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PULLBACKS OF EISENSTEIN SERIES; APPLICATIONS

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Introduction.

This is a more detailed version of a lecture given at Katata, and replaces some earlier preprint versions, with some insights added after H. Klingen's lecture on work of S. Böcherer, and after some helpful discussions with S. Kudla and M. Harris.

The purpose here is to compute the pullback of a Siegel's Eisenstein series via a map

$$H_m \times H_n \ni (z,w) \longrightarrow \begin{pmatrix} z & o \\ o & w \end{pmatrix} \in H_{m+n} ;$$

$$Sp(m) \times Sp(n) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} a & o & b & o \\ o & a' & o & b' \\ c & o & d & o \\ o & c' & o & d' \end{pmatrix} \in Sp(m+n) .$$

The "explicit" formula for this is in §5. The only real obstacle to the calculation is determination of coset representatives for

$$\{\begin{pmatrix} * & * \\ o & * \end{pmatrix} \in Sp(m+n, \mathbb{Z})\} \backslash Sp(m+n,\mathbb{Z}) / Sp(m,\mathbb{Z})Sp(n,\mathbb{Z}) ,$$

which is worked-out in §2 (and related coset computations occur in §3). S. Kudla has indicated to this author a "coordinate-free" way of doing the coset computation, at least over a field, but we prefer the present method for the computation over $\mathbb{Z}$.

Occurring in the "Main Formula" of §5 are eigenvalues of a "symmetric square" Hecke operator S_n, introduced in §4, defined in a de facto manner from the coset decompositions. In the case of $Sp(1) \approx SL(2)$, this operator is the "usual" one for elliptic modular forms (see [Sh 2]).

In §6, using some arithmetic of Siegel modular forms, and using the rationality of the Fourier coefficients of Siegel's Eisenstein series, we find that the Main Formula for the collection of <u>all</u> such equivariant imbeddings has some surprising impact on algebraicity properties of special values of symmetric-square Dirichlet series, and of Fourier coefficients of Eisenstein series made by lifting cuspforms (with algebraic Fourier coefficients) from the (standard) rational boundary components.

This "special value" issue has been considerably studied: see [Sh 4], [Sh 5], [Sh 7], [H 1], [St 1], [St 2].

The other part of the algebraicity assertion was new, at the time: a lifting Ef of a cuspform f with algebraic Fourier coefficients has algebraic Fourier coefficients <u>if</u> a certain value $S_f(*)$ of an associated symmetric-square Dirichlet series S_f is non-zero. This author observed that in the case that f is on a (complex) 1-dimensional rational boundary component, then the Euler product (from [Sh 2]) for S_f, f an eigenform, yields this non-vanishing. It was natural to conjecture that these special values <u>never</u> vanish, and hence one would obtain a general result about arithmeticity of Eisenstein series.

After this was communicated to M. Harris, he gave a direct Hecke-operator proof of the arithmeticity of these "generalized" Eisenstein series ([H 2], [H 3]).

Recently, S. Böcherer has shown that, indeed, these special values <u>are</u> $\neq 0$. (See [B] and [K]). Further, he found, as corollary, a proof of the "Basis Problem" using the Main Formula. His proof of the non-vanishing is critical for this. Also, the Siegel-Weil formula is an important ingredient. Some of the formalisms can be conceptually simplified in the context of the "see-saw" dual reductive pairs theory: the lecture [Ku] of S. Kudla may contain some discussion of this.

Despite the encouragement of Professor G. Shimura, these calculations of the pullback seemed like an ineffable coincidence to this author. It was unclear (to this author) what either the significance or the general version of this computation might be. For example, exactly analogous computations go through for <u>certain</u> congruence subgroups of $Sp(n,o_F)$, when F is a totally real number field, and o_F has class number one, but for arbitrary congruence subgroups or for more general o_F things rapidly become confusing. Also, one may carry out analogous computations for <u>certain</u> classical groups, but in what seems to be a very ad hoc manner.

Given the impetus especially of Böcherer's work, and of some remarks of Kudla, this author now can claim that both the coset decompositions and the pullback formula are, indeed, part of a general phenomenon (see [G 3] for some discussion of the coset decomposition). What seems to be appropriate is the following situation. Let G be a semi-simple (linear) algebraic group over $\mathbb{Q}$, of rational tube type, for example. Let F be a ("standard") rational boundary component, with associated $\mathbb{Q}$-parabolic P_F, containing a subgroup G_F which is "essentially" the automorphism group of F. Let G_F' be the centralizer of G_F in G. Let P_0 be the parabolic associated to a "standard" 0-dimensional rational boundary component. Then, believing that we should do things "adelically" anyway, we consider the double coset space

$$P_0(\mathbb{Q}) \backslash G(\mathbb{Q}) / G_F'(\mathbb{Q})G_F(\mathbb{Q}).$$

This coset space has "nice" representatives $\{R\}$ in the unipotent radical of the opposite parabolic to P_0. Further, for such representative R, and for $g \in G_F(\mathbb{Q})$, $g' \in G_F'(\mathbb{Q})$

$$P_0(\mathbb{Q})Rg'g = P_0(\mathbb{Q})R$$

iff g' lies in a $\mathbb{Q}$-parabolic P_R' of G_F' (determined by R), g lies in a $\mathbb{Q}$-parabolic P_R of G_F (determined by R), and certain straightforward additional conditions are satisfied. Thus, in some generality, the pullback of a holomorphic "classical" Eisenstein series will always involve the automorphic forms related to all the "standard" $\mathbb{Q}$-parabolics of G_F, G_F', but with some coefficients which are special values, and may vanish in a generic situation.

The proof of the above (and slightly more general versions) does begin from a Bruhat decomposition, shows that every coset $P_0(\mathbb{Q})gG_F'(\mathbb{Q})G_F(\mathbb{Q})$ has a representative in the "big cell", and then normalizes this representative essentially by "elementary divisor reduction".

We note that, in the terminology of [G 2], computation of such pullbacks amounts to calculation of the "0-th normalized division-point values" of the Fourier-Jacobi expansion over the rational boundary component F. In fact, it was such a viewpoint that gave some motivation to the original computation.

Still, we will reproduce here the original version, as thus we have more direct access to the "symmetric-square" operator.

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§1. Siegel Modular Forms

Without further explicit references, we rely upon [Go] and [S] for the following.

Let H_n be the Siegel upper half-space of "genus" n, and $Sp(n)$ the symplectic group of 2n-by-2n matrices, with the usual action of $Sp(n,\mathbb{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (az+b)(cz+d)^{-1} \quad .$$

The "canonical" automorphy factor is

$$\mu(g,z) = \det(cz+d)^{-2} \qquad (\text{with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \quad .$$

A Siegel modular form of weight k is a holomorphic function f on H_n so that for $g \in Sp(n,\mathbb{Z})$, $z \in H_n$,

$$f(gz)\mu(g,z)^k = f(z) \quad .$$

If $n=1$, we add the usual growth condition. Such f has a Fourier expansion

$$f(z) = \Sigma_T \, a_T \, e^{2\pi i \,\mathrm{tr}\, Tz} \quad ,$$

where "tr" is trace, and T runs through positive semi-definite semi-integral n-by-n matrices. Then f is a cuspform if $a_T \neq 0$ implies T is positive definite.

The Siegel's Eisenstein series of weight k (on H_n) is

$$E_k^{(n)}(z) = \sum_{\{c,d\}} \det(cz+d)^{-2k} \quad ,$$

where, as usual, $\{c,d\}$ indicates summation over (n-by-n) "symmetric coprime" pairs (c,d) left modulo $GL(n,\mathbb{Z})$. This series is nicely convergent for $2k > n+1$.

Now we describe the "standard" maximal parabolics of $Sp(n)$. For $0 \leq r \leq n$, put

$$P_{n,r} = Z_{n,r} G_{n,r} \quad ,$$

where $Z_{n,r}$ is the subgroup of $Sp(n)$ of elements of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & 1_r & * & 0_r \\ 0 & 0 & * & 0 \\ 0 & 0 & * & 1_r \end{pmatrix} ,$$

and $G_{n,r}$ consists of elements of $Sp(n)$ of the form

$$\begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & * & 0 & * \end{pmatrix} .$$

Note that $G_{n,r} \approx Sp(r)$ in the obvious way. We may identify $G_{n,r}$ with $Sp(r)$ when convenient.

We have the geodesic projections, for $0 \leq r \leq n$:

$$pr^n_r : H_n \longrightarrow H_r$$

by

$$pr^n_r: z = \begin{pmatrix} * & * \\ * & z_{22} \end{pmatrix} \longmapsto z_{22} ,$$

where z_{22} is r-by-r , etc. For a cuspform f of weight k on H_r , define a "generalized" Eisenstein series (a lifting of f to H_n):

$$E^r_n f(z) = \Sigma_g \mu(g,z)^k f(pr^n_r(g(z))) ,$$

where g is summed over $P_{n,r}(\mathbb{Z}) \backslash Sp(n,\mathbb{Z})$. Note that $E^0_n 1(z) = E^{(n)}_k(z)$. We will have an indirect proof in §5 that this is nicely convergent for $2k > n+r+1$. One may see [Bo] for a general discussion of such convergence issues.

The reproducing kernel $K^{(n)}_k$ for the space of weight-k cuspforms on H_n is

$$K_k^{(n)}(z,w) = c_k^{(n)} \sum_{g \in Sp(n,\mathbb{Z})} \mu(g,z)^k \det(g(z) - \bar{w})^{-2k} \quad ,$$

with some constant $c_k^{(n)}$ which can be explicitly calculated (it is a power of π times a rational). For a cuspform f, we have

$$f(z) = \int_{Sp(n,\mathbb{Z}) \setminus H_n} f(w) K_k^{(n)}(z,w) \quad (\det \operatorname{Im} w)^k \, d^{\cdot}w \quad ,$$

where $d^{\cdot}w$ is the $Sp(n,R)$-invariant measure on H_n. Equivalently,

$$K_k^{(n)}(z,w) = \Sigma_{j=1}^{d(n)} f_j(z) \overline{f_j(w)} \quad ,$$

where $\{f_j\}$ is an orthonormal basis for the space of cuspforms.

§2. The Double Coset Decomposition

We consider $Sp(m) \times Sp(n)$ imbedded in $Sp(m+n)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \quad ,$$

and when convenient will identify these groups with their images.

Theorem: $P_{m+n,0}(\mathbb{Z}) \setminus Sp(m+n,\mathbb{Z}) / Sp(m,\mathbb{Z})\, Sp(n,\mathbb{Z})$ has an irredundant set of coset representatives

$$g_{\tilde{M}} = \begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{M} & 1_m & 0 \\ {}^t\tilde{M} & 0 & 0 & 1_n \end{pmatrix} \quad ,$$

where

$$\tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \quad ,$$

<u>and</u>

$$M = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_r \end{pmatrix}$$

<u>is in elementary divisor form</u>: <u>each</u> $M_j > 0$, $M_j \in \mathbb{Z}$, $M_1 | M_2 | \cdots | M_r$. (<u>Refer to</u> r <u>as the "rank" of</u> $g_{\tilde{M}}$, $\tilde{M}$, <u>or</u> M) .

<u>Proof</u>: This is a tedious bit of linear algebra. It depends just on the fact that $\mathbb{Z}$ is a noetherian principal ideal domain, and exactly the same conclusion follows in that generality. Over a field, the computation is much simpler, and gives $\min(m,n)$ representatives.

It is well-known that

$$P_{m+n,0}(\mathbb{Z}) \backslash Sp(m+n, \mathbb{Z})$$

is in one-to-one correspondence with

$$GL(m+n, \mathbb{Z}) \backslash \{\text{symmetric coprime pairs of size } n\text{-by-}n\} .$$

And $Sp(m,\mathbb{Z}) \times Sp(n,\mathbb{Z})$ still acts on the right on such pairs.

Let $SC(n)$ be the set of n-by-n symmetric coprime pairs, and put

$$L_q = \{g \in Sp(q,\mathbb{Z}) : \; g = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \} ,$$

$\Delta_{q,j}$ = subgroup of $Sp(q,\mathbb{Z})$ of the elements of the form

$$\begin{pmatrix} 1_{j-1} & 0 & 0 & 0_{j-1} & 0 & 0 \\ 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & 1_{q-j} & 0 & 0 & 0_{q-j} \\ 0_{j-1} & 0 & 0 & 1_{j-1} & 0 & 0 \\ 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0_{q-j} & 0 & 0 & 1_{q-j} \end{pmatrix} .$$

We have $\Delta_{q,j} \cong SL(2,\mathbb{Z})$ in an obvious way.

Lemma: Every coset in $P_{m+n,0}(\mathbb{Z})\backslash SC(m+n)/Sp(m,\mathbb{Z})\,Sp(n,\mathbb{Z})$ contains an element of the form

$$(c \quad d) = \begin{pmatrix} c_{11}\,d_{11} & c_{12} & d_{11} & d_{12} \\ c_{21}\,d_{11} & c_{22} & 0 & d_{22} \end{pmatrix} ,$$

where c_{11}, d_{11} are m-by-m , c_{22}, d_{22} are n-by-n , and d_{11} is in elementary divisor form

$$d_{11} = \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_m \end{pmatrix} ,$$

$D_j \geq 0$, $D_1 | D_2 | \cdots | D_m$.

Proof of Lemma: We use the action of $GL(m+n,\mathbb{Z})$ on the left, and of $\Delta_{m,1} \times \cdots \times \Delta_{m,m} \subseteq Sp(m,\mathbb{Z})$ and $L_m \subseteq Sp(m,\mathbb{Z})$ on the right. Use a block decomposition as in the statement

First, by acting on the left by $GL(m+n,\mathbb{Z})$, and on the right by $\Delta_{m,j} \subseteq Sp(m) \subseteq Sp(m+n,\mathbb{Z})$, we may put the j-th and $(m+n+j)$-th columns of any $(c \quad d) \in SC(m+n)$ into an elementary form

$$\begin{pmatrix} D_j p_1 & D_o j \\ \cdots & \cdots \\ D_j p_{m+n} & 0 \end{pmatrix} ,$$

with $D_j \geq 0$.

Second, we may put the submatrix $\begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix}$ of $(c \quad d)$ into an elementary form

$$\begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_m \\ & 0 & \end{pmatrix} ,$$

with $D_j \geq 0$, $D_1 | D_2 | \cdots | D_m$, by acting on the left by $GL(m+n,\mathbb{Z})$, and on the right by $L_m \subseteq Sp(m,\mathbb{Z})$.

Consider the effect of performing the former reduction consecutively to the first and $(m+n+1)$-th columns, ..., m th and $(2m+n)$-th columns, and then applying the latter reduction. Refer to this compound reduction as "X" .

Let

$$\begin{pmatrix} D_1^{(p)} & & 0 \\ & \cdots & \\ 0 & & D_m^{(p)} \\ & 0 & \end{pmatrix}$$

be the elementary form of $\begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix}$ obtained after iterating X p times. It is easy to see that

$$D_m^{(p+1)} \mid D_m^{(p)} \ .$$

Thus, eventually the $D_m^{(p)}$'s stabilize. After that, we have

$$D_{m-1}^{(p+1)} \mid D_{m-1}^{(p)} \ .$$

So eventually the $D_{m-1}^{(p)}$'s stabilize. Likewise, eventually all the $D_j^{(p)}$'s stabilize. Then once more performing X puts (c d) into the desired form. ///

Lemma: The double coset representative of the previous lemma may be further normalized to the form

$$\begin{pmatrix} c_{11}\, d_{11} & c_{12} & d_{11} & d_{12} \\ c_{21}\, d_{11} & 0 & 0 & d_{22} \end{pmatrix}$$

with both d_{11} and d_{22} in the elementary form of the previous lemma.

Proof of Lemma: Take

$$(c \ \ d) = \begin{pmatrix} c'_{11} d'_{11} & c'_{12} & d'_{11} & d'_{12} \\ c'_{21} d'_{11} & c'_{22} & 0 & d'_{22} \end{pmatrix}$$

as in the previous lemma. The "symmetry" $c\,{}^td = d\,{}^tc$ of $(c\ \ d)$ yields $c'_{22}\,{}^td'_{22} = d'_{22}\,{}^tc'_{22}$. It is well-known (or follows by linear algebra) that for such c'_{22}, d'_{22} there is $g \in Sp(n,\mathbb{Z})$ so that

$$(c'_{22}\ d'_{22})g = (0\ \ d''_{22}) \quad .$$

Thus,

$$(c\ \ d)g = \begin{pmatrix} c'_{11} & d'_{11} & * & d'_{11} & * \\ c'_{21} & d'_{11} & 0 & 0 & d''_{22} \end{pmatrix} .$$

By acting on the left by an element

$$\begin{pmatrix} 1_m & 0 \\ 0 & * \end{pmatrix} \in GL(m+n,\mathbb{Z}) \quad ,$$

and on the right by an element of L_m , we can put d''_{22} into the indicated elementary form. This gives the desired form, and does not change the block $\begin{pmatrix} d'_{11} \\ 0 \end{pmatrix}$, nor the right-d'_{11}-divisibility of the left m columns.///

<u>Lemma</u>: <u>In the special coset representative chosen as in the previous lemma</u>, $d_{22} = 1_n$.

<u>Proof of Lemma</u>: Take

$$d_{11} = \begin{pmatrix} D_1 & & \\ & \cdots & \\ & & D_m \end{pmatrix} ,$$

$$d_{22} = \begin{pmatrix} E_1 & & 0 \\ & \cdots & \\ 0 & & E_n \end{pmatrix}$$

as in the previous lemma. From the "symmetry" of $(c\ \ d)$,

$$c_{21}d_{11}^2 = {}^t(c_{12}d_{22}) = d_{22}\,{}^tc_{12} \ ;$$

and by the "coprimeness" (i.e., each row of $(c\ \ d)$ has greatest common divisor 1),

$$c_{21}d_{11}x + d_{22}y = 1_n$$

for some integral matrices x,y. Let A_j be the j-th entry of the bottom row of c_{21}, and B_j the j-th entry of the bottom row of ${}^t c_{12}$. Then

$$A_j D_j^2 = E_n B_j \qquad (j = 1,\dots,m) \quad ,$$

and the greatest common divisor of

$$A_1 D_1, \dots, A_m D_m, E_n$$

is 1. It is easy to conclude that $E_n = 1$, and, hence, all E_j's are 1, and $d_{22} = 1_n$. ///

Lemma: We can further normalize the representative of the previous lemma to the form

$$\begin{pmatrix} c_{11}d_{11} & d_{11}^2\,{}^t c_{21} & d_{11} & 0 \\ c_{21}d_{11} & 0 & 0 & 1_n \end{pmatrix} ,$$

with d_{11} still in the same elementary form.

Proof of Lemma: Starting with the form of the previous lemma, left multiply by

$$\begin{pmatrix} 1_m & -d_{12} \\ 0 & 1_n \end{pmatrix} \in GL(m+n, \mathbb{Z})$$

to annihilate the $(1,4)$-th block. Then the "symmetricness" gives the form of the $(1,2)$-block. ///

Lemma: In the normalized form of the previous lemma, $d_{11} = 1_m$.

Proof of Lemma: Let A_j be the j-th entry of the bottom row of c_{11}, and B_j the j-th entry of the bottom row of ${}^tc_{21}$. The "coprimeness" implies that the common divisor of

$$A_1D_1, \ldots, A_mD_m, D_m^2B_1, \ldots, D_m^2B_n, D_m$$

is 1. The "symmetricness" implies $c_{11}d_{11}^2 = d_{11}^2\,{}^tc_{11}$, so that $D_m | A_jD_j$, for all j. But then D_m, and hence all D_j's, must be 1, and $d_{11} = 1_m$. ///

Now we can quickly finish the existence part of the proof. From "symmetricness", the c_{11} in the normal form of the last lemma is symmetric. Thus, we can right multiply by

$$\begin{pmatrix} 1_m & 0 \\ -c_{11} & 1_m \end{pmatrix} \in Sp(m,\mathbb{Z}) \subseteq Sp(m+n,\mathbb{Z})$$

to obtain

$$\begin{pmatrix} 0 & {}^tc_{21} & 1_m & 0 \\ c_{21} & 0 & 0 & 1_n \end{pmatrix}.$$

Then left multiplication by elements

$$\begin{pmatrix} (\text{m-by-m}) & 0 \\ 0 & (\text{n-by-n}) \end{pmatrix}$$

of $GL(m+n,\mathbb{Z})$, and right multiplication by L_mL_n can put c_{21}, ${}^tc_{21}$ in the elementary form of the statement of the theorem.

For uniqueness, suppose that $g \in GL(m+n,\mathbb{Z})$, $g' \in Sp(m,\mathbb{Z})$, $g'' \in Sp(n,\mathbb{Z})$, $\tilde{M}$ and $\tilde{M}'$ are such that

$$g\begin{pmatrix} 0 & \tilde{M} & 1_m & 0 \\ {}^t\tilde{M} & 0 & 0 & 1_n \end{pmatrix} g'g'' = \begin{pmatrix} 0 & \tilde{M}' & 1_m & 0 \\ {}^t\tilde{M}' & 0 & 0 & 1_n \end{pmatrix}.$$

Then we have

$$g\begin{pmatrix} 0 & 1_m \\ {}^t\tilde{M} & 0 \end{pmatrix} \quad g' = \begin{pmatrix} 0 & 1_m \\ {}^t\tilde{M}' & 0 \end{pmatrix} ,$$

$$g\begin{pmatrix} \tilde{M} & 0 \\ 0 & 1_n \end{pmatrix} \quad g'' = \begin{pmatrix} \tilde{M}' & 0 \\ 0 & 1_n \end{pmatrix} .$$

By uniqueness of elementary divisor forms, $\tilde{M} = \tilde{M}'$.

So, given such $\tilde{M}$, clearly

$$g_m = \begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{M} & 1_m & 0 \\ {}^t\tilde{M} & 0 & 0 & 1_n \end{pmatrix}$$

is in $Sp(m+n,\mathbb{Z})$. This proves the theorem. ///

§3. <u>The Twisted Coset Decomposition</u>

Here we finish the group-theoretic calculations.

<u>Theorem</u>: <u>With</u> $g_{\tilde{M}}$ <u>as in</u> §2 , $M = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$, $\tilde{M}$ <u>of rank</u> r ,

$$P_{m+n,0} \backslash P_{m+n,0}\, g_{\tilde{M}}\ Sp(m,\mathbb{Z})\, Sp(n,\mathbb{Z})$$

<u>has an irredundant set of coset representatives</u>

$$g_{\tilde{M}}\, g_0'\, g'\, g_1''\, g'' ,$$

<u>where</u> $g_0' \in G_{m,r}(\mathbb{Z})$,

$g' \in P_{m,r}(\mathbb{Z}) \backslash Sp(m,\mathbb{Z})$,

$g_1'' \in \Gamma_r(M) \backslash G_{n,r}(\mathbb{Z})$,

$g'' \in P_{n,r}(\mathbb{Z}) \backslash Sp(n,\mathbb{Z})$, *where* $G_{n,r}(\mathbb{Z})$ *is identified with* $Sp(r,\mathbb{Z})$, *and* $\Gamma_r(M)$ *is the congruence subgroup of* $Sp(r,\mathbb{Z})$ *of elements* g *so that*

$$\begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} g \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} \in Sp(r,\mathbb{Z}) \quad .$$

Proof: It suffices to show that

$$P_{m+n,0}(\mathbb{Z})g_{\tilde{M}}g'g'' = P_{m+n,0}(\mathbb{Z})g_{\tilde{M}}$$

iff

$$g' = g'_0 g'_Z \quad , \qquad g'' = g''_0 g''_Z$$

with $g'_0 \in G_{m,r}(\mathbb{Z})$, $g'_Z \in Z_{m,r}(\mathbb{Z})$,

$$g''_0 \in G_{n,r}(\mathbb{Z}) \,, \quad g''_Z \in Z_{n,r}(\mathbb{Z}) \quad ,$$

and

$$g''_0 = \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} g'_0 \begin{pmatrix} 0 & M^{-1} \\ M & \end{pmatrix} ,$$

identifying $G_{m,r}$ and $G_{n,r}$ with $Sp(r)$. We look at the condition

$$g_{\tilde{M}}\, g'g''\, g_{\tilde{M}}^{-1} \in P_{m+n,0}(\mathbb{Z}) \quad .$$

Now this just follows by multiplying-out. Put

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad , \qquad g'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \quad ,$$

and further decompose by

$$a' = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix} \quad , \text{ etc.},$$

where a'_{22} is r-by-r , a'_{11} is (m-r)-by-(m-r) ;

$$a'' = \begin{pmatrix} a''_{11} & a''_{12} \\ a''_{21} & a''_{22} \end{pmatrix}, \text{ etc.,}$$

where a''_{22} is r-by-r, a''_{11} is (n-r)-by-(n-r). When we multiply-out, we find that equality of cosets is equivalent to

$$\begin{pmatrix} c'_{11}, & c'_{12}, & c'_{21}, & d'_{12}M, & Ma'_{21} \\ c''_{11}, & c''_{12}, & c''_{21}, & d''_{12}M, & Ma''_{21} \end{pmatrix},$$

are all 0 (of appropriate sizes), and

$$\begin{pmatrix} a''_{22} & b''_{22} \\ c''_{22} & d''_{22} \end{pmatrix} = \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix} \begin{pmatrix} a'_{22} & b'_{22} \\ c'_{22} & d'_{22} \end{pmatrix} \begin{pmatrix} 0 & M^{-1} \\ M & 0 \end{pmatrix}.$$

This translates directly into the assertion of the Theorem. ///

§4. The Symmetric-Square Operator

For a Siegel modular form f of weight k on H_n, and for a non-singular integral n-by-n diagonal matrix M in elementary form, define

$$(T_M f)(z) = \Sigma_g f(Mg(z)M)\mu(g,z)^k \quad ,$$

where g is summed over $\Gamma_n(M)\backslash Sp(n,\mathbb{Z})$, with $\Gamma_n(M)$ as in the Theorem of §3. We define the symmetric-square operator $S_n = S_n^{(k)}$ by

$$S_n = \Sigma_M T_M$$

where M runs over all matrices as above.

Proposition: (Assuming the convergence of the series $S_n f$ for a cuspform f) $S_n = S_n^{(k)}$ is a hermitian operator on the space of weight-k cuspforms on H_n, with respect to the Petersson inner product. Further, the eigenspaces of S_n are spanned by cuspforms with algebraic coefficients.

Proof: First, it is elementary to see that

$$(\det M)^{2k} T_M = T'_M \quad ,$$

where T'_M is

$$(T'_M f)(z) = \Sigma_g f(g(z))\mu(g,z)^k \quad ,$$

where g is summed over

$$Sp(n,\mathbb{Z}) \backslash Sp(n,\mathbb{Z}) \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} Sp(n,\mathbb{Z}) \quad .$$

Then one can check by standard methods (see [Sh 1] ch. 3) that T'_M is hermitian. Hence, S_n is a sum of hermitian operators, so is hermitian if the series converges.

For the second assertion, recall from [Sh 3] and [G 1] that the space of cuspforms has a basis of cuspforms with rational Fourier coefficients, and that the operators T_M or T'_M map cuspforms with algebraic Fourier coefficients to cuspforms with algebraic Fourier coefficients. If we can show that all the T_M's commute , then the second assertion would follow by linear algebra (and the finite-dimensionality of the space of cuspforms).

By the criterion of [Sh 1] ch. 3 Prop. (3.8), if we can find an anti-involution $*$ on $Sp(n,\mathbb{Q})$ so that

$$\left(Sp(n,\mathbb{Z}) \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} Sp(n,\mathbb{Z})\right)^* =$$

$$= Sp(n,\mathbb{Z}) \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} Sp(n,\mathbb{Z}) \quad ,$$

then we have the commutativity. It is easy to check that

$$g^* = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \cdot g^{-1} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

works. This proves the Proposition. ///

§5. The Main Formula

Theorem: ("*Main Formula*") *Let* $E_k^{(m+n)}$ *be Siegel's Eisenstein series of weight* k *on* H_{m+n}. *Let* $z \in H_m$, $w \in H_n$. *Then for* $2k > m+n+1$,

$$E_k^{(m+n)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = E_k^{(m)}(z)E_k^{(n)}(w) + \sum_{1 \le r \le \min(m,n)} c_k^{(r)-1} \sum_{j=1}^{d(r)} S_{r,j}(E_m^r f_{r,i})(z)(E_n^r f_{r,i}^{\theta})(w) ,$$

where $c_k^{(r)}$ *is the constant in* §1, $\{f_{r,i}: i\}$ *is an orthonormal basis for cuspforms on* H_r, *consisting of eigenvectors for* S_r (*see* §4), *with eigenvalues* $\{S_{r,i}: i\}$, *respectively, and* $E_m^r f_{r,i}$ *is an Eisenstein series as in* §1. (*Implicitly, we assert that the generalized Eisenstein series converge nicely, as does the series for each* $S_r f_{r,i}$). *Finally, the* θ*-operator complex-conjugates the Fourier coefficients of the* $f_{r,i}$'s.

Remark: In about 1980 this author conjectured that it might happen that no eigenvalue $S_{r,i}$ vanished, thus giving unconditional version of the theorem. M. Harris, in [H 2], [H 3], proved a general version of assertion (iii), by a more direct use of Hecke operators.

Already in the case $n = 1$, m arbitrary, if one writes out the eigenvalue $S_{r,j}$ corresponding to an eigenform, one finds that this Dirichlet series is that occurring in [Sh 2]. There it was shown that such Dirichlet series have an Euler product with rational factors with trivial numerators. This assures non-vanishing.

In [B], Böcherer showed that (for this case of $Sp(n,\mathbb{Z})$) there is *no* 0-eigenvector of the symmetric-square operator. Not only that, but he went on to use this fact, together with the "Main Formula", to solve the Basis Problem for large-enough even weight. See also the lecture of Klingen [K].

Proof: This really just amounts to putting together our previous material, especially using the coset decompositions of §2, §3, and the "cocycle formula" for μ.

First, for $\tilde{M}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1_r \end{pmatrix}$,

$$\mu(g_{\tilde{M}_0}, \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) =$$

$$= \det(1_r - (pr_r^m z)(pr_r^n w))^{-2} .$$

Then

$$\sum_{g_0' \in G_{m,r}(\mathbb{Z})} \mu(g_{\tilde{M}_0} g_0', \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix})^k =$$

$$= \sum_{g \in Sp(r,\mathbb{Z})} \mu(g, pr_r^m z)^k \det(1_r - g(pr_r^m z)(pr_r^n w))^{-2k}$$

$$= \Sigma_g \, \mu(g, pr_r^m z)^k \det(g(pr_r^m z) + pr_r^n w)^{-2k} \quad ,$$

since

$$\begin{pmatrix} 0 & -1 \\ 1_r & 0 \end{pmatrix} : \; \xi \longrightarrow -\xi^{-1}$$

is in $Sp(r,\mathbb{Z})$. But this expression is just

$$c_k^{(r)^{-1}} K_k^{(r)}(pr_r^m z \; , \; -\overline{pr_r^n w} \;) \quad ,$$

where $K_k^{(r)}$ is the kernel function of §1 , and $c_k^{(r)}$ is the constant there. This is, then,

$$c_k^{(r)^{-1}} \sum_{j=1}^{d(r)} f_{r,j}(pr_r^m z) \; \overline{f_{r,j}(-\overline{pr_r^n w})} =$$

$$= c_k^{(r)^{-1}} \Sigma_j \; f_{r,j}(pr_r^m z) \; f_{r,j}^{\theta}(pr_r^n w) \quad ,$$

where θ is as in the statement of the Theorem, and $\{f_{r,j}: j = 1,\ldots,d(r)\}$ is any orthonormal basis for cuspforms on H_r .

For fixed $\tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$ of rank r , one similarly computes that

$$\sum_{g_0' g_1''} \mu(g_{\tilde{M}} g_0' g_1'' , \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix})^k =$$

$$= \sum_{g''} \mu(g'', pr_r^n(w)) \; c_k^{(r)^{-1}} K_k^{(r)}(pr_r^m z, -\overline{Mg''(pr_r^n w)M}) \; ,$$

where g_0', g_1'' are summed as in the Theorem of section 3, and

$g'' \in \Gamma_r(M) \setminus Sp(r,\mathbb{Z})$. Then, by the previous, this is

$$c_k^{(r)^{-1}} \sum_{j=1}^{d(r)} f_{r,j}(pr_r^m z) \quad (T_M f_{r,j})^{\theta}(pr_r^n w) \quad .$$

Now by the Proposition of §4, we may take $\{f_{r,j}\}$ to be eigenvectors for S_r, with real eigenvalues $S_{r,j}$, as S_r is hermitian. Then the above becomes, when summed over M of rank r,

$$c_k^{(r)^{-1}} \Sigma_j \, S_{r,j} \, f_{r,j}(pr_r^m z) \quad f_{r,j}^{\theta}(pr_r^n w) \quad .$$

Then the sum over $g' \in P_{m,r}(\mathbb{Z}) \setminus Sp(m,\mathbb{Z})$ $g'' \in P_{n,r}(\mathbb{Z}) \setminus Sp(n,\mathbb{Z})$ for fixed r gives

$$c_k^{(r)^{-1}} \Sigma_j S_{r,j} (E_m^r f_{r,i})(z)(E_n^r f_{r,i}^{\theta})(w) \quad .$$

Summing over $r = 0,1,\ldots,\min(m,n)$ gives the asserted formula, noting that for $r=0$ we take $f_{0,1} = 1$, and $E_m^0(z) = E_k^{(m)}(z)$, the Siegel's Eisenstein series.

With regard to convergence, we see that the series defining $S_r f_{r,j}$, $E_m^r f_{r,j}$, etc., actually occur as subseries of the series $E_k^{(m+n)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$, so are absolutely convergent uniformly for (z,w) in compact subsets of $H_m \times H_n$.

Thus, the Theorem is proven. ///

§6. Algebraicity Applications

Theorem. Let $X_s = X_s^{(n)}$ be the s-eigenspace of S_n on cuspforms of weight k on H_n. Suppose that $s \neq 0$. Then

i) There is an orthogonal basis for X_s of cuspforms with algebraic Fourier coefficients;

ii) For f in X_s with algebraic Fourier coefficients, $c_k^{(n)^{-1}} s\langle f,f\rangle^{-1}$ is algebraic;

iii) With f as in ii), for $2k > m+n+1$, $m \geq n$, the generalized Eisenstein series $E_m^n f$ on H_m has algebraic Fourier coefficients.

Proof: All this will follow, surprisingly enough, from the rationality of the Fourier coefficients of the Siegel's Eisenstein series, and the arithmetic of Siegel modular forms encapsulated in the Proposition of §4 .

Clearly, in light of the Proposition of §4 , it suffices to prove, instead of (i) ,

(i') For f_1, f_2 in X_s , with algebraic Fourier coefficients,

$$\langle f_1,f_2\rangle \langle f_1,f_1\rangle^{-1/2} \langle f_2,f_2\rangle^{-1/2}$$

is algebraic.

Then we could do a "Gram-Schmidt process", without losing algebraic Fourier coefficients, to obtain an orthogonal basis.

Lemma: $E_k^{(m+n)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$ has algebraic Fourier coefficients in (z,w) .

Proof: We will show that the Fourier coefficients of the pullback are finite sums of those of $E_k^{(m+n)}$, which are rational, from [S] .

As remarked in §1 , the T-th Fourier coefficient a_T is 0 unless T is positive semi-definite (and semi-integral). Decompose

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} ,$$

where T_{11} is m-by-m , etc. Then

$$e^{2\pi i \, \mathrm{tr}\, T\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}} = e^{2\pi i \, \mathrm{tr}\, T_{11} z} \, e^{2\pi i \, \mathrm{tr}\, T_{22} w} .$$

Thus, the (T_{11},T_{22})-th Fourier coefficient of the pullback is the sum of a_T where T has (1,1)-block T_{11} and (2,2)-block T_{22} . This sum is finite. ///

Now let $P(n_0,m_0)$ denote the assertion of the truth of (i) or (i'), (ii), (iii) for $0 \leq n \leq n_0$, $1 \leq m \leq m_0$. We will do an induction, using the Main Formula for varying m,n .

The first part of the induction is completed by observing that $P(0,m_0)$ only asserts that the Siegel's Eisenstein series on H_{m_0} , of weight k , has algebraic Fourier coefficients, which we know already (see [S]).

Suppose that we know $P(n_0-1,m_0)$. We will prove $P(n_0,m_0)$ (here $n_0 \geq 1$). This will yield the Theorem.

For $1 \leq r \leq n_0$, let $\{F_{r,j}\}$ be an orthogonal basis for cuspforms on H_r, consisting of eigenvectors for S_r. Let $\{f_{n_0,j}\}$ be an orthonormal basis for cuspforms on H_{n_0}, of eigenvectors for S_{n_0}. Now take $m = n = n_0$ in the Main Formula of §5 :

$$E_k^{(2n_0)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = E_k^{(n_0)}(z)E_k^{(n_0)}(w) +$$

$$+ \sum_{1 \leq r < n_0} c_k^{(r)^{-1}} \Sigma_j s_{r,j} \langle F_{r,j}, F_{r,j} \rangle^{-1} E_m^r F_{r,j}(z)\, E_n^r F_{r,j}^{\theta}(w) +$$

$$+ c_k^{(n_0)-1} \Sigma_j s_{n_0,j} f_{n_0,j}(z) f_{n_0,j}^{\theta}(w) ,$$

where we note that θ must be a $\mathbb{C}$-antilinear isometry of the space of cuspforms. By the induction hypothesis $P(n_0-1,m_0)$, all terms but possibly the last sum have algebraic Fourier coefficients. Hence, by the first lemma above,

$$c_k^{(n_0)^{-1}} \Sigma_j s_{n_0,j} f_{n_0,j}(z) f_{n_0,j}^{\theta}(w)$$

has algebraic Fourier coefficients.

Now let $\{F_{n_0,j}^{(s)} : j = 1,\ldots,d_s(n_0)\}$ be a basis for $X_s^{(n_0)}$, consisting of cuspforms with algebraic Fourier coefficients, as in §4 . Let $A_{ij}^{(s)}$ be complex numbers so that

$$f_{n_0,i}^{(s)} = \Sigma_j A_{ij}^{(s)} F_{n_0,j}^{(s)} \langle F_{n_0,j}^{(s)}, F_{n_0,j}^{(s)} \rangle^{-1/2}$$

(as j varies) gives an orthonormal basis for X_s . Then from above

$$c_k^{(n_0)^{-1}} \Sigma_s s \sum_{i,j} F^{(s)}_{n_0,j}(z) F^{(s)\theta}_{n_0,i}(w) \langle F^{(s)}_{n_0,j}, F^{(s)}_{n_0,j} \rangle^{-1/2} \cdot$$

$$\cdot \langle F^{(s)}_{n_0,i}, F^{(s)}_{n_0,j} \rangle^{-1/2} \Sigma_h A^{(s)}_{hj} \overline{A^{(s)}_{hi}} ,$$

where again we use the fact that θ is a $\mathbb{C}$-anti-linear isometry. As the functions $F^{(s)}_{n_0,j}(z) F^{(s)\theta}_{n_0,i}(w)$ are linearly independent , and have algebraic Fourier coefficients, and by the first lemma above, we find that

$$c_k^{(n_0)^{-1}} s \, \Sigma_h A^{(s)}_{hj} \overline{A^{(s)}_{hi}} \langle F^{(s)}_{n_0,j}, F^{(s)}_{n_0,j} \rangle^{-1/2} \langle F^{(s)}_{n_0,i}, F^{(s)}_{n_0,i} \rangle^{-1/2}$$

is algebraic, for each i,j,s . We surely could have taken the matrix $\{A^{(s)}_{ij}\}$ (for each s) to be upper triangular, and $A_{11} = 1$. Thus,

$$c_k^{(n_0)-1} s \langle F^{(s)}_{n_0,1} , F^{(s)}_{n_0,1} \rangle^{-1}$$

is algebraic. But this applies as well to any $F \neq 0$ in X_s with algebraic Fourier coefficients. This proves (ii) for our induction step.

For F, F' in X_s , we may apply the previous to, F, F' , $F+F'$, $F+iF'$ to find that

$$\langle F, F' \rangle \langle F, F \rangle^{-1/2} \langle F', F' \rangle^{-1/2}$$

is algebraic. This proves (i') for the induction step.

To prove (iii), take $n = n_0$, $m = m_0$ in the Main Formula. Arguing as before, by the induction hypothesis

$$c^{(n_0)^{-1}} \Sigma_j s_{n_0,j} E^{n_0}_{m_0} f_{n_0,j}(z) \, f^{\theta}_{n_0,j}(w)$$

has algebraic Fourier coefficients. By (i') or (i) of the induction hypothesis, we may take

$$f_{n_0,j} = F_{n_0,j} \langle F_{n_0,j}, F_{n_0,j} \rangle^{-1/2}$$

for some $F_{n_0,j}$ with algebraic Fourier coefficients. By linear independence of the $F^{\theta}_{n_0^*,j}$, $E^{n_0}_{m_0} f_{n_0,j}$ must have algebraic Fourier coefficients. This proves (iii) for the induction step.

Thus, the Theorem is proven. ///

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