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Dirac and Casimir operators

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An important contrast between a Dirac operator $\mathbb{D}$ and Casimir operator Ω attached to a semi-simple Lie group G, as elements of $C\mathfrak{p} \otimes U\mathfrak{g}$ and $U\mathfrak{g}$, respectively, is that the action of G on functions on G attached to right translation *does not* directly involve the enhanced coefficients $C\mathfrak{p}$.

Abstract equivariance properties are manifest differently. Significantly, while Ω is left-and-right G invariant as a right (differential) operator on functions on G, $\mathbb{D}$ (acting as a differential operator on the right) is certainly left G-invariant, but *not* right invariant at all, although it has *intelligible* behavior under right translation by K. Specifically, it does not map the space of right K-invariant functions on G to itself, but, rather, to a space of vector-valued functions transforming equivariantly under right translation by K. Yes, as an element of $C\mathfrak{p} \otimes U\mathfrak{g}$ with K acting diagonally on *both* factors, $\mathbb{D}$ is K-invariant, but the natural right-translation action *does not* have K acting on the first factor $C\mathfrak{p}$.

In particular, in the case of $G = SL_2(\mathbb{R})$, since $\mathbb{D}$ does not map the space of right K-invariant functions to itself, it cannot be an operator on spaces of functions on $G/K \approx \mathfrak{H}$.

Background: [Dirac 1928] expressed Laplacians as *squares* of linear operators by extending scalars to non-commutative algebras. For several reasons, it is useful to similarly express Laplace-Beltrami operators on quotients G/K and $\Gamma\backslash G/K$ of semi-simple Lie groups G by maximal compact subgroups K and discrete subgroups Γ as squares of linear operators. Naturally, the non-abelian-ness of these groups and Lie algebras creates complications.

We work out some mundane features of Dirac operators on symmetric spaces G/K, using only the most basic ideas from [Parthasarathy 1972], [Atiyah-Schmid 1977/79], [Huang-Pandžić 2006], *et alia*.

The set-up here is merely a mild rearrangement of other excellent accounts, especially [Huang-Pandžić 2006]. The goal of the latter is construction of *discrete series* representations of semi-simple Lie groups, especially proofs of conjectures of Vogan in seminar talks in 1999 at MIT. We stay in a simpler context than [Kostant 1999].

In a much broader geometric context, [Atiyah-Singer 1963] announced proof of the Index Theorem for elliptic operators using Dirac operators. See [Palais 1965] and [Lawson-Michelsohn 1989] for exposition and references to further work in the general geometric setting.

[Parthasarathy 1972] used Dirac operators to construct discrete series representations of semi-simple Lie groups. The latter application was systematized by [Atiyah-Schmid 1977/79], and in Vogan's 1999 conjectures, the latter proven in [Huang-Pandžić 2002].

See [Sands 2020] for an application to automorphic forms.

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1. *Characterization of Dirac operator* $\mathbb{D}$

Let G be a semi-simple real Lie group, $\mathfrak{g}$ its Lie algebra, K a maximal compact, Killing form $\langle , \rangle$. Let $C\mathfrak{p}$ be the Clifford algebra of the -1 eigenspace $\mathfrak{p}$ of a Cartan involution whose $+1$-eigenspace is the Lie algebra $\mathfrak{k}$ of K, with quadratic form given by the restriction of the Killing form. The Dirac operator, as an element of

$C\mathfrak{p}\otimes U(\mathfrak{g})$, is the image of the identity operator $1_\mathfrak{p}$ on $\mathfrak{p}$ under the chain of natural maps

$$1_\mathfrak{p} \in \mathrm{End}\,\mathfrak{p} \xrightarrow{\approx} \mathfrak{p}\otimes\mathfrak{p}^* \xrightarrow{\approx} \mathfrak{p}\otimes\mathfrak{p} \xrightarrow{\text{inc}} C\mathfrak{p}\otimes U\mathfrak{g} \ni \mathbb{D}$$

The standard way for $\mathbb{D}$ to act on functions f on G is by the natural incarnation of $U\mathfrak{g}$ as differential operators induced by right multiplication on G, with coefficients in $C\mathfrak{p}$.

The pattern is considerably analogous to the characterization of the Casimir element $\Omega \in U\mathfrak{g}$:

$$1_\mathfrak{g} \in \mathrm{End}\,\mathfrak{g} \xrightarrow{\approx} \mathfrak{g}\otimes\mathfrak{g}^* \xrightarrow{\approx} \mathfrak{g}\otimes\mathfrak{g} \xrightarrow{\text{inc}} \bigotimes\nolimits^\bullet \mathfrak{g} \xrightarrow{\text{quot}} U\mathfrak{g} \ni \Omega$$

The desired effect is $\mathbb{D}^2 = -\Omega \bmod U\mathfrak{k}$: in coordinates, letting $\{v_i\}$ be an orthonormal basis of $\mathfrak{p}$, in $C\mathfrak{p}\otimes U\mathfrak{g}$,

$$\mathbb{D}^2 \;=\; \Big(\sum_i v_i\otimes v_i\Big)^2 \;=\; \sum_i v_i^2\otimes v_i^2 + \sum_{i\neq j} v_iv_j\otimes v_iv_j \;=\; -1\otimes\big(\sum_i v_i^2\big) + \sum_{i<j}\big(v_iv_j\otimes v_iv_j + v_jv_i\otimes v_jv_i\big)$$

$$=\; -1\otimes\big(\sum_i v_i^2\big) + \sum_{i<j}\big(v_iv_j\otimes v_iv_j - v_iv_j\otimes v_jv_i\big) \;=\; -1\otimes\big(\sum_i v_i^2\big) + \sum_{i<j} v_iv_j\otimes[v_i,v_j]$$

$$=\; -1\otimes\Omega_\mathfrak{g} + 1\otimes\Omega_\mathfrak{k} + \sum_{i<j} v_iv_j\otimes[v_i,v_j] \;=\; -1\otimes\Omega_\mathfrak{g} \quad \bmod U\mathfrak{k}$$

Happily, the $C\mathfrak{p}$ coefficients on $\Omega_\mathfrak{g}$ collapse to $-1\in\mathbb{R}\subset C\mathfrak{p}$. ///

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2. G-equivariance/invariance of Casimir Ω

As recalled below, by design, $\mathrm{Ad}(g)(\Omega)=\Omega$, with the (extended) Adjoint action of G on $U\mathfrak{g}$. Thus, when Ω acts by differential operators attached to the right translation action of G on functions on G, Ω commutes with the right-translation action of G. (With $U\mathfrak{g}$ acting by *right* translation, of course every element of it commutes with the *left* translation action of G.) In particular, Ω preserves right K-invariance.

Let $T_{v\otimes w}$ be the element of $\mathrm{End}\mathfrak{g}$ attached to $v,w\in\mathfrak{g}$, under $\mathfrak{p}\otimes\mathfrak{p}\longrightarrow \mathrm{End}\,\mathfrak{g}$, by $T_{v\otimes w}(x) = v\cdot\langle x,w\rangle$. For notational compactness, let $A=\mathrm{Ad}g$. Then

$$\Big(A\circ T_{v\otimes w}\circ A^{-1}\Big)(x) \;=\; A\Big(v\cdot\langle A^{-1}x,w\rangle\Big) \;=\; Av\cdot\langle x,Aw\rangle\Big) \;=\; T_{Av\otimes Aw}(x)$$

since A respects $\langle Ay, Az,\rangle = \langle y,z\rangle$ for all $y,z\in\mathfrak{g}$. Thus,

$$T_{A\Omega} \;=\; A\circ T_\Omega\circ A^{-1} \;=\; A\circ 1_\mathfrak{g}\circ A^{-1} \;=\; 1_\mathfrak{g} \;=\; T_\Omega$$

By Poincaré-Birkhoff-Witt, $\mathfrak{g}\otimes\mathfrak{g}$ *injects* to $U\mathfrak{g}$, so $A\Omega=\Omega$. ///

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3. K-equivariance/invariance of $\mathbb{D}$

With the diagonal Adjoint action of K on *both* factors $\mathfrak{p}$ and $\mathfrak{p}^*$, since $1_\mathfrak{p}$ commutes with this action, $\mathbb{D}$ commutes with this action, since all the maps are K-equivariant/invariant.

However, in contrast to the standard use of Ω, the action of K that make $\mathbb{D}$ K-invariant is *not* just the right-translation action $(k \cdot f)(g) = f(gk)$, but also must include the action of $k \in K$ on the *values* in $\mathfrak{p}$ of $\mathbb{D}f$. In particular, letting $(R_k f)(g) = f(gk)$, and letting $U\mathfrak{g}$ act by differential operators (on the right), we claim that the *right translation* R_k has the effect

$$(R_k(\mathbb{D}f))(g) \;=\; (\mathrm{Ad}k)^{-1}((\mathbb{D}f)(g))$$

where $(\mathrm{Ad}k)^{-1}$ acts on the *values* of $(\mathbb{D}f)(g)$. In particular, this action of $\mathbb{D}$ does *not* map right K-invariant scalar-valued functions to right K-invariant $\mathfrak{p}$-valued functions, but to $\mathfrak{p}$-valued functions whose values transform by Ad under right translation by K.

Proof: Here the action of $K \times K$ on $\mathfrak{p} \otimes \mathfrak{p}$ is relevant, rather than the diagonal action of a single copy of K, so it may be notationally simplest to do this computation in coordinates. Let $\{v_i\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form, so

$$\mathbb{D} \;=\; \sum_i v_i \otimes v_i \;\in\; C\mathfrak{p} \otimes U\mathfrak{g}$$

Letting $U\mathfrak{g}$ act by differential operators on the *right* on G,

$$(R_k(\mathbb{D}f))(g) \;=\; (\mathbb{D}f)(gk) \;=\; \sum_i \big((v_i \otimes v_i)f\big)(gk) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \sum_i v_i \otimes f(gke^{tv_i})$$

$$=\; \frac{\partial}{\partial t}\Big|_{t=0} \sum_i v_i \otimes f(ge^{t\cdot kv_ik^{-1}}k) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \sum_i v_i \otimes f(ge^{t\cdot \mathrm{Ad}(k)(v_i)}k)$$

Replacing the orthonormal basis v_i by the orthonormal basis $\mathrm{Ad}(k)^{-1}(v_i)$, this becomes

$$(R_k(\mathbb{D}f))(g) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \sum_i \mathrm{Ad}(k)^{-1}(v_i) \otimes f(ge^{tv_i}k) \;=\; \mathrm{Ad}(k)^{-1} \sum_i v_i \otimes (v_i(R_kf))(g)$$

$$=\; \mathrm{Ad}(k)^{-1}\Big(\big(\sum_i v_i \otimes v_i\big)(R_kf)(g)\Big) \;=\; \mathrm{Ad}(k)^{-1}\Big(\mathbb{D}(R_kf)(g)\Big)$$

as claimed. ///

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4. *Spinors and spinor representations*

From above, for f on G/K, the image $\mathbb{D}f$ takes values in $\mathfrak{p}$, and under right translation by K transforms by Ad acting on $\mathfrak{p}$. This is an instance of a *spinor representation* on spinors $\mathfrak{p}$, as follows.

Let V be a *non-degenerate* quadratic space over $\mathbb{R}$. There are copies of the Lie algebra $\mathfrak{so}(V)$ of the special orthogonal group $SO(V)$ of V inside the Clifford algebra CV. A *canonical* copy can be distinguished by taking trace-zero elements of the Lie subalgebra $\mathfrak{a}$ in the following claim.

[4.1] Claim: Let $\mathfrak{a}$ be the linear subspace of CV spanned by products uv for u, v in V. Under the Lie bracket $[a, b] = ab - ba$ in CV, $\mathfrak{a}$ is a Lie algebra, and the action $\theta \cdot w = \theta w - w\theta$ on $w \in V \subset CV$ gives a Lie isomorphism of $\mathfrak{a}$ *modulo constants* to $\mathfrak{so}(V)$.

[4.2] Remark: The Lie subgroup of CV associated to $\mathfrak{a}$ is the *Spin group* associated to $SO(V)$. In this context, V is *spinors*, and the action of the Lie group on V is a *spinor representation.*

Proof: First, $\mathfrak{a}$ stabilizes V: for $u, v, x \in V$,

$$[uv, x] \;=\; uvx - xuv \;=\; uvx - (-ux - \langle u,x\rangle)v \;=\; uvx + uxv + \langle u,x\rangle v$$

$$=\; uvx + u(-vx - \langle v,x\rangle) + \langle u,x\rangle v \;=\; uvx - uvx - \langle v,x\rangle u + \langle u,x\rangle v \;=\; -\langle v,x\rangle u + \langle u,x\rangle v$$

Thus, the image of V in CV is stabilized by $\mathfrak{a}$. The latter computation also gives a useful commutation rule. Second, show that $\mathfrak{a}$ is closed under brackets: for $x, y, u, v \in V$, repeatedly using the two-step commutation rule just demonstrated,

$$[xy, uv] \;=\; (xy)(uv) - (uv)(xy) \;=\; x(yuv - uvy) + x(uvy) - uvxy \;=\; -x(uvy - yuv) - (uvx - xuv)y$$

$$=\; -x(-\langle v,y\rangle u + \langle u,y\rangle v) - (-\langle v,x\rangle u + \langle u,x\rangle v)y \;=\; \langle v,y\rangle xu - \langle u,y\rangle xv + \langle v,x\rangle uy - \langle u,x\rangle vy$$

which is back in $\mathfrak{a}$, as claimed. To show that the action of $\mathfrak{a}$ on V preserves $\langle,\rangle$ is to show that

$$\langle [uv,x], y\rangle + \langle x, [uv,y]\rangle \;=\; 0 \qquad\qquad (\text{for } x,y,u,v \in V)$$

From the earlier computation,

$$\langle [uv,x], y\rangle \;=\; \Big\langle -\langle v,x\rangle u + \langle u,x\rangle v,\; y\Big\rangle \;=\; -\langle v,x\rangle\langle u,y\rangle + \langle u,x\rangle\langle v,y\rangle$$

while

$$\Big\langle x, [uv,y]\Big\rangle \;=\; \Big\langle x,\, -\langle v,y\rangle u + \langle u,y\rangle v\Big\rangle \;=\; -\langle v,y\rangle\langle u,x\rangle + \langle u,y\rangle\langle x,v\rangle$$

showing that the action of $\mathfrak{a}$ preserves $\langle,\rangle$.

Certainly constants act by 0 by the bracket on V. To prove that $\mathfrak{a}$ modulo constants maps isomorphically to $\mathfrak{so}(V)$ dimension-counting seems necessary, so let $e_1, \ldots, e_n$ be an orthogonal basis of V, and claim that the images of e_ie_j with $i < j$ are linearly independent as linear endomorphisms of V. This would prove injectivity, and then surjectivity by dimension-count. Suppose $\sum_{i<j} c_{ij}e_ie_j$ is a shortest linear combination acting by 0 on V. From the two-step commutativity above, for all k,

$$0 \;=\; [0, e_k] \;=\; \Big[\sum_{i<j} c_{ij}e_ie_j,\, e_k\Big] \;=\; \sum_{i<j} c_{ij}\Big(-\langle e_j,e_k\rangle e_i + \langle e_i,e_k\rangle e_j\Big) \qquad (\text{in the copy of } V \text{ in } CV)$$

Let i_o be the lowest index such that $c_{i_oj} \neq 0$ for some $j > i_o$, and take $k = i_o$, so

$$0 \;=\; \sum_{i<j} c_{ij}\Big(-\langle e_j, e_{i_o}\rangle e_i + \langle e_i, e_{i_o}\rangle e_j\Big) \;=\; \sum_{j>i_o} c_{i_oj}e_j$$

which implies that $c_{i_oj} = 0$ for all $j > i_o$, contradiction. This proves injectivity. ///

[4.3] Remark: The proof shows that a choice of basis e_i for V gives a choice of a copy of $\mathfrak{so}(V)$, by taking the span of e_ie_j for $i < j$. Changing the basis will change this span, in general, but will *not* change the action on the copy of V in CV. A trace zero condition also disambiguates.

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