}} \mathcal{K}(h,m)\varphi(h) W_f(hk)\Phi_{\pi' j}(h)|\det h|^{it}\overline{W}_f(mk)\overline{\Phi}_{\pi' j}(m)|\det m|^{-it} dm\, dh\, d\pi'\, dk\, dt$$

Normalize the volume of $N_k\backslash N_{\mathbb{A}}$ to 1. For a left N_k-invariant function Φ on $H_{\mathbb{A}}$, using the left $N_{\mathbb{A}}$-equivariance of W by ψ, and the left $N_{\mathbb{A}}$-invariance of φ,

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(nh)\, \Phi(nh)\, W_f(nhk)\, dn \;=\; \varphi(h)\, W_f(h) \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\, \Phi(nh)\, dn \;=\; \varphi(h)\, W_f(hk)\, W_\Phi(h)$$

where

$$W_\Phi(h) = \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\, \Phi(nh)\, dn$$

(The integral is not against $\overline{\psi}(n)$, but $\psi(n)$.) That is, the integral over $N_k\backslash H_{\mathbb{A}}$ is equal to an integral against (up to an alteration of the character) the Whittaker function W_Φ of Φ, which factors over primes for suitable Φ. Thus, the whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2 \, dg$$

$$= \int_{\mathbb{R}} \int_{\Xi^o} \sum_j \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{K_\infty} \mathcal{K}(h,m) W_f(hk) W_{\Phi_{\pi' j}}(h)|\det h|^{it}\overline{W}_f(mk)\overline{W}_{\Phi_{\pi' j}}(m)|\det m|^{-it} dm\, dh\, d\pi'\, dk\, dt$$

For fixed π', j, t, the integral over m, h, k is a product of two Euler products, since the Whittaker functions factor over primes, normalized by global constants ρ_f and $\rho_{\Phi_{\pi' j}}$. The functions $\{\Phi_{\pi',j} : j\}$ correspond to an orthonormal basis $\{\varepsilon_{\pi' j}\}$ in the local archimedean part π'_∞ of π', so, as noted earlier, by Schur's lemma the global constant $\rho_{\Phi_{\pi' j}}$ is independent of j. For each π', let $F_{\pi'}$ be the finite-prime spherical automorphic form corresponding to distinguished vectors at archimedean places. The $\Phi_{\pi' j}$'s are normalized spherical at all finite places. Thus, for each π' and j,

$$\int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{K_\infty} \varphi(h) W_f(hk) W_{\Phi_{\pi' j}}(h)|\det h|^{it}\overline{W}_f(mk)\overline{W}_{\Phi_{\pi' j}}(m)|\det m|^{-it} dm\, dh\, dk$$

$$= \; |\rho_f|^2 \cdot |\overline{\rho}_{F_{\pi'}}|^2 \cdot L(\tfrac{1}{2} + it + z, \pi\times\pi')\, L(\tfrac{1}{2} - it, \pi\times\pi')$$

$$\times \int_{N_\infty\backslash H_\infty} \int_{N_\infty\backslash H_\infty} \int_{K_\infty} \int_{K_\infty} e_{\pi_\infty}(huk)\varepsilon_{\pi' j}(h)|\det h|^{it}\overline{\varepsilon}_{\pi' j}(m)\overline{e}_{\pi_\infty}(muk)|\det m|^{-it} dm\, dh\, dk$$

This gives the assertion of the theorem. ///

3.4 Remark: Automorphic forms not admitting Whittaker models do not enter this expansion.

4. *Spectral expansion of Poincaré series*

The Poincaré series admits a spectral expansion facilitating its meromorphic continuation. The only cuspidal data appearing in this expansion is from GL_2, right K_v-invariant everywhere locally.

In the Poincaré series $\mathfrak{P}$, let φ_∞ be the archimedean data, and z, w the two complex parameters. For a spherical GL_2 cuspform F, let

$$\Phi_{s,F}\Big(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \cdot \theta \Big) \;=\; |\det A|^{2s} \cdot |\det D|^{-(r-2)s} \cdot F(D) \qquad\qquad (\text{where } \theta \in K_{\mathbb{A}})$$

and define an Eisenstein series

$$E^{r-2,2}_{s,F}(g) \;=\; \sum_{\gamma \in P^{r-2,2}_k \backslash G_k} \Phi_{s,F}(\gamma \cdot g)$$

Also, for a Hecke character $\overline{\chi}$, with

$$\Phi_{s_1,s_2,s_3,\chi}(\begin{pmatrix} A & * & * \\ 0 & m_2 & * \\ 0 & 0 & m_3 \end{pmatrix} \cdot \theta) = \; |\det A|^{s_1} \cdot |m_2|^{s_2} \chi(m_2) \cdot |m_3|^{s_3} \overline{\chi}(m_3) \qquad \text{(for } \theta \in K_{\mathbb{A}},\ A \in GL_{r-2})$$

define an Eisenstein series

$$E^{r-2,1,1}_{s_1,s_2,s_3,\chi}(g) \;=\; \sum_{\gamma \in P^{r-2,1,1}_k \backslash G_k} \Phi_{s_1,s_2,s_3,\chi}(\gamma g)$$

4.1 Theorem: With Eisenstein series as just above, the Poincaré series $\mathfrak{P}$ has a spectral expansion

$$\mathfrak{P} \;=\; \Big(\int_{N_\infty} \varphi_\infty \Big) E^{r-1,1}_{z+1} \;+\; \sum_F \Big(\int_{PGL_2(k_\infty} \widetilde{\varphi}_\infty W_{\overline{F},\infty} \Big) \cdot \rho_{\overline{F}} \cdot L(\tfrac{rz+r-2}{2} + \tfrac{1}{2}, \pi_{\overline{F}}) \cdot E^{r-2,2}_{\frac{z+1}{2},F}$$

$$+ \sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \Bigg(\Big(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W_{E_{1-s,\overline{\chi}},\infty} \Big)$$

$$\times \frac{L(\frac{rz+r-2}{2} + 1 - s, \overline{\chi}) \cdot L(\frac{rz+r-2}{2} + s, \chi)}{\Lambda(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rz+r-2}{2} + s - \frac{1}{2})} \cdot E^{r-2,1,1}_{z+1, s - \frac{(r-2)(z+1)}{2}, -s - \frac{(r-2)(z+1)}{2}, \chi} \Bigg)\, ds$$

where F runs over an orthonormal basis for everywhere-spherical cuspforms for GL_2, $\overline{\rho}_F$ is the GL_2 leading Fourier coefficient of $\overline{F}$, χ runs over unramified grossencharacters, $\mathfrak{d}$ is the differental ideal of k, κ is the residue of $\zeta_k(s)$ at $s = 1$, $W_{F,\infty}$ and $W_{E_{s,\chi}}$ are the normalized archimedean Whittaker functions for GL_2, $\pi_{\overline{F}}$ is the representation generated by $\overline{F}$, $L(s, \chi)$ is the usual grossencharacter L–function, and $\Lambda(s, \chi)$ is the grossencharacter L–function with its gamma factor.

4.2 Remark: Notably, the spectral expansion of $\mathfrak{P}$ contains nothing beyond the main term, the cuspidal GL_2 part induced up to GL_r, and the continuous GL_2 part induced up to GL_r.

Proof: Rewrite the Poincaré series as summed in two stages, and apply Poisson summation, namely

$$\mathfrak{P}(g) = \sum_{Z_k H_k \backslash G_k} \varphi(\gamma g) = \sum_{Z_k H_k U_k \backslash G_k} \; \sum_{\beta \in U_k} \varphi(\beta \gamma g) = \sum_{Z_k H_k U_k \backslash G_k} \; \sum_{\psi \in (U_k \backslash U_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi)$$

where

$$\widehat{\varphi}_g(\psi) = \int_{U_{\mathbb{A}}} \overline{\psi}(u)\, \varphi(ug)\, du \qquad \text{(for } g \in G_{\mathbb{A}})$$

The inner summand for ψ *trivial* gives the leading term in the spectral expansion of the Poincaré series. Specifically, it gives a vector from which a degenerate Eisenstein series for the $(r-1, 1)$ parabolic $P^{r-1,1} = ZHU$ is formed by the outer sum. That is,

$$g \to \int_{U_{\mathbb{A}}} \varphi(ug)\, du$$

is left equivariant by a character on $P^{r-1,1}_{\mathbb{A}}$, and is left invariant by $P^{r-1,1}_k$, namely,

$$\int_{U_{\mathbb{A}}} \varphi(upg)\, du = \int_{U_{\mathbb{A}}} \varphi(p \cdot p^{-1} up \cdot g)\, du = \delta_{P^{r-1,1}}(m) \cdot \int_{U_{\mathbb{A}}} \varphi(m \cdot u \cdot g)\, du$$

$$= \left| \frac{\det A}{d^{r-1}} \right|^{z+1} \int_{U_{\mathbb{A}}} \varphi(ug)\, du \qquad \text{(where } p = \begin{pmatrix} A & * \\ 0 & d \end{pmatrix},\ m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix},\ A \in GL_{r-1})$$

The normalization is explicated by setting $g = 1$:

$$\int_{U_\mathbb{A}} \varphi(u)\, du = \int_{U_\infty} \varphi_\infty \cdot \int_{U_{\rm fin}} \varphi_{\rm fin} = \int_{U_\infty} \varphi_\infty \cdot {\rm meas}\,(U_{\rm fin} \cap K_{\rm fin}) = \int_{U_\infty} \varphi_\infty$$

A natural normalization is that this be 1, so the Eisenstein series includes the archimedean integral and finite-prime measure constant as factors:

$$\int_{U_\infty} \varphi_\infty \quad\cdot\quad E^{r-1,1}_{z+1}(g) \;=\; \sum_{\gamma \in P^{r-1,1}_k \backslash G_k} \left(\int_{U_\mathbb{A}} \varphi(u\gamma g)\, du \right)$$

The group H_k is transitive on non-trivial characters of $U_k \backslash U_\mathbb{A}$. For fixed non-trivial character ψ_0 on $k\backslash\mathbb{A}$, let

$$\psi^\xi(u) = \psi_0(\xi \cdot u_{r-1,r}) \qquad\qquad (\text{for } \xi \in k^\times)$$

The spectral expansion of $\mathfrak{P}$ with its leading term removed is

$$\sum_{\gamma \in P^{r-1,1}_k \backslash G_k} \;\; \sum_{\alpha \in P^{r-2,1}_k \backslash H_k} \left(\sum_{\xi \in k^\times} \widehat{\varphi}_{\alpha\gamma g}(\psi^\xi) \right)$$

where $P^{r-2,1}$ is the corresponding parabolic subgroup of $H \approx GL_{r-1}$. Let

$$U' \;=\; \{ \begin{pmatrix} 1_{r-2} & & * \\ & 1 & \\ & & 1 \end{pmatrix} \} \qquad U'' \;=\; \{ \begin{pmatrix} 1_{r-2} & & \\ & 1 & * \\ & & 1 \end{pmatrix} \}$$

Let

$$\Theta \;=\; \{ \begin{pmatrix} 1_{r-2} & & \\ & * & * \\ & * & * \end{pmatrix} \} \;\approx\; GL_2$$

Regrouping the sums, the expansion of the Poincaré series with its leading term removed is

$$\sum_{\gamma \in P^{r-2,1,1}_k \backslash G_k} \left(\sum_{\xi \in k^\times} \int_{U''_\mathbb{A}} \overline{\psi}^\xi(u'') \int_{U'_\mathbb{A}} \varphi(u'u''\gamma g)\, du'\, du'' \right)$$

$$= \sum_{\gamma \in P^{r-2,2}_k \backslash G_k} \;\; \sum_{\alpha \in P^{1,1} \backslash \Theta_k} \left(\sum_{\xi \in k^\times} \int_{U''_\mathbb{A}} \overline{\psi}^\xi(u'') \int_{U'_\mathbb{A}} \varphi(u'u''\alpha\gamma g)\, du'\, du'' \right)$$

Letting

$$\widetilde{\varphi}(g) = \int_{U'_\mathbb{A}} \varphi(u'g)\, du'$$

the expansion becomes

$$\sum_{\gamma \in P^{r-2,2}_k \backslash G_k} \;\; \sum_{\alpha \in P^{1,1} \backslash \Theta_k} \;\; \sum_{\xi \in k^\times} \int_{U''_\mathbb{A}} \overline{\psi}^\xi(u'')\, \widetilde{\varphi}(u''\alpha\gamma g)\, du''$$

We claim the equivariance

$$\widetilde{\varphi}(pg) = |\det A|^{z+1} \cdot |a|^z \cdot |d|^{-(r-1)z-(r-2)} \cdot \widetilde{\varphi}(g) \qquad\qquad (\text{for } p = \begin{pmatrix} A & * & * \\ & a & \\ & & d \end{pmatrix} \in G_\mathbb{A}, \text{ with } A \in GL_{r-2})$$

This is verified by changing variables in the defining integral: let $x \in \mathbb{A}^{r-2}$ and compute

$$\begin{pmatrix} 1_{r-2} & & x \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c+xd \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} \begin{pmatrix} 1_{r-2} & & A^{-1}xd \\ & 1 & \\ & & 1 \end{pmatrix}$$

Thus, $|\det A|^z \cdot |a|^z \cdot |d|^{-(r-1)z}$ comes out of the definition of φ, and another $|\det A| \cdot |d|^{2-r}$ from the change-of-measure in the change of variables replacing x by Ax/d in the integral defining $\widetilde{\varphi}$ from φ. Note that

$$|a|^z \cdot |d|^{-(r-1)z-(r-2)} = |\det \begin{pmatrix} a & \\ & d \end{pmatrix}|^{-\frac{(r-2)}{2}\cdot(z+1)} \cdot |a/d|^{\frac{rz+(r-2)}{2}}$$

Thus, letting

$$\Phi(g) \;=\; \sum_{\alpha \in P_k^{1,1}\backslash \Theta_k} \Big(\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\, \widetilde{\varphi}(u''\alpha g)\, du'' \Big)$$

we can write

$$\mathfrak{P}(g) \;-\; \sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \int_{U_{\mathbb{A}}} \varphi(u\gamma g)\, du = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi(\gamma g)$$

The right-hand side of the latter equality is not an Eisenstein series for $P^{r-2,2}$ in the strictest sense.

Define a GL_2 kernel $\varphi^{(2)}$ for a Poincaré series as follows. As in the general case, we require right invariance by the maximal compact subgroups locally everywhere, and left equivariance

$$\varphi^{(2)}(\begin{pmatrix} a & * \\ & d \end{pmatrix} \cdot D) \;=\; |a/d|^{\beta} \cdot \varphi^{(2)}(D)$$

The remaining ambiguity is the archimedean data $\varphi^{(2)}_\infty$, completely specified by giving its values on the archimedean part of the standard unipotent radical, namely,

$$\varphi^{(2)}_\infty \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;=\; \widetilde{\varphi} \begin{pmatrix} 1_{r-2} & & \\ & 1 & x \\ & & 1 \end{pmatrix} \qquad\qquad (\widetilde{\varphi} \text{ as above})$$

Let $U^{1,1}$ be the unipotent radical of the standard parabolic $P^{1,1}$ in GL_2. Express $\varphi^{(2)}$ in its Fourier expansion along $U^{1,1}$, and remove the constant term: let

$$\varphi^*(\beta, D) \;=\; \varphi^{(2)}(\beta, D) - \int_{U^{1,1}_{\mathbb{A}}} \varphi^{(2)}(\beta, uD)\, du \;=\; \sum_{\xi \in k^\times} \int_{U^{1,1}_{\mathbb{A}}} \overline{\psi}^{\xi}(u)\, \varphi^{(2)}(\beta, uD)\, du$$

The corresponding GL_2 Poincaré series with leading term removed is

$$\mathfrak{Q}(\beta, D) = \sum_{\alpha \in P_k^{1,1}\backslash GL_2(k)} \varphi^*(\beta, \alpha D)$$

Thus, for

$$g \;=\; \begin{pmatrix} A & * \\ & D \end{pmatrix} \qquad\qquad (\text{with } A \in GL_{r-2}(\mathbb{A}) \text{ and } D \in GL_2(\mathbb{A}))$$

the inner integral

$$g \to \int_{U''_{\mathbb{A}}} \overline{\psi}(u'')\, \widetilde{\varphi}(u'' g)\, du''$$

is expressible in terms of the kernel φ^* for $\mathfrak{Q}$, namely,

$$\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'')\, \widetilde{\varphi}(u''g)\, du'' \;=\; |\det A|^{z+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(z+1)} \cdot \varphi^*\left(\tfrac{rz+r-2}{2}, D\right)$$

Thus,

$$\sum_{\alpha \in P_k^{1,1}\backslash \Theta_k} \sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'')\, \widetilde{\varphi}(u''\alpha g)\, du'' \;=\; |\det A|^{z+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(z+1)} \cdot \mathfrak{Q}\left(\tfrac{rz+r-2}{2}, D\right)$$

Thus, letting

$$\Phi\begin{pmatrix} A & * \\ & D \end{pmatrix} \;=\; |\det A|^{z+1} \cdot |\det D|^{-(r-2)\cdot\frac{z+1}{2}} \cdot \mathfrak{Q}(\tfrac{rz+r-2}{2}, D) \qquad (\text{with } A \in GL_{r-2} \text{ and } D \in GL_2)$$

we have

$$\mathfrak{P}(g) \;=\; \Big(\int_{U_\infty} \varphi_\infty\Big) \cdot E^{r_1,1}_{z+1}(g) \;+\; \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi(\gamma g)$$

To obtain a spectral decomposition of the Poincaré series $\mathfrak{P}$ for GL_r, we first recall from [Diaconu-Garrett 2009] the spectral decomposition of $\mathfrak{Q}$ for $r = 2$, and then form $P^{r-2,2}$ Eisenstein series from the spectral fragments.

As in [Diaconu-Garrett 2009], a direct computation shows that the spectral expansion of the GL_2 Poincaré series with constant term removed is

$$\mathfrak{Q}(\beta, D) \;=\; \sum_F \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W_{\overline{F},\infty}\right) \cdot \overline{\rho}_F \cdot L(\beta + \tfrac{1}{2}, \pi_{\overline{F}}) \cdot F$$

$$+ \;\sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W_{E_{1-s,\overline{\chi}},\infty}\right) \frac{L(\beta+1-s, \overline{\chi}) \cdot L(\beta+s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\beta+s-1/2)} \cdot E_{s,\chi}(D)\, ds$$

where F runs over an orthonormal basis of everywhere-spherical cuspforms, $\overline{\rho}_F$ is the general GL_2 analogue of the leading Fourier coefficient, $\pi_{\overline{F}}$ is the cuspidal automorphic representation generated by $\overline{F}$, $W_{\overline{F},\infty}$ and $W_{E_{s,\chi},\infty}$ are the normalized spherical vectors in the corresponding archimedean Whittaker models, $\Lambda(s,\chi)$ is the standard L–function completed by adding the archimedean factors, and $\mathfrak{d}$ is the differental idele. Thus, the individual spectral components of Φ are of the form

$$\Phi_{\frac{z+1}{2},\Psi}\Big(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \cdot \theta\Big) = (\text{constant}) \cdot |\det A|^{z+1} \cdot |\det D|^{-(r-2)\frac{z+1}{2}} \cdot \Psi(D) \qquad\qquad (\text{where } \theta \in K_{\mathbb{A}})$$

where Ψ is either a spherical GL_2 cuspform or a spherical GL_2 Eisenstein series, in either case with trivial central character.

For Ψ a spherical GL_2 cuspform F averaging over $P_k^{r-2,2}\backslash G_k$ produces a half-degenerate Eisenstein series

$$E^{r-2,2}_{\frac{z+1}{2},F}(g) \;=\; \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{z+1}{2},F}(\gamma \cdot g)$$

As in the appendix, the half-degenerate Eisenstein series $E^{r-2,2}_{s,F}$ has *no poles* in $\mathrm{Re}\,(s) \geq 1/2$. With $s = (z+1)/2$ this assures absence of poles in $\mathrm{Re}\,(z) \geq 0$.

The continuous spectrum part of $\mathfrak{Q}$ produces degenerate Eisenstein series on G, as follows. With $\Psi = E_{s,\chi}$ the usual spherical, trivial central character, Eisenstein series for GL_2, define an Eisenstein series

$$E^{r-2,2}_{\frac{z+1}{2},E_{s,\chi}}(g) \;=\; \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{z+1}{2},E_{s,\chi}}(\gamma g)$$

As usual, for $\mathrm{Re}(s) \gg 0$ and $\mathrm{Re}(z) \gg 0$, this iterated formation of Eisenstein series is equal to a single-step Eisenstein series. That is, let

$$\Phi_{s_1,s_2,s_3,\chi}(\begin{pmatrix} A & * & * \\ 0 & m_2 & * \\ 0 & 0 & m_3 \end{pmatrix} \cdot \theta) = \ |\det A|^{s_1} \cdot |m_2|^{s_2}\chi(m_2) \cdot |m_3|^{s_3}\overline{\chi}(m_3) \qquad (\text{for } \theta \in K_{\mathbb{A}},\ A \in GL_{r-2})$$

and

$$E^{r-2,1,1}_{s_1,s_2,s_3,\chi}(g) \ = \sum_{\gamma \in P_k^{r-2,1,1}\backslash G_k} \Phi_{s_1,s_2,s_3,\chi}(\gamma g)$$

Taking $s_1 = 2 \cdot \frac{z+1}{2}$, $s_2 = s - \frac{(r-2)(z+1)}{2}$, and $s_3 = -s - \frac{(r-2)(z+1)}{2}$,

$$E^{r-2,2}_{\frac{z+1}{2},E_{s,\chi}} \ = \ E^{r-2,1,1}_{z+1,s-\frac{(r-2)(z+1)}{2},-s-\frac{(r-2)(z+1)}{2},\chi}$$

Adding up these spectral components yields the spectral expansion of the Poincaré series. ///

4.3 Remark: Suitable archimedean data to give the Poincaré series a functional equation is best described in the context of the spectral expansion, and, due to the form of the spectral expansion, essentially reduces to GL_2. It is useful to describe the data via a *differential equation*, since this explains the outcome of the computation more transparently. Since each archimedean place affords its own opportunity for data choices, we simplify this aspect of the situation by taking groundfield $k = \mathbb{Q}$.

First, for $G = GL_2(\mathbb{Q})$, let Δ be the usual invariant Laplacian on the upper half-plane $\mathfrak{H}$, and consider the partial differential equations

$$(\Delta - s(s-1))^\nu\, u^\beta_{s,\nu} \ = \ \text{the distribution } f \to \int_0^\infty y^\beta \cdot f\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \qquad (1 \le \nu \in \mathbb{Z} \text{ and } s,\beta \in \mathbb{C})$$

on $\mathfrak{H}$. Further, require that $u^\beta_{s,\nu}$ have the same equivariance as the target distribution, namely,

$$u^\beta_{s,\nu}(t \cdot z) \ = \ t^\beta \cdot u^\beta_{s,\nu}(z) \qquad (\text{for } t > 0 \text{ and } z \in \mathfrak{H})$$

Then $u^\beta_{s,\nu}(x+iy) = y^\beta \cdot \varphi^\beta_{s,\nu}(x/y)$ for a function $\varphi^\beta_{s,\nu}$ on $\mathbb{R}$ satisfying the corresponding differential equation

$$\Big((1+x^2)\frac{\partial^2}{\partial x^2} + 2x(1-\beta)\frac{\partial}{\partial x} + \big(\beta(\beta-1) - s(s-1)\big)\Big)^\nu f \ = \ \delta \qquad (\text{with Dirac } \delta \text{ at } 0)$$

The generalized function δ is in the L^2 Sobolev space on $\mathbb{R}$ with index $-\frac{1}{2} - \varepsilon$ for every $\varepsilon > 0$. By elliptic regularity, solutions f to this differential equation are in the local Sobolev space with index $2\nu - \frac{1}{2} - \varepsilon$, and by Sobolev's lemma are locally at least $C^{2\nu-1-2\varepsilon} \subset C^{2\nu-2}$. That is, by increasing ν solutions are made as differentiable as desired, and their Fourier transforms will have corresponding decay, giving convergence of the Poincaré series (for suitable s, β), as in [Diaconu-Garrett 2009].

The spectral expansion of the GL_2 Poincaré series $\mathfrak{P}^\beta_{s,\nu}$ formed with this archimedean data $\varphi^\beta_{s,\nu}$ is special case of the computation in [Diaconu-Garrett 2009], recalled above, but in fact gives a much simpler outcome. For example, the *cuspidal* components are directly computed by unwinding, *integrating by parts*, and applying the characterization of $\varphi^\beta_{s,\nu}$ by the differential equation:

$$\langle \mathfrak{P}^\beta_{s,\nu},\, F\rangle \ = \ \frac{\overline{\rho}_F(1) \cdot \Lambda(\beta + \frac{1}{2}, \overline{F})}{\big(s_F(s_F-1) - s(s-1)\big)^\nu} \qquad (\text{where } \Delta F = s_F(s_F-1))$$

where $\Lambda(\cdot, F)$ is the L-function completed with its gamma factors. Thus,

$$\mathfrak{P}^\beta_{s,\nu} \ = \ \sum_F \frac{\overline{\rho}_F(1) \cdot \Lambda(\beta + \frac{1}{2}, \overline{F}) \cdot F}{\big(s_F(s_F-1) - s(s-1)\big)^\nu} \ + \ (\text{non-cuspidal})$$

summing over an orthonormal basis of cuspforms F. Granting convergence for ν sufficiently large and $\mathrm{Re}(s), \mathrm{Re}(\beta)$ large, the cuspidal part has a meromorphic continuation in s with poles at the values s_F, as expected. Visibly, the *cuspidal* part of $\mathfrak{P}^\beta_{s,\nu}$ is invariant under $s \leftrightarrow 1-s$, and in these coordinates the map $\beta \to -\beta$ maps F to $\overline{F}$ (whether or not F is self-contragredient).

The *leading term* of the spectral expansion of $\mathfrak{P}^\beta_{s,\nu}$, via Poisson summation, is a constant multiple $C^\beta_{s,\nu} \cdot E_{\beta+1}$ of the spherical Eisenstein series $E_{\beta+1}$. This happens regardless of the precise choice of archimedean data, simply due to the homogeneity we have required of the archimedean data throughout.

Similarly, the *continuous* part of this Poincaré series on GL_2 is

$$\frac{1}{4\pi i}\int_{\mathrm{Re}(s_e)=\frac{1}{2}} \frac{\xi(\beta+s_e)\,\xi(\beta+1-s_e)\cdot E_{s_e}}{\xi(2s_e)\cdot\big((s_e(s_e-1)-s(s-1)\big)^\nu}\,ds_e$$

where ξ is the ζ-function completed with its gamma factor. In analogy with the cuspidal discussion, the product $\xi(\beta+s_e)\cdot\xi(\beta+1-s_e)$ is invariant under $\beta\to-\beta$, since $\xi(1-z)=\xi(z)$. The *visual* symmetry in $s\leftrightarrow 1-s$ is slightly deceiving, since the meromorphic continuation (in s) through the critical line $\mathrm{Re}(s_e)=\frac{1}{2}$ (over which the Eisenstein series is integrated) introduces extra terms from residues at $s_e=s$ and $s_e=1-s$. Indeed, parts of these extra terms cancel a pole in the leading term $C^\beta_{s,\nu}\cdot E_{\beta+1}$ at $\beta=0$. Despite this subtlety in the continuous spectrum, the special choice of archimedean data makes meromorphic continuation in s,β visible.

In summary, for $GL_2(\mathbb{Q})$, the special choice of archimedean data makes the *cuspidal* part of the Poincaré series have a visible meromorphic continuation, and satisfy obvious functional equations. The *continuous* part of the Poincaré series satisfies functional equations modulo explicit leftover terms.

The ideal choice of archimedean data φ_∞ for the Poincaré series for $GL_r(\mathbb{Q})$ is such that the *averaged* version of it, denoted $\widetilde{\varphi}_\infty$ in the proof above, restricts to the function $\varphi^\beta_{s,\nu}$ for GL_2 just discussed: we want

$$\int_{\mathbb{R}^{r-2}} \varphi_\infty\begin{pmatrix}1_{r-2} & 0 & u\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} du \;=\; \widetilde{\varphi}_\infty\begin{pmatrix}1_{r-2} & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} \;=\; \varphi^\beta_{s,\nu}(x) \qquad\qquad (\text{for } x\in\mathbb{R})$$

It is not obvious that, given a reasonable (even) function f on $\mathbb{R}$, there is a rotationally symmetric function u on $\mathbb{R}^{r-2}$ such that

$$\int_{\mathbb{R}^{r-2}} u(y+xe_{r-1})\,dy \;=\; f(x) \qquad\qquad (e_i \text{ the standard basis for } \mathbb{R}^{r-1})$$

with $\mathbb{R}^{r-2}$ sitting in the first $r-2$ coordinates in $\mathbb{R}^{r-1}$. Fourier inversion clarifies this, as follows. Supposing the integral identity just above holds, integrate further in the $(r-1)^{th}$ coordinate x, against $e^{2\pi i\xi x}$, to obtain

$$\widehat{u}(\xi e_{r-1}) \;=\; \widehat{f}(\xi) \qquad\qquad (\text{for } \xi\in\mathbb{R})$$

where the Fourier transform on the left-hand side is on $\mathbb{R}^{r-1}$, on the right-hand side is on $\mathbb{R}$. For u rotationally invariant, $\widehat{u}$ is also rotationally invariant, and the latter equality can be rewritten as

$$\widehat{u}(\xi) \;=\; \widehat{f}(|\xi|) \qquad\qquad (\text{for } \xi\in\mathbb{R}^{r-1})$$

By Fourier inversion,

$$u(x) \;=\; \int_{\mathbb{R}^{r-1}} e^{2\pi i\langle\xi,x\rangle}\;\widehat{f}(|\xi|)\,d\xi \qquad\qquad (\text{for } x\in\mathbb{R}^{r-1})$$

That is, given an even function f on $\mathbb{R}$, the latter formula yields a rotationally invariant function on $\mathbb{R}^{r-1}$, whose averages along $\mathbb{R}^{r-2}$ are the given f. This proves existence of an essentially unique φ_∞ yielding the prescribed $\varphi^\beta_{s,\nu}$.

Then the functional equation of the most-cuspidal part of the special-data Poincaré series on GL_r is inherited from the functional equation of the cuspidal part of the special-data Poincaré series on GL_2.

5. *Appendix: half-degenerate Eisenstein series*

Take $q > 1$, and let f be a cuspform on $GL_q(\mathbb{A})$, in the strong sense that f is in $L^2(GL_q(k)\backslash GL_q(\mathbb{A})^1)$, and f meets the Gelfand-Fomin-Graev conditions

$$\int_{N_k\backslash N_{\mathbb{A}}} f(ng)\, dn = 0 \qquad\qquad \text{(for almost all } g)$$

and f generates an irreducible representation of $GL_q(k_v)$ locally at all places v of k. For a Schwartz function Φ on $\mathbb{A}^{q\times r}$ and Hecke character χ, let

$$\varphi(g) = \varphi_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(\mathbb{A})} f(h^{-1})\, \chi(\det h)^r\, \Phi(h\cdot [0_{q\times(r-q)}\ 1_q]\cdot g)\, dh$$

This function φ has the same central character as f. It is left invariant by the adele points of the unipotent radical

$$N = \{\begin{pmatrix} 1_{r-q} & * \\ & 1_r \end{pmatrix}\} \qquad\qquad \text{(unipotent radical of } P = P^{r-q,q})$$

The function φ is left invariant under the *k-rational* points M_k of the standard Levi component of P,

$$M = \{\begin{pmatrix} a & \\ & d \end{pmatrix}\ :\ a \in GL_{r-q},\ d\in GL_r\}$$

To understand the normalization, observe that

$$\xi(\chi^r, f, \Phi(0,*)) = \varphi(1) = \int_{GL_q(\mathbb{A})} f(h^{-1})\, \chi(\det h)^r\, \Phi(h\cdot [0_{q\times(r-q)}\ 1_q])\, dh$$

is a zeta integral as in [Godement-Jacquet 1972] for the standard L–function attached to the cuspform f. Thus, the Eisenstein series formed from φ includes this zeta integral as a factor, so write

$$\xi(\chi^r, f, \Phi(0,*))\ \cdot\ E^P_{\chi,f,\Phi}(g)\ =\ \sum_{\gamma\in P_k\backslash GL_r(k)} \varphi(\gamma\, g) \qquad\qquad \text{(convergent for } \mathrm{Re}(\chi)\gg 1)$$

The meromorphic continuation follows by Poisson summation:

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g)$$

$$= \chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{\alpha\in GL_q(k)} \Phi(h^{-1}\cdot [0\ \alpha]\cdot g)\, dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{full rank}} \Phi(h^{-1}\cdot y\cdot g)\, dh$$

The Gelfand-Fomin-Graev condition on f fits the full-rank constraint. Anticipating that we can drop the rank condition suggests that we define

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

As in [Godement-Jacquet 1972], the non-full-rank terms integrate to 0:

5.1 Proposition: For f a cuspform, less-than-full-rank terms integrate to 0, that is,

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{rank} < q} \Phi(h^{-1}\cdot y\cdot g)\, dh = 0$$

Proof: Since this is asserted for arbitrary Schwartz functions Φ, we can take $g = 1$. By linear algebra, given $y_0 \in k^{q\times r}$ of rank ℓ, there is $\alpha \in GL_q(k)$ such that

$$\alpha \cdot y_0 = \begin{pmatrix} y_{\ell\times r} \\ 0_{(q-\ell)\times r} \end{pmatrix} \qquad \text{(with } \ell\text{-by-}r \text{ block } y_{\ell\times r} \text{ of rank } \ell)$$

Thus, without loss of generality fix y_0 of the latter shape. Let Y be the orbit of y_0 under left multiplication by the rational points of the parabolic

$$P^{\ell,q-\ell} = \{\begin{pmatrix} \ell\text{-by-}\ell & * \\ 0 & (q-\ell)\text{-by-}(q-\ell) \end{pmatrix}\} \subset GL_q$$

This is some set of matrices of the same shape as y_0. Then the subsum over $GL_q(k)\cdot y_0$ is

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in GL_q(k)\cdot y_0} \Phi(h^{-1}\cdot y)\,dh = \int_{P_k^{\ell,q-\ell}\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in Y} \Phi(h^{-1}\cdot y)\,dh$$

Let N and M be the unipotent radical and standard Levi component of $P^{\ell,q-\ell}$,

$$N = \begin{pmatrix} 1_\ell & * \\ 0 & 1_{q-\ell} \end{pmatrix} \qquad M = \begin{pmatrix} \ell\text{-by-}\ell & 0 \\ 0 & (q-\ell)\text{-by-}(q-\ell) \end{pmatrix}$$

Then the integral can be rewritten as an iterated integral

$$\int_{N_k M_k\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in Y} \Phi(h^{-1}\cdot y)\,dh$$

$$= \int_{N_{\mathbb{A}} M_k\backslash GL_q(\mathbb{A})} \sum_{y\in Y} \int_{N_k\backslash N_{\mathbb{A}}} f(nh)\,\chi(\det nh)^{-r}\,\Phi((nh)^{-1}\cdot y)\,dn\,dh$$

$$= \int_{N_{\mathbb{A}} M_k\backslash GL_q(\mathbb{A})} \sum_{y\in Y} \chi(\det h)^{-r}\,\Phi(h^{-1}\cdot y) \left(\int_{N_k\backslash N_{\mathbb{A}}} f(nh)\,dn\right) dh$$

since all fragments but $f(nh)$ in the integrand are left invariant by $N_{\mathbb{A}}$. The inner integral of $f(nh)$ is 0, by the Gelfand-Fomin-Graev condition, so the whole is 0. ///

Let ι denote the transpose-inverse involution. Poisson summation gives

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

$$= |\det(h^{-1})^\iota|^r\,|\det g^\iota|^q \sum_{y\in k^{q\times r}} \widehat{\Phi}((h^\iota)^{-1}\cdot y\cdot g^\iota) \;=\; |\det(h^{-1})^\iota|^r\,|\det g^\iota|^q\,\Theta_{\widehat{\Phi}}(h^\iota, g^\iota)$$

As with Θ_Φ, the lower-rank summands in $\Theta_{\widehat{\Phi}}$ integrate to 0 against cuspforms. Thus, letting

$$GL_q^+ = \{h \in GL_q(\mathbb{A}) \;:\; |\det h| \ge 1\} \qquad GL_q^- = \{h \in GL_q(\mathbb{A}) \;:\; |\det h| \le 1\}$$

we have

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh + \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \quad \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$+ \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} |\det(h^{-1})^\iota|^r \, |\det g^\iota|^q \, f(h) \, \chi(\det h)^{-r} \, \Theta_{\widehat{\Phi}}(h^\iota, g^\iota) \, dh$$

By replacing h by h^ι in the second integral, convert it to an integral over $GL_q(k)\backslash GL_q^+$, and the whole is

$$\xi(\chi^r, f, \Phi(0,*)) \cdot E^P_{\chi,f,\Phi}(g) \;=\; \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h) \, \chi(\det h)^{-r} \, \Theta_\Phi(h, g) \, dh$$

$$+ \chi^{-1}(\det g^\iota)^q \int_{GL_q(k)\backslash GL_q^+} f(h^\iota) \;\; \nu\chi^{-1}(\det h^\iota)^{-r} \, \Theta_{\widehat{\Phi}}(h, g^\iota) \, dh$$

Since $f \circ \iota$ is a cuspform, the second integral is entire in χ. Thus, we have proven

$$\xi(\chi^r, f, \Phi(0,*)) \;\cdot E^P_{\chi,f,\Phi} \quad \text{is entire}$$

References

[Bruggeman 1978] R.W. Bruggeman, *Fourier coefficients of cusp forms*, Inv. Math. **45** no. 1 (1978), 1-18.

[Bruggeman-Motohashi 2001] R.W. Bruggeman, Y. Motohashi, *Fourth power moment of Dedekind zeta-functions of real quadratic number fields with class number one*, Funct. Approx. Comment. Math. **29** (2001), 41-79.

[Bruggeman-Motohashi 2003] R.W. Bruggeman, Y. Motohashi, *Sum formula for Kloosterman sums and fourth moment of the Dedekind zeta-function over the Gaussian number field*, Funct. Approx. Comment. Math. **31** (2003), 23-92.

[Casselman 1973] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **206** (1973), 311-318.

[Casselman-Shalika 1980] W. Casselman, J. Shalika, *The unramified principal series of p-adic groups, II, the Whittaker function*, Comp. Math. **41** (1980), 207-231.

[Cogdell 2002] J. Cogdell, *L–functions and Converse Theorems*, in *Automorphic Forms and Applications*, P. Sarnak, F. Shahidi eds, IAS/Park City Mathematics Series, vol. 12, AMS, Providence, 2007, 95-178.

[Cogdell 2003] J. Cogdell, *Analytic theory of L–functions for GL_n*, in *An Introduction to the Langlands Program*, J. Bernstein and S. Gelbart eds, Birhauser, 2003, 197-228.

[Cogdell 2004] J. Cogdell, *Lectures on L–functions, Converse Theorems, and Functoriality for GL_n*, Fields Institute Notes, in *Lectures on Automorphic L–functions*, Fields Institute Monographs no. 20, AMS, Providence, 2004.

[Cogdell-PS 2003] J. Cogdell, I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions*, in *Contributions to Automorphic Forms, Geometry, and Number Theory (Shalikafest 2002)*, H. Hida, D. Ramakrishnan, and F. Shahidi eds., Johns Hopkins University Press, Baltimore, 2005.

[CFKRS 2005] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, N.C. Snaith, *Integral Moments of L–functions*, Proc. London Math. Soc. **91** (2005), 33-104.

[Diaconu-Garrett 2009] A. Diaconu, P. Garrett, *Integral moments of automorphic L–functions*, J. Math. Inst. Jussieu **8** no. 2 (2009), 335-382.

[Diaconu-Garrett 2010] A. Diaconu, P. Garrett, *Subconvexity bounds for automorphic L–functions*, J. Math. Inst. Jussieu **9** no. 1 (2010), 95-124.

[Diaconu-Garrett-Goldfeld 2008] A. Diaconu, P. Garrett, D. Goldfeld, *Natural boundaries and integral moments of L-functions*, to appear, Proceedings of 2008 Edinburgh Conference on Multiple Dirichlet Series.

[Diaconu-Goldfeld 2006] A. Diaconu, D. Goldfeld, *Second moments of quadratic Hecke L–series and multiple Dirichlet series, I*, Proc. Symp. Pure Math. **75**, AMS, Providence, 2006, 59-89.

[Diaconu-Goldfeld 2007] A. Diaconu, D. Goldfeld, *Second moments of GL_2 automorphic L–functions*, in proceedings of Gauss-Dirichlet Conference (Göttingen, 2005), Clay Math. Proc., AMS (2007), 77-105.

[Diaconu-Goldfeld-Hoffstein 2005] A. Diaconu, D. Goldfeld, J. Hoffstein, *Multiple Dirichlet series and moments of zeta and L–functions*, Comp. Math. **139** no. 3 (2003), 297-360.

[Donnelly 1982] H. Donnelly, *On the cuspidal spectrum for finite volume symmetric spaces*, J. Diff. Geom. **17** (1982), 239-253.

[Godement-Jacquet 1972] R. Godement, H. Jacquet, *Zeta functions of simple algebras*, SLN 260, Springer-Verlag, Berlin, 1972.

[Goldfeld 2006] D. Goldfeld, *Automorphic forms and L–functions for the group $GL(n, \mathbb{R})$*, Cambridge Studies in Adv. Math. vol. 99, Cambridge Univ. Press, 2006.

[Good 1981] A. Good, *Cusp forms and eigenfunctions of the Laplacian*, Math. Ann. **255** (1981), 523-548.

[Good 1983] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika **29** (1983), 278-95.

[Good 1986] A. Good, *The convolution method for Dirichlet series*, in *Selberg Trace Formula and Related Topics (Brunswick, Maine, 1984)*, Contemp. Math. **53**, AMS, Providence, 1986, 207-214.

[Hardy-Littlewood 1918] G.H. Hardy, J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math **41** (1918), 119-196.

[Heath-Brown 1979] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. **38** no. 3 (1979), 385-422.

[Hoffstein-Lockhart 1994] J. Hoffstein, P. Lockhart, *Coefficients of Maass forms and the Siegel zero*, with appendix *An effective zero-free region*, by D. Goldfeld, J. Hoffstein, D. Lieman, Ann. of Math. **140** (1994), 161-181.

[Hoffstein-Ramakrishnan 1995] J. Hoffstein and D. Ramakrishnan, *Siegel zeros and cuspforms*, Int. Math. Research Notices **6** (1995), 279-308.

[Ingham 1926] A.E. Ingham, *Mean-value theorems in the theory of the Riemann zeta function*, Proc. London Math. Soc. **27** no. 2 (1926), 273-300.

[Iwaniec-Sarnak 2000] H. Iwaniec and P. Sarnak, *Perspectives on the analytic theory of L–functions*, GAFA special volume (2000), 705-741.

[Jacquet 1983] H. Jacquet, *On the residual spectrum of GL_n*, in *Lie Group Representations, II*, SLN 1041.

[Jacquet-PS-Shalika 1979] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Automorphic forms on GL_3, I, II*, Ann. of Math. **109** (1979), 169-212, 213-258.

[Jacquet-PS-Shalika 1981] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981),199-214.

[Jacquet-PS-Shalika 1983] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367-464.

[Jacquet-Shalika 1981] H. Jacquet, J. Shalika, *On Euler products and the classification of automorphic representations I, II*, Amer. J. Math. **103** (1981), 499-588, 777-815.

[Jacquet-Shalika 1990] H. Jacquet, J. Shalika, *Rankin-Selberg convolutions: archimedean theory*, in *Festschrift in Honor of I.I. Piatetski-Shapiro, I*, Weizmann Science Press, Jerusalem, 1990, 125-207.

[Jutila 1997] M. Jutila, *Mean values of Dirichlet series via Laplace transforms*, London Math. Soc. Lecture Notes, vol. 247, Cambridge Univ. Press, 1997, 169-207.

[Langlands 1976] R. Langlands, *On the functional equations satisfied by Eisenstein series*, SLN 544, 1975.

[Li 2009] X. Li, *Bounds for $GL_3 \times GL_2$ L–functions and GL_3 L–functions*, to appear, Ann. of Math.

[Lindenstrauss-Venkatesh 2007] E. Lindenstrauss and A. Venkatesh *Existence and Weyl's law for spherical cusp forms*, GAFA **17** vol 1 (2007), 220-251.

[Michel 2007] P. Michel, *Analytic number theory and families of automorphic L–functions*, in IAS/Park City Summer Institute Series Vol 12 *Automorphic forms and applications*, (2007), 179-296.

[Moeglin-Waldspurger 1989] C. Moeglin, J.-L. Waldspurger, *Le spectre résiduel de GL_n*, with appendix *Poles des fonctions L de pairs pour GL_n*, Ann. Sci. École Norm. Sup. **22** (1989), 605-674.

[Moeglin-Waldspurger 1995] C. Moeglin, J.-L. Waldspurger, *Spectral Decompositions and Eisenstein series*, Cambridge Univ. Press, Cambridge, 1995.

[Motohashi 1997] Y. Motohashi, *Spectral theory of the Riemann zeta function*, Cambridge Tracts in Mathematics 127, Cambridge Univ. Press, 1997.

[Petridis-Sarnak 2001] Y. Petridis and P. Sarnak, *Quantum unique ergodicity for $SL_2(\mathscr{O})\backslash H^3$ and estimates for L–functions*, J. Evol. Equ. **1** (2001), 277-290.

[Sarnak 1985] P. Sarnak, *Fourth moments of Grössencharakteren zeta functions*, Comom. Pure Appl. Math. **38** no. 2 (1985), 167-178.

[Shalika 1974] J. Shalika, *The multiplicity-one theorem for GL_n*, Ann. of Math. **100** (1974), 171-193.

[Shintani 1976] T. Shintani, *On an explicit formula for class-one Whittaker functions on GL_n over p-adic fields*, Proc. Japan Acad. **52** (1976), 180-182.

[Stade 2001] E. Stade, *Mellin transforms of $GL_n(\mathbb{R})$ Whittaker functions*, Amer. J. Math. **123** (2001), 121-161.

[Stade 2002] E. Stade, *Archimedean L-factors on $GL_n \times GL_n$ and generalized Barnes integrals*, Israel J. Math. **127** (2002), 201-209.

[Wallach 1992] N.R. Wallach, *Real reductive groups, II*, Pure and Applied Math. vol. 132, Academic Press, Boston, 1992.

[Ye 2000] Y. Ye, *A Kuznetsov formula for Kloosterman sums on GL_n*, Ramanujan J. **4** no. 4 (2000), 385-395.

[Young 2009] M. Young, *The second moment of $GL(3) \times GL(2)$ L–functions, integrated*, preprint, arXiv:0903.1575v1.