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Quasi-completeness theorem

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Making explicit and concrete some facts that seem to have been known for at least 60 years, with proof:

[1.1] Theorem: For X a Fréchet space or LF-space, and Y quasi-complete and locally convex, the space $\mathrm{Hom}(X, Y)$ of continuous linear maps $X \to Y$, with any locally convex topology fine enough so that evaluation $T \to Tx$ is a continuous map $\mathrm{Hom}(X, Y)$ for every $x \in X$, is *quasi-complete*.

[1.2] Remark: For $Y = \mathbb{C}$, this space of continuous linear maps is the continuous dual X^*. The restriction on topologies on X^* includes every (locally convex) topology as fine as the weak dual (*finite-to-open*) topology on X^*, which has basis

$$N_{S,U} = \{T \in \mathrm{Hom}(X, Y) : T(S) \subset U\} \qquad \text{(for finite } S \subset X \text{ and open } U \subset Y)$$

For example, it includes the strong *bounded-to-open* topology [1] with basis given consisting of sets

$$N_{S,U} = \{T \in \mathrm{Hom}(X, Y) : T(S) \subset U\} \qquad \text{(for bounded } S \subset X \text{ and open } U \subset Y)$$

There is also the intermediate-strength *compact-to-open* topology with basis given at 0 consisting of sets

$$N_{S,U} = \{T \in \mathrm{Hom}(X, Y) : T(S) \subset U\} \qquad \text{(for compact } S \subset X \text{ and open } U \subset Y)$$

In strength, the compact-to-open topology is intermediate between the finite-to-open and bounded-to-open.

Proof: As usual, a set E of continuous linear maps from $X \to Y$ is *equicontinuous* when, for every neighborhood U of 0 in Y, there is a neighborhood N of 0 in X so that $T(N) \subset U$ for every $T \in E$.

[1.3] Claim: Let locally convex V be a strict colimit of closed subspaces V_i. Let Y be locally convex. A set E of continuous linear maps from V to Y is *equicontinuous* if and only if for each index i the collection of continuous linear maps $\{T|_{V_i} : T \in E\}$ is equicontinuous.

Proof: Given open $U \ni 0$ in Y, shrink U if necessary so that U is convex and balanced. For each index i, let N_i be a convex, balanced neighborhood of 0 in V_i so that $TN_i \subset U$ for all $T \in E$. Let N be the image in the colimit of the convex hull of the union of the images of the N_i's in the coproduct. By the convexity of N, still $TN \subset U$ for all $T \in E$. By the construction of the colimit as a quotient of the coproduct topology given by the diamond topology, N is an open neighborhood of 0 in the colimit. This gives the equicontinuity of E. The opposite implication is easier. ///

Recall

[1.4] Claim: (*Banach-Steinhaus*) Let X be a Fréchet space or LF-space and Y locally convex. A set E of linear maps $X \to Y$, such that every set of images $Ex = \{Tx : T \in E\}$ is *bounded* in Y, is *equicontinuous*.

Proof: First consider X Fréchet. Given a neighborhood U of 0 in Y, let $A = \bigcap_{T \in E} T^{-1}\overline{U}$. By assumption, $\bigcup_n nA = X$. By the Baire category theorem, the complete metric space X is not a countable union of nowhere dense subsets, so at least one of the closed sets nA has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, A itself has non-empty interior, containing some $x + N$ for a neighborhood N of 0 and $x \in A$. For every $T \in E$,

$$TN \subset T\{a - x : a \in A\} \subset \{u_1 - u_2 \,:\, u_1, u_2 \in \overline{U}\} = \overline{U} - \overline{U}$$

[1] Here *boundedness* of a set E is meant in the topological vector sense, namely, that for any open $U \ni 0$ in X, there is t_o such that for every $z \in \mathbb{C}$ with $|z| \geq t_o$ we have $E \subset zU$.

By continuity of addition and scalar multiplication in Y, given an open neighborhood U_o of 0, there is U such that $\overline{U} - \overline{U} \subset U_o$. Thus, $TN \subset U_o$ for every $T \in E$, and E is equicontinuous.

For $X = \bigcup_i X_i$ an LF-space, this argument shows that E restricted to each X_i is equicontinuous. As in the previous claim, this gives equicontinuity on the strict colimit. ///

For proof of the theorem, let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}(X, Y)$, with directed set I. Attempt to define the limit of the net by $Tx = \lim_i T_i x$. For any topology as in the statement of the theorem, for each fixed $x \in X$ the net $T_i x$ is bounded and Cauchy in Y. By the quasi-completeness of Y, $T_i x$ converges to an element of Y suggestively denoted Tx.

To prove *linearity* of T, fix x_1, x_2 in X, $a, b \in \mathbb{C}$ and fix a neighborhood U_o of 0 in Y. Since T is in the closure of E, for any open neighborhood N of 0 in $\mathrm{Hom}(X, Y)$, there exists $T_i \in E \cap (T + N)$. In particular, for any neighborhood U of 0 in Y, take

$$N \;=\; \{S \in \mathrm{Hom}(X,Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Since T_i is linear,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2)$$
$$=\; (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2))$$

The latter expression is

$$T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2) \;\in\; U + aU + bU$$

By choosing U small enough so that $U + aU + bU \subset U_o$, $T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \in U_o$. This hold for every neighborhood U_o of 0 in Y, so $T(ax_1 + bx_2) - aT(x_1) - bT(x_2) = 0$, proving linearity of T.

Continuity of the limit operator T exactly requires *equicontinuity* of $E = \{T_i x : i \in I\}$. Indeed, for each $x \in X$, $\{T_i x : i \in I\}$ is *bounded* in Y, so by Banach-Steinhaus, $\{T_i : i \in I\}$ is equicontinuous.

Fix a neighborhood U of 0 in Y. Invoking the equicontinuity of E, let N be a small enough neighborhood of 0 in X so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. By the characterization of the topology on $\mathrm{Hom}(X, Y)$, $Tx - T_i x \in U$ for large enough i. Then $Tx \in U + T_i x \subset U + U$. Replacing U by U' such that $U' + U' \subset U$, T is continuous. ///
