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Integral Representations of Certain L-functions attached to One, Two, and Three Modular Forms

Paul B. Garrett, University of Minnesota

Here we demonstrate that certain 'product' L-functions attached in a natural way to either

i) three modular forms on GL(2,F) for a number field F, or

ii) a modular form on GL(2,F) and a modular form on GL(2,M) for a quadratic extension M of F, or

iii) a modular form on GL(2,M) for a cubic extension M of F,

have integral representations as integrals against the restriction of an Eisenstein series on GSp(3,F), under certain conditions. As corollaries, of course one obtains explicit results on analytic continuations, functional equations, and special values results at certain points. This result includes as a very special case the 'triple product L-function' treated in [G2]. In effect, we are considering GL(2,B), where B is a semi-simple algebra of dimension three over a base number field F.

It should be noted that some of the factors of the Hasse-Weil zeta

erratum: throughout, replace "split/inertial/etc. in B/F" by "split/inertial/etc. in B/F and absolutely unramified" function of Hilbert modular three-folds are L-functions of this type. (The argument is essentially identical to that in [Sh1] for modular curves).

The motivation for these investigations and for that of [G2] comes from [G1], where it is shown that Siegel's Eisenstein series on symplectic groups, when 'decomposed' along a pair of smaller diagonally imbedded symplectic groups, have explicit expressions in terms of Eisenstein series attached to cuspforms on rational boundary components and special values of related symmetric-square L-functions. Then the readily-understood arithmetic nature of (holomorphic) Eisenstein series can be made to yield arithmetic information about subtler types of automorphic forms and some associated L-functions.

First (Theorem (1.7)), we have a rather general but inexplicit Euler factorization result, under an assumption of "adaptability" of the right representation. This assumption includes an "equal weight" assumption (see (1.4)). Then, we consider explicitly certain more special automorphic forms which are a straightforward adelic analogue of newforms (with character) for $\Gamma_0(N)$, as treated in chapter 3 of [Sh1]. We compute the Euler factors at (finite) primes which are not ramified in B/F (Theorem (1.9)). For primes unramified in B/F, but for which the character ramifies, we also have an explicit expression for the Euler factors (see Theorem (1.10)). We explicitly evaluate the archimedean factors only in the case of holomorphic modular forms, requiring that only totally-real number fields appear (see Theorem (1.11)). A special-value result is obtained for holomorphic eigencuspforms of equal weight, level one, and trivial central character, in Theorem (1.12). (A much more general case of the special-value result will be treated in a forthcoming paper by this author and M. Harris). A direct calculation of the infinite-prime integral for waveforms to express them in terms of ordinary gamma functions does not seem to have been done at this time, though F. Shahidi has indicated to this author that his methods show that the integral should be expressible in terms of gamma functions (e.g., see [S1], [S2]).

After communication of the results of [G2] concerning the integral representation of the triple product, I.I. Piatetskii-Shapiro and S. Rallis also began work on integral representations of such L-functions via "twisted D_4's", which will appear in a forthcoming paper. (Their results and the present result were announced at Oberwolfach in 1985). This type of result apparently fits in with their methodology of "compactifying" reductive groups inside larger groups, which they had developed considerably in [PR1]. Indeed, a question asked of this author by I. I. Piatetskii-Shapiro during the development of [G2] about the possibility of treating cubic fields was part of the motivation for the present generalization of that result. This author thanks the previously-mentioned authors for some clarifying remarks regarding their viewpoints on these matters. It is anticipated that their forthcoming paper will treat the analytic aspects of such integral representations in greater detail, by more general representation-theoretic methods.

We now roughly describe the result in the case of (adelic) holomorphic eigencuspforms f of level one. Let B be a cubic (semi-simple) algebra over a (totally real) number field F, and suppose that the number fields occuring in B are all totally real. Let f be a holomorphic eigencuspform of weight $\tilde{\kappa}=(2\kappa, \ldots, 2\kappa)$ (with $\kappa\in\mathbb{Z}$), of level one, on $G^{\#}(\mathbb{A})=GL(2,\mathbb{A}\otimes_{\mathbb{Q}}B)$, with trivial central character, where $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. Let G be the $\mathbb{Q}$-subgroup of G defined by

$$G(R) = \{g\in G^{\#}(R): \det(g)\in R\otimes_{\mathbb{Q}}F\},$$

for a (commutative) $\mathbb{Q}$-algebra R. For an alternating B-valued B-bilinear form J on B^2, we have an alternating F-valued F-bilinear form J' on B^2 given by $J'=\mathrm{trace}_{B/F}J$. Let G' be the $\mathbb{Q}$-group (after restriction of scalars) of F-linear symplectic similitudes of (B^2,J'). Thus, we have a natural homomorphism ι of $\mathbb{Q}$-groups $\iota : G \rightarrow G'$. Let $E(g;s)$ be the (adelic) Eisenstein series on $G'(\mathbb{A})$ which is attached to the standard maximal parabolic with abelian unipotent radical, which is 'adapted' to weight 2κ, and to the right representation and central character.

The integral we consider is

$$\int \overline{E}(\iota(g);\overline{s})\ f(g)\ dg,$$

taken over $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, where Z is the center of G. One finds that this integral has an Euler product over primes of F. Further, the Euler factors can be very explicitly described, as follows.

There are several distinct types of $\wp$-factors which occur, depending on the behavior of the prime in the extension, and depending on the nature of the central character and right representation at $\wp$. The five types of behavior of a prime $\wp$ in the extension B/F are:

i) $\wp$ is "inertial": $B \otimes_F F_\wp$ is the unramified cubic field extension of $F_\wp$;

ii) $\wp$ is "completely split": $\wp\mathcal{O}=\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$: $B \otimes_F F_\wp \approx F_\wp \times F_\wp \times F_\wp$;

iii) $\wp$ is "totally ramified": $\wp\mathcal{O}=\mathfrak{P}^3$: $B \otimes_F F_\wp$ is a totally ramified cubic field extension of $F_\wp$;

iv) $\wp$ is "partially ramified": $B \otimes_F F_\wp \approx F_\wp \times$(ramified quadratic extension of $F_\wp$);

v) $\wp$ is "partially split": $B \otimes_F F_\wp \approx F_\wp \times$(unramified quadratic extension of $F_\wp$).

Note that in the case that $B \approx F \times F \times F$, all primes are "completely split".

We then have the following Euler factors arising from the integral representation (for any prime $\wp$ of F unramified in B/F) (see (1.7),(1.9), and (1.10) for general and precise statements):

<u>Suppose that for each</u> i, α_i <u>and</u> β_i <u>are the roots of an equation</u>

$$Y^2-b(\mathfrak{P}_i)Y+\chi(\mathfrak{P}_i)\mathbb{N}\mathfrak{P}_i^{-1}=0,$$

<u>where</u> $b(\mathfrak{P}_i)$ <u>is the</u> $\mathfrak{P}_i$-th <u>Hecke eigenvalue of</u> f <u>multiplied by the obvious a power of</u> $\mathbb{N}\mathfrak{P}_i$, <u>and</u> f <u>is a level-one holomorphic eigencuspform of weight</u> 2κ. <u>Let</u> $X=\mathbb{N}\wp^{-s}$. <u>Then the Euler</u> $\wp$-<u>factor of the above integral is</u>

$$(1-X^2) \times (1-N\mathfrak{p}^2X^4) \times L(f,s-2,B/F)_{\mathfrak{p}}$$

<u>where the Euler factor</u> $L(f,s,B/F)_{\mathfrak{p}}$ <u>is given by:</u>

i) <u>for</u> $\mathfrak{p}$ <u>inertial</u>:

$$L(f,s,B/F)_{\mathfrak{p}}^{-1} =$$
$$=(1-\alpha_1X)\,(1-\beta_1X)\,(1-\alpha_1N\mathfrak{p}^{-3}X^3)\,(1-\beta_1N\mathfrak{p}^{-3}X^3);$$

ii) <u>for</u> $\mathfrak{p}$ <u>partially split</u>:

$$L(f,s,B/F)_{\mathfrak{p}}^{-1} =$$
$$=(1-\alpha_1\alpha_2X)\,(1-\alpha_1\beta_2X)\,(1-\beta_1\alpha_2X)\,(1-\beta_1\beta_2X) \times$$
$$\times\,(1-\alpha_1^2N\mathfrak{p}^{-2}X^2)\,(1-\beta_1^2N\mathfrak{p}^{-2}X^2);$$

iii) <u>for</u> $\mathfrak{p}$ <u>completely split</u>:

$$L(f,s,B/F)_{\mathfrak{p}}^{-1} =$$
$$= (1-\alpha_1\alpha_2\alpha_3X)\,(1-\alpha_1\alpha_2\beta_3X)\,(1-\alpha_1\beta_2\alpha_3X)\,(1-\alpha_1\beta_2\beta_3X) \times$$
$$\times\,(1-\beta_1\alpha_2\alpha_3X)\,(1-\beta_1\alpha_2\beta_3X)\,(1-\beta_1\beta_2\alpha_3X)\,(1-\beta_1\beta_2\beta_3X).$$

We also treat the case that χ is ramified at $\mathfrak{p}$, but $\mathfrak{p}$ is not ramified in B/F. See (1.10). We note that the two inverse zeta factors cancel with those occuring in the normalization of the Eisenstein series. (E.g., see [L] regarding Eisenstein series. The level-one case of Eisenstein series on symplectic groups is explicitly treated in an appendix in [L]).

The first immediate corollary one obtains is the (meromorphic) analytic continuation and functional equation of $L(f,s,B/F)$ from those of the Eisenstein series. Second, on the arithmetic side, it is well known that the holomorphic Eisenstein series $E(g;\kappa)$ is "arithmetic over $\mathbb{Q}$", for example in the classical sense of having Fourier coefficients in $\mathbb{Q}$ when the rings of integers of F and B have class number one. (We refer to [H1] and [H2] for general and systematic versions of this notion). From some relatively elementary arithmetic properties of the Hecke operators (in this example), one finds directly that the normalized special value

$$L^{\#}(f,\kappa-2,B/F) =$$

$$= \zeta_F(2\kappa)^{-1}\,\zeta_F(4\kappa-2)^{-1} \times \langle f,f\rangle^{-1} \times L(f,\kappa-2,B/F)$$

is in $\mathbb{Q}$, and that for an automorphism τ of $\mathbb{C}/\mathbb{Q}$,

$$L^{\#}(f,\kappa-2,B/F)^{\tau} = L^{\#}(f^{\tau},\kappa-2,B/F),$$

where $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on "Fourier coefficients" of f (i.e., on global sections of a certain line bundle on the associated Shimura variety), and $\langle f,f\rangle$ is the Petersson inner product. From the explicit evaluation of the archimedean Euler factor, one has a further paraphrase of this, as given in (1.12). As Shimura has pointed out to this author, the other special values may be obtained as well from this integral representation by using results of [Sh3], [Sh4] on the arithmetic nature of such Eisenstein series at points other than merely $s=\kappa$.

A key point in the verification of the integral formula is the determination of 'nice' representatives for a particular coset space of the general form

$$P'(\mathbb{Q})\backslash G'(\mathbb{Q})/G(\mathbb{Q}),$$

where G and G' are reductive $\mathbb{Q}$-groups, G is imbedded in G' (over $\mathbb{Q}$), and P' is a $\mathbb{Q}$-parabolic in G'. In the case of interest, we take G and G' to be as above, and P' to be the maximal proper $\mathbb{Q}$-parabolic in G' mentioned earlier. Then this coset space has just three representatives, and only one will make a non-trivial 'cuspidal' contribution. Such a calculation was done in [G2] in the simplest possible case, in a somewhat different manner.

Then the global integral presents itself as an Euler product in a fairly natural manner, even for somewhat more general types of automorphic forms. We find it convenient in this matter to describe "eigencuspforms" by the local Whittaker functions which determine them (by an adelic Fourier expansion). The only obstacle is that the Euler factors are expressed as sums of rational functions, with denominators including several extraneous factors. Fortunately, the most optimistic 'guess' regarding factors of the numerator is correct, under the hypotheses mentioned earlier concerning the central character and right representation, and one obtains the results above, after some calculation.

The archimedean factor in the product is readily evaluated in terms of classical gamma functions when the cuspforms are holomorphic, giving a complete result in this case. However, in general the direct determination of the analytic nature of this factor is not so clear. If the general archimedean factor can be determined sufficiently explicitly, then it is possible (according to conversations with Rallis and Piatetskii-Shapiro) to use 'converse theorems' to obtain the meromorphic continuation of quadruple and quintuple products.

Likewise, only in the holomorphic case does the arithmetic nature of the corresponding Eisenstein series yield arithmetic results on the special values.

At the moment, it seems unlikely that there is a directly analogous result in more complicated situations (e.g., to obtain 'higher' product L-functions), due to obstacles in the nature of the analogous coset decompositions and related matters. This is in contrast to the occurence of 'symmetric square' L-functions for all the families of classical groups: see [PR1]. Even mere consideration of dimension can illustrate this.

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1. Statement of Results

First, we establish some conventions and elementary results automorphic forms for congruence subgroups of GL(2,B), where B is a commutative semi-simple algebra over a number field F (so is a product of number fields). We will consider these items explicitly only for an analogue of the groups $\Gamma_0(N)$, following [Sh1] chapter 3, in essence. Second, we will introduce an Eisenstein series on a symplectic group 'adapted to' the central character and right representation type of a given automorphic form f 'of equal weight' on GL(2,B). Then the integral formula will be stated, along with some consequences. The proofs will be given in subsequent sections.

(1.1) Automorphic forms on GL(2). Fix a number field F, with ring of integers $\mathfrak{o}$. Let B be a commutative semi-simple F-algebra over F. Then B is isomorphic to a direct product of number fields M_i. We identify B with such a product. Let $\mathcal{O}$ be the integral closure of $\mathfrak{o}$ in B. $\mathcal{O}$ is just the direct product of the rings of integers $\mathcal{O}_i$ in the simple factors of B, and ideals of $\mathcal{O}$ are just direct products of ideals in the rings $\mathcal{O}_i$. In a similar manner one has $\mathfrak{P}$-adic completions $B_{\mathfrak{P}}$ and $\mathcal{O}_{\mathfrak{P}}$ for a prime $\mathfrak{P}$ in $\mathcal{O}$. Generally, a finite or infinite prime $\mathfrak{P}$ of B lying over a prime $\mathfrak{p}$ of F will be construed to be an absolute value (or valuation) on the image of B under a non-trivial $F_{\mathfrak{p}}$-algebra homomorphism of B into an algebraic closure of $F_{\mathfrak{p}}$. Define a linear algebraic group $G^{\#}$ over $\mathcal{O}$ by

$$G^{\#}(R) = GL(2,R),$$

for a commutative $\mathcal{O}$–algebra R. Let $\mathbb{A}$ be the adeles of F, and $\mathbb{A}_0$ the finite adeles of F. Let $P^{\#}$ be the parabolic subgroup of $G^{\#}$ consisting of upper–triangular matrices, $U^{\#}$ its unipotent radical, and $T^{\#}$ the group of diagonal matrices in $G^{\#}$. Let $K^{\#}$ be the standard subgroup of $G^{\#}(\mathbb{R}\otimes_{\mathbb{Q}}B)$ isomorphic to a number of copies of SO(2) and U(2). For α,β in a ring R, we will denote by $\delta(\alpha,\beta)$ the 2–by–2 matric with diagonal entries α and β, and zero off–diagonal entries. Let $\mathbb{K}^{\#}$ be the subgroup $\prod_{\mathfrak{p}} G^{\#}(\mathcal{O}_{\mathfrak{p}})$ of $G^{\#}(\mathbb{A}_0\otimes B)$. Let $Z^{\#}$ be the center of $G^{\#}$. We identify $Z^{\#}(\mathbb{A}\otimes_F B)$ with $(\mathbb{A}\otimes_F B)^{\times}$. For a grossencharacter χ of $(B\otimes_F\mathbb{A})^{\times}$ (which is just a product of ideles of the simple factors of B), we view χ as being a character on $Z^{\#}(\mathbb{A}\otimes_F B)$. For an integral ideal $\mathfrak{f}$ of F, let

$$\mathbb{K}^{\#}(\mathfrak{f}) = \{\, g\in\mathbb{K}^{\#}:\ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ modulo } \mathfrak{f}\mathcal{O}\}.$$

In general, for a group G and subgroup H, with a one–dimensional (complex) representation ρ of H, we will say that a $\mathbb{C}$–valued function f on G is left (resp., right) (H,ρ)–equivariant if for $h\in H$, $g\in G$, we have $f(hg)=\rho(h)f(g)$ (resp., $f(gh)=\rho(h)f(g)$). Say that f is left (resp., right) H–finite if the collection of translates $\{f(hg):h\in H\}$ (resp., $\{f(gh):h\in H\}$) spans a finite–dimensional vectorspace of functions on G.

For each prime $\mathfrak{p}$ (infinite or finite) of a number field M, we have a pairing $\langle,\rangle_{\mathfrak{p}}$ of $M_{\mathfrak{p}}\times M_{\mathfrak{p}}$ into $\mathbb{Q}/\mathbb{Z}$ defined as follows. If $\mathfrak{p}$ is infinite and real, then for $\alpha,\beta\in M_{\mathfrak{p}}$ define

$$\langle\alpha,\beta\rangle_{\mathfrak{p}} = \alpha\beta \bmod \mathbb{Z}.$$

If $\mathfrak{p}$ is infinite and complex, then put

$$\langle\alpha,\beta\rangle_{\mathfrak{p}} = (\alpha\beta+\overline{\alpha\beta}) \bmod \mathbb{Z}.$$

Now suppose that $M_{\mathfrak{p}}$ is a finite extension of $\mathbb{Q}_p$, with p finite. We have the natural injection

$$\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{Q}/\mathbb{Z},$$

which we use to write

$$\langle\alpha,\beta\rangle_{\mathfrak{p}} = -\mathrm{trace}(\alpha\beta) \bmod \mathbb{Z},$$

where the trace is that from $M_{\mathfrak{p}}$ to $\mathbb{Q}_p$. For a prime $\mathfrak{p}$ of M, we have an additive character $\tau_{\mathfrak{p}}$ on $M_{\mathfrak{p}}$ given by

$$\tau_{\mathfrak{p}}(x) = \langle x,1\rangle_{\mathfrak{p}}.$$

Define an additive character τ on the adeles of M by

$$\tau(x) = \prod_{\mathfrak{p}} \tau_{\mathfrak{p}}(x_{\mathfrak{p}}),$$

where $x_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of x. Every M-invariant additive character on the adeles of M is of the form

$$\prod_{\mathfrak{p}} \mathbb{e}(\langle x,\xi\rangle_{\mathfrak{p}})$$

for some $\xi\in M$, where, for $\alpha\in\mathbb{C}$, $\mathbb{e}(\alpha)=e^{2\pi i\alpha}$. We can define pairings $\langle,\rangle_p$ on $B_p\times B_p$ as the sums of the pairings on the simple factors of B_p, and additive characters

$$x \rightarrow \tau(\xi x) = \prod_p \mathbb{e}(\langle x,\xi\rangle_p)$$

for x and ξ in $\mathbb{A}\otimes B$. Again, all B-invariant characters on $\mathbb{A}\otimes B$ are of this form for some $\xi \in B$.

For a left $U^{\#}(B)$-invariant continuous function f on $G^{\#}(\mathbb{A}\otimes B)$, we have Fourier coefficients

$$f_{\xi}(g) = \int \bar{\tau}(\xi x)\ f(u(x)g)\ dx$$

where $g\in G^{\#}(\mathbb{A}\otimes B)$, $\xi\in B$, and we write

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Here dx is the (additive) Haar measure on $\mathbb{A}\otimes B$ which is a product of the usual local Haar measures $dx_{\mathfrak{p}}$ on the factors $B_{\mathfrak{p}}$. (I.e., for a finite prime $\mathfrak{P}$, $\mathcal{O}_{\mathfrak{p}}$ has measure $(\mathbb{N}\mathfrak{D}_{\mathfrak{p}})^{-1/2}$, where $\mathfrak{D}_{\mathfrak{p}}$ is the local different (in an obvious sense), and $\mathbb{N}$ is the ideal norm. We take the usual Lebesgue measure for real places, and twice the Lebesgue measure for complex places). With this normalization of measure, for $x\in\mathbb{A}\otimes B$, $y_1,y_2\in(\mathbb{A}\otimes B)^{\times}$,

$$f(u(x)\delta(y_1,y_2)) = \sum_{\xi} f_{\xi}(\delta(y_1,y_2))\ \tau(\xi x),$$

(at least in an L^2-sense) with $\delta(y_1,y_2)\in T^{\#}(\mathbb{A}\otimes B)$, $u(x)\in U^{\#}(\mathbb{A}\otimes B)$ as above.

Fix a grossencharacter χ on $Z^{\#}(\mathbb{A}\otimes B)$ with conductor dividing $\mathfrak{f}\mathcal{O}$ (with $\mathfrak{f}$ an ideal of F), and a $\mathbb{C}^{\times}$-valued representation ρ_{∞} of $K^{\#}$. Let ρ_0 be a $\mathbb{C}^{\times}$-valued representation of $\mathbb{K}^{\#}(\mathfrak{f})$. Put $\rho=\rho_{\infty}\otimes\rho_0$. Let f be a continuous, $\mathbb{C}$-valued, $(Z^{\#}(\mathbb{A}\otimes B),\chi)$-equivariant, left $G^{\#}(B)$-invariant, right $(K^{\#}\mathbb{K}^{\#}(\mathfrak{f}),\rho)$-equivariant function on $G^{\#}(\mathbb{A}\otimes B)$, supported on $P^{\#}(\mathbb{A}\otimes B)K^{\#}\mathbb{K}^{\#}(\mathfrak{f})$. We require that there is a function $W=W_f$ on $T^{\#}(\mathbb{A}\otimes B)\approx((\mathbb{A}\otimes B)^{\times})^2$ so that for $\xi\in B^{\times}$, $\varepsilon\in(\mathbb{A}\otimes B)^{\times}$,

$$f_\xi(\delta(\varepsilon,1)) = W(\xi\varepsilon,1),$$

and we require that $f_\xi = 0$ for $\xi \notin B^\times$. (Note that necessarily W is right $(T^\# \cap \mathbb{K}^\# \mathbb{K}^\#(\mathfrak{f}), \rho)$–equivariant, and $(Z^\#(\mathbb{A}\otimes B), \chi)$–equivariant). Further, we require that for each prime $\mathfrak{P}$ of B there is a function $W_\mathfrak{p}$ on $T^\#(B_\mathfrak{p})$ so that

$$W = \prod_\mathfrak{p} W_\mathfrak{p}.$$

If also f is square integrable on

$$Z^\#(\mathbb{A}\otimes B)G^\#(B)\backslash G^\#(\mathbb{A}\otimes B),$$

and is an eigenfunction for the Casimir operator on $G^\#(F\otimes\mathbb{R})$, we say that f is <u>an eigencuspform with central character</u> χ, <u>and with right representation</u> ρ, <u>of level</u> $\mathfrak{f}$. The functions $W_\mathfrak{p}$ attached to f we refer to as the (local) <u>Whittaker functions associated to</u> f.

(1.2) Explicit Examples. Of special interest is a certain class of the local (Whittaker) functions $W_\mathfrak{p}$. We consider a class of (adelizations of) holomorphic Hilbert modular cuspforms. Suppose that the simple factors M_i of B are totally real number fields. Let χ be a grossencharacter of finite order on $(\mathbb{A}\otimes B)^\times$, with conductor dividing $\mathfrak{f}\mathfrak{O}$, trivial on $B_\mathfrak{p}^\times$ for $\mathfrak{P}$ infinite, where $\mathfrak{f}$ is an ideal of F. Let ρ_0 be a $\mathbb{C}^\times$–valued representation of $\mathbb{K}^\#(\mathfrak{f})$ which restricts to χ on $Z(\mathbb{A})\cap G^\#(\mathfrak{O}_\mathfrak{p})$ for $\mathfrak{P}$ dividing $\mathfrak{f}$, and which is of the form (modulo $\mathfrak{f}$)

$$\rho_0\left(\begin{pmatrix} \lambda a & * \\ 0 & a^{-1} \end{pmatrix}\right) = \chi(a)^{-1}$$

on $G^{\#}(\mathcal{O}_{\mathfrak{p}})\cap\mathbb{K}^{\#}(\mathfrak{f})$ for $\mathfrak{P}$ dividing $\mathfrak{f}$. Let f be an eigencuspform with central character χ, supported on $P^{\#}(\mathbb{A}\otimes B)K^{\#}\mathbb{K}^{\#}(\mathfrak{f})$, with right representation $\rho=\rho_{\infty}\otimes\rho_0$. We will describe a certain explicit class of local Whittaker functions which occur when f is the adelization of a holomorphic Hilbert modular eigencuspform.

Let $\mathfrak{P}$ be an infinite (real) prime of B. (Then we may identify $\mathfrak{P}$ with an infinite prime $\mathfrak{P}$ of some M_i). We see that ρ_{∞} restricted to the $\mathfrak{P}$–factor of $K^{\#}$ must be of the form

$$(1.2.1)\qquad \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} \to \mathbb{e}(-\kappa(\mathfrak{P})\alpha)$$

for some integer $\kappa(\mathfrak{P})=\kappa_i(\mathfrak{P})$. We say that $\kappa=\{\kappa(\mathfrak{P}):\mathfrak{P}\text{ infinite}\}$ is the <u>weight</u> of ρ_{∞}, or of ρ, or of the eigencuspform f. Partition κ as $\kappa=(\kappa_1,\ldots)$, where

$$\kappa_i=\{\kappa(\mathfrak{P}):\ \mathfrak{P}\text{ is an infinite prime of the simple factor } M_i \text{ of } B\}.$$

Write

$$|\kappa_i|=\sum\kappa(\mathfrak{P})\qquad\qquad\text{(over infinite primes } \mathfrak{P} \text{ of } M_i).$$

For $\mathfrak{P}$ a finite prime of a simple factor M_i of B, $\mathfrak{P}$ not dividing $\mathfrak{f}$, and $y\in B_{\mathfrak{p}}^{\times}$ with $\mathrm{ord}_{\mathfrak{p}}(y)=m$, we consider local Whittaker functions of the form

$$(1.2.2)\qquad W_{\mathfrak{p}}(y,1)=\begin{cases}(\alpha_{\mathfrak{p}}^{m+1}-\beta_{\mathfrak{p}}^{m+1})/(\alpha_{\mathfrak{p}}-\beta_{\mathfrak{p}}) & (m\geq 0)\\ 0 & (m<0),\end{cases}$$

where $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ are the roots of an equation

$$Y^2 - b(\mathfrak{P})Y + \chi(\mathfrak{P})\mathbb{N}\mathfrak{P}^{-1} = 0,$$

with $b(\mathfrak{P})$ an algebraic number. For a finite prime $\mathfrak{P}$ of M_i dividing $\mathfrak{f}$, take

$$(1.2.3) \qquad W_{\mathfrak{p}}(y,1) = \begin{cases} \alpha(\mathfrak{P})^m & (m \geq 0) \\ 0 & (m < 0), \end{cases}$$

with an algebraic number $\alpha(\mathfrak{P}) = b(\mathfrak{P})$. For an infinite (real) prime $\mathfrak{P}$, put

$$(1.2.4) \qquad W_{\mathfrak{p}}(y,1) = |y|^{\kappa(\mathfrak{P})/2} \exp(-2\pi|y|),$$

with $\kappa(\mathfrak{P})$ as above.

This formalism applies to the situation where

$$a(\mathfrak{P}) = b(\mathfrak{P})/(\mathbb{N}\mathfrak{p})^{\kappa(\mathfrak{P})/2}$$

$$\kappa(\mathfrak{P}) = |\kappa_i| \qquad (\mathfrak{P} \text{ a prime of } M_i)$$

and $a(\mathfrak{P})$ is the eigenvalue for the (suitably-chosen, normalized) $\mathfrak{P}$-th Hecke operator on 'adelic holomorphic Hilbert modular cuspforms' of weight κ for congruence subgroups of type $\Gamma_0(\mathfrak{f})$ with primitive character χ having conductor $\mathfrak{f}$. Note: our $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ are the usual eigenvalues divided by the obvious power of $\mathbb{N}\mathfrak{P}$. See [Sh1] chapter 3, sections 5 and 6 for treatment of $\Gamma_0(\mathfrak{f}) \subset SL(2,\mathbb{Z})$. The methods of [Sh1] apply, with simple modifications, to this more general setting. See, e.g., [JL] for the representation-theoretic treatment, and [GGP] or [Ge] for a description of the transition from the classical to adelic/local Whittaker function treatment.

The simplest case of this is where $B = F = \mathbb{Q}$, $\mathfrak{f} = 1$, and

$a(p)=b(p)/(p^{\kappa/2})$ is the p-th Fourier coefficient of a holomorphic cuspform of weight $\kappa \in 2\mathbb{Z}$ for $SL(2,\mathbb{Z})$ which is an eigenfunction for the classical Hecke operators.

A similar formalism also applies to Maass' waveforms, with all $\kappa(\wp)=0$, and with a modification at the infinite primes. Now we need not require that the simple factors of B be totally real. For an infinite prime $\wp$, we now put

(1.2.5) $$W_{\wp}(y,1) = \begin{cases} |y|^{\nu} \int e(-x)\,(x^2+y^2)^{-\nu}\,dx & (\wp \text{ real}) \\ |y|^{\nu} \int e(-(x+\bar{x}))(|x|^2+|y|^2)^{-\nu}dx & (\wp \text{ complex}) \end{cases}$$

where the first integral is over $\mathbb{R}$, the second integral is over $\mathbb{C}$, and ν is a suitable complex number possibly depending on $\wp$.

(1.3) The Imbedding of Groups. Now we will imbed a certain subgroup G of $G^{\#}$ in a symplectic group. Let J be the standard B-valued B-bilinear alternating form on B^2:

$$J(\,(x_1,y_1)\,,\,(x_2,y_2)\,) = x_1y_2-x_2y_2.$$

Let $\sigma_1,\ldots$ be the F-algebra homomorphisms of B into an algebraic closure of F. We define an F-linear trace map

$$tr : B \longrightarrow F$$

by

$$tr(\beta) = \textstyle\sum_j \sigma_j(\beta).$$

Let J' be the F-bilinear F-valued alternating form on B^2 defined by $J'(v_1,v_2)=tr(J(v_1,v_2))$. Let G' be the linear algebraic group over ϑ of

ϑ-similitudes of $(\mathcal{O}^2, J')$, i.e., for a commutative ϑ-algebra R, the group of R-valued points of G' is given by

$$G'(R) = \{g \text{ an } R\text{-linear automorphism of } R\otimes_{\vartheta}\mathcal{O}^2 \text{ such that, for } v_1, v_2 \text{ in } R\otimes\mathcal{O}^2,\ J'(gv_1, gv_2) = q(g)\, J'(v_1, v_2), \text{ with some } q(g)\in R^{\times}\},$$

where we take the obvious extension of J'. We may view G'(F) as a subgroup of GL(2, $End_F(B)$), acting on the left on 2-by-1 matrices with components in B. Let P' be the parabolic subgroup of G' whose R-valued points are

$$P'(R) = \{g\in G'(R) : g \text{ fixes the } (R\otimes_{\vartheta}\mathcal{O})\text{-line in } R\otimes_{\vartheta}B^2 \text{ consisting of points of the form } (x,0)^T,\ x\in R\otimes_{\vartheta}B\}.$$

Here x^T is x-transpose. That is,

$$P'(F) = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G'(F) \} \subset GL(2, End_F B).$$

Let U' be the unipotent radical of P'. Let $\mathbb{K}'$ be the group

$$\mathbb{K}' = \prod_{\wp} G'(\vartheta_{\wp}) \subset G'(\mathbb{A}_0),$$

where $\vartheta_{\wp}$ is the $\wp$-adic completion of ϑ, for a (finite) prime $\wp$ of ϑ. Over F, G' is isomorphic to the F-group usually denoted by GSp($\dim_F B$,F) (i.e., to the $2\dim_F B$-by-$2\dim_F B$ standard symplectic similitudes over F). Let Z' be the center of G'.

Clearly <u>some</u> of the elements of $G^{\#}$=GL(2,B) naturally sit inside G'. However, there is a non-trivial restriction on the 'torus part'. Let G be the

ϑ-subgroup of $G^{\#}$ whose R-valued points (for a commutative ϑ-algebra R) are

$$G(R) = \{g \in G^{\#}(R \otimes_{\vartheta} \mathcal{O}) : g \in G'(R)\} =$$
$$= \{g \in G^{\#}(R \otimes_{\mathbb{Z}} \mathcal{O}) : \det(g) \in R^{\times}\}.$$

Thus, the 'semi-simple part' of G is the same as that of $G^{\#}=GL(2,B)$, and G is a normal subgroup. Let

$$\iota : G \longrightarrow G'$$

be the natural (inclusion) group homomorphism. Then ι gives a morphism of linear algebraic groups over F, for example. Let P be the subgroup of upper-triangular matrices in G, U its unipotent radical (which may clearly be identified with the analogous subgroup of GL(2,B) except for issues of rationality), T the group of diagonal matrices in H, and Z the center. Let

$$\mathbb{K} = G(\mathbb{A} \otimes B) \cap \mathbb{K}^{\#},$$
$$K = K^{\#} \cap G(\mathbb{R} \otimes B),$$

where $K^{\#}$ and $\mathbb{K}^{\#}$ are as in (1.1).

Let K' be the maximal compact subgroup of $G'(F \otimes \mathbb{R})$ consisting of elements

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in G'(F \otimes \mathbb{R}) \subset GL(2, \mathrm{End}_F B \otimes \mathbb{R}),$$

where the bar denotes complex conjugation at complex imbeddings of F, and the 'transpose' is the adjoint with respect to the pairing $(x,y) \longrightarrow tr(xy)$. Let $\mathbb{K}' = \prod_{\wp} G'(\vartheta_{\wp})$.

By construction,

$\iota : \mathbb{K} \to \mathbb{K}'$

$\iota : K \to K'$

$\iota : P \to P'$

$\iota : U \to U'$

$\iota(Z) = Z'$

in addition to the fact that ι gives (upon 'restriction of scalars') a morphism of algebraic groups defined over the number field F.

Fix an integral ideal $\mathfrak{f}$ of F. Let

$$\mathbb{K}'(\mathfrak{f}) = \{g\in\mathbb{K}' : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ modulo } \mathfrak{f}\}.$$

Let $\mathbb{K}(\mathfrak{f})=\mathbb{K}\cap\mathbb{K}^{\#}(\mathfrak{f})=\mathbb{K}\cap\mathbb{K}'(\mathfrak{f})$.

(1.4) Adaptable characters and representations. Suppose that χ, ρ_∞, ρ_0 are such that there are a grossencharacter χ' on $Z'(\mathbb{A})$, a $\mathbb{C}^\times$-valued representation ρ_∞ of K', and a $\mathbb{C}^\times$-valued representation ρ'_0 of $\mathbb{K}'(\mathfrak{f})$ so that:

$$\chi = \chi' \circ \iota \qquad \text{(on } Z(\mathbb{A})\text{)}$$

$$\rho_\infty = \rho'_\infty \circ \iota \qquad \text{(on } K\text{)}$$

$$\rho_0 = \rho'_0 \circ \iota \qquad \text{(on } \mathbb{K}(\mathfrak{f})\text{)}.$$

(Note that such χ' always exists). Suppose also that the restriction η of $\rho'=\rho'_\infty\otimes\rho'_0$ to $K'\mathbb{K}'(\mathfrak{f})\cap P'(\mathbb{A})$ has an extension to a $\mathbb{C}^\times$-valued representation $\tilde{\eta}$ of $P'(\mathbb{A})$ which agrees with χ' on $Z'(\mathbb{A})$. (If η is trivial locally at $\mathfrak{p}$, then we take $\tilde{\eta}$ to be trivial locally at $\mathfrak{p}$). Further, suppose that ρ' and $\tilde{\eta}$ are trivial on G'(F) and P'(F), respectively. Then say that χ and ρ are adaptable over F, and that χ' and ρ' are adapted to χ and ρ. It is elementary to verify that if such χ' and ρ' exist, then they are unique.

In the examples of (1.2), the requirement of adaptability can be made more explicit. For $\wp$ dividing $\mathfrak{f}$, ρ'_0 is of the form (modulo $\mathfrak{f}$)

$$\rho_0'(\begin{pmatrix} \lambda a^T & * \\ 0 & a^{-1} \end{pmatrix}) = \psi(\det(a))^{-1}$$

where ψ is a finite-order character modulo $\mathfrak{f}$ on $F_\wp^\times$, and $\lambda \in \mathfrak{o}_\wp^\times$. We must have $\psi(\mathrm{Norm}(\beta)) = \chi(\beta)$ for $\beta \in B^\times$. (Here the norm is that from $B \otimes F_\wp$ to $F_\wp$).

Let $\wp$ be a real prime of F which has only real primes $\mathfrak{P}_i$ lying over it in B. Let $\kappa(\wp) = \kappa(\mathfrak{P}_i)$, where $\kappa(\mathfrak{P})$ is as in (1.2), and $\kappa(\mathfrak{P}_i)$ must be independent of i since ρ_∞ is adaptable over F. Then on the factor $G'(\mathfrak{o}_\wp)$ of K' we have

$$\rho'_\infty \begin{pmatrix} A & B \\ -B & \bar{A} \end{pmatrix} = \det(iB + A)^{\kappa(\wp)}.$$

(1.5) Some Symplectic Eisenstein Series. Let $2n+1 = \dim_F B$ be odd. Let χ and ρ be adaptable over F, with χ' and ρ' be adapted to χ and ρ, as in (1.4).

For primes $\wp$ of $\mathfrak{o}$ (finite or infinite), let $\| \ \|_\wp$ be the usual $\wp$-adic valuation on $F_\wp$ so that the product formula holds. In particular, if π is a local parameter at a finite prime $\wp$, then $\|\pi\|_\wp = \mathbb{N}\wp^{-1}$, where $\mathbb{N}$ is the ideal norm.

For a complex number s, define a $\mathbb{C}^\times$-valued representation $\mu_\wp$ on $P'(F_\wp)$ (for $\wp$ finite or infinite) by

$$\mu_\wp\left(\begin{pmatrix} \lambda a^T & b \\ 0 & a^{-1} \end{pmatrix}\right) = \|\lambda^{2n+1}\det(a)^2\|_\wp{}^s \times \tilde{\eta}\left(\begin{pmatrix} \lambda a^T & b \\ 0 & a^{-1} \end{pmatrix}\right),$$

where $\lambda \in F_\wp{}^\times$, $\tilde{\eta}$ is attached to ρ' as in (1.4), and $a \longrightarrow a^T$ is the transpose with respect to the $F_\wp$-valued pairing $\alpha \times \beta \longrightarrow \mathrm{trace}(\alpha\beta)$ on $B \otimes F_\wp$. Note that $\mu_\wp$ is equal to χ' on $Z'(\mathbb{A})$, is $(Z'(\mathbb{A}),\chi')$-equivariant, is necessarily left $U'(\mathbb{A})$-invariant, and right $(K'\mathbb{K}'(\mathfrak{f}) \cap P'(\mathbb{A}),\rho')$-equivariant. Further, $\mu_\wp$ is left invariant under the subgroup P'' of $P'(\mathbb{A})$ described by

$$P'' = \left\{ \begin{pmatrix} a^T & * \\ 0 & a^{-1} \end{pmatrix} \in P'(\mathbb{A}) : \det(a)=1 \right\},$$

(This is from elementary properties of topological groups, using the fact that $\mu_\wp$ is $\mathbb{C}^\times$-valued).

Define a function $\nu_\wp$ on $G'(\mathbb{A})$ (depending on $s \in \mathbb{C}$), supported on $P'(\mathbb{A})K'\mathbb{K}'(\mathfrak{f})$, by decomposing $g=pk$, $p \in P'(\mathbb{A})$, $k \in K'\mathbb{K}'(\mathfrak{f})$, and putting

$$\nu_\wp(g;\chi',\rho',s) = \begin{cases} \mu_\wp(p)\ \rho'_\infty(k) & (\wp \text{ infinite}) \\ \mu_\wp(p)\ \rho'_0(k) & (\wp \text{ finite}) \end{cases}$$

Here we suppress reference to $\mathfrak{f}$. For g in $G'(\mathbb{A})$, define

$$\nu(g;\chi,\rho',s) = \prod_{\wp} \nu_{\wp}(g;\chi,\rho',s)$$

where the product is over all primes of F. From the definitions (and from the product formula), $\nu(g;\chi,\rho',s)$ is left $P'(F)$–invariant, right $(K'\mathbb{K}'(\mathfrak{f}),\rho')$–equivariant, and $(Z'(\mathbb{A}\otimes F),\chi')$–equivariant.

We have an adelic Eisenstein series

$$E(g;\chi',\rho',s) = \sum_{\gamma} \nu(\gamma g;\chi',\rho',s) \qquad (\gamma \in P'(F)\backslash G'(F)\ ).$$

As usual, for fixed χ and ρ' this series is convergent for real part s large–enough, and defines a left $G'(F)$–invariant, left $(Z'(\mathbb{A}),\chi')$–equivariant, right $(K'\mathbb{K}'(\mathfrak{f}),\rho')$–equivariant continuous function on $G'(\mathbb{A})$.

Let χ be of finite order, and trivial on the infinite factors of the ideles. Let $\wp$ be an infinite prime of F. Then for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in SL(2,F_{\wp}\otimes \mathrm{End}_F B) \cap G'(F_{\wp})$$

one has

$$\nu_{\wp}(g;\chi',\rho',s) = \left|\det(c\bar{c}^{T}+d\bar{d}^{T})\right|^{-s}$$

if any place of B extending $\wp$ is complex, and

$$\nu_{\wp}(g;\chi',\rho',s) =$$
$$= \left|\det(cc^{T}+dd^{T})\right|^{-s} \det(ci+d)^{-\kappa(\wp)/2} \det(-ci+d)^{\kappa(\wp)/2}$$

if $\wp$ splits as only real primes, where $\kappa(\wp)$ is as (1.2). Note that χ' does not appear here, as follows from the assumption that χ (and, hence, χ') is trivial on infinite prime factors $F_{\mathfrak{p}}^{\times}$.

(1.6) A special group element. Define $e \in \mathrm{End}_F B$ by $e(\beta) = \mathrm{tr}(\beta) \in F \subset B$, where tr is the trace from B to F, as in (1.1). Let

$$\theta = \begin{pmatrix} 1+e/3 & -e/9 \\ e & 1-e/3 \end{pmatrix} \in G'(F).$$

The proofs of the following results will be given in subsequent sections.

(1.7) Theorem. Let B be of dimension three over F. Let f be an eigencuspform on GL(2,B) in the sense of (1.1), with associated Whittaker functions $W_{\mathfrak{P}}$, central character χ of conductor dividing $\mathfrak{f}$, and right representation ρ. We assume that χ and ρ are adaptable over F. For a prime $\mathfrak{p}$ of F, write

$$W_{\mathfrak{p}} = \prod_{\mathfrak{P}} W_{\mathfrak{P}} \qquad \text{(product over primes } \mathfrak{P} \text{ of B dividing } \mathfrak{p}\text{).}$$

Take a central character χ' and right representation ρ' for G' adapted to χ and ρ, as in (1.4). For a prime $\mathfrak{p}$ of F, let $\tau_{\mathfrak{p}}$ be the character on $\mathbb{A}$ as in (1.1). Then, integrating over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$,

$$\int \bar{E}(g;\chi',\rho',\bar{s})\, f(g)\, dg = \prod_{\mathfrak{p}} \tilde{L}(f,s,B/F)_{\mathfrak{p}},$$

where

$$\tilde{L}(f,s,B/F)_{\mathfrak{p}} =$$

$$= \prod_{\mathfrak{p}} \int \Big[\tau_{\mathfrak{p}}(x)\ \bar{\nu}_{\mathfrak{p}}(\theta u(x)\delta(y\tilde{y},\tilde{y}^{-1};\chi',\rho',\bar{s})\ \ W_{\mathfrak{p}}(y_1 y_2, y_2^{-1}) \Big]\, \|y^3 N(\tilde{y}^2)\|_{\mathfrak{p}}^{-1} d^{\cdot}y\, d^{\cdot}\tilde{y}\, dx$$

where for $\wp$ a prime of F the integral is over $x \in F_\wp$, $y \in F_\wp^\times$, $\tilde{y} \in (B \otimes F_\wp)^\times / F_\wp^\times$. (Here $u(x)$, and $\delta(y_1, y_2, y_2^{-1})$ are as in (1.1), $d'y$ is the usual normalization of a multiplicative Haar measure and N is the norm from B to F). (The proof will be completed in section 3).

(1.8) Corollary. The integral expression in (1.7) has an analytic continuation to a meromorphic function of $s \in \mathbb{C}$.

Proof. From [L], one knows generally that the Eisenstein series as above have meromorphic analytic continuations, and that for $s \in \mathbb{C}$ away from the poles of the Eisenstein series $E(g;\chi',\rho',s)$ is of 'moderate growth' in g. Since a cuspform $f(g)$ is always of 'rapid decay' in g, the above integral is finite.///

Remark. We emphasize that the preceding and following results indeed do apply to holomorphic eigencuspforms (newforms) for groups analogous to $\Gamma_0(N)$, with characters χ modulo N.

(1.9) Theorem. Let $\wp$ be a finite prime of F, and let χ be a finite-order grossencharacter of B which is of the form $\chi = \psi \circ \mathrm{Norm}_{B/F}$ for some grossencharacter ψ of F. Suppose that ψ is unramified at $\wp$, and that $\wp$ is unramified in B/F. For each prime $\mathfrak{P}_i$ lying over $\wp$ in B suppose that the local Whittaker function $W_\wp$ is of the form (1.2.2):

$$W_{\wp_i}(y,1) = \begin{cases} (\alpha_i^{m+1} - \beta_i^{m+1}) / (\alpha_i - \beta_i) & (m = \mathrm{ord}_{\wp_i}(y) \geq 0), \\ 0 & (m < 0), \end{cases}$$

where, for each i, α_i and β_i are the roots of an equation

$$Y^2 - b_i Y + \kappa(\wp_i) N\wp_i^{-1} = 0,$$

where κ does not depend on i. Let π be a local parameter at $\wp$. Then the the $\wp$-factor $\tilde{L}(f,s,B/F)_\wp$ of (1.7) is given by:

$$\tilde{L}(f,s,B/F)_\wp = (1-\psi(\pi)X^2)(1-\psi(\pi^{-1})N\wp^2X^4)\ L(f,s-2,B/F)_\wp$$

where the Euler factor $L(f,s,B/F)_\wp$ is given by:

i) for $\wp$ inertial:

$$L(f,s,B/F)_\wp^{-1} =$$

$$= (1-\alpha_1\psi(\pi^{-1})X)\ (1-\beta_1\psi(\pi^{-1})X)\ (1-\alpha_1 N\wp^{-3}X^3)\ (1-\beta_1 N\wp^{-3}X^3);$$

ii) for $\wp$ partially split:

$$L(f,s,B/F)_\wp^{-1} =$$

$$= (1-\alpha_1\alpha_2\psi(\pi^{-1})X)\ (1-\alpha_1\beta_2\psi(\pi^{-1})X)\ (1-\beta_1\alpha_2\psi(\pi^{-1})X)\ (1-\beta_1\beta_2\psi(\pi^{-1})X) \times$$

$$\times\ (1-\alpha_1^2 N\wp^{-2}X^2)\ (1-\beta_1 N\wp^2 X^2);$$

iii) for $\wp$ completely split:

$$L(f,s,B/F)_\wp^{-1} =$$

$$= (1-\alpha_1\alpha_2\alpha_3\psi(\pi^{-1})X)\ (1-\alpha_1\alpha_2\beta_3\psi(\pi^{-1})X) \times$$

$$\times\ (1-\alpha_1\beta_2\alpha_3\psi(\pi^{-1})X)\ (1-\alpha_1\beta_2\beta_3\psi(\pi^{-1})X) \times$$

$$\times\ (1-\beta_1\alpha_2\alpha_3\psi(\pi^{-1})X)\ (1-\beta_1\alpha_2\beta_3\psi(\pi^{-1})X) \times$$

$$\times\ (1-\beta_1\beta_2\alpha_3\psi(\pi^{-1})X)\ (1-\beta_1\beta_2\beta_3\psi(\pi^{-1})X).$$

(Proof will be completed in section 4).

(1.10) Theorem. Take f as in (1.9). Let $\mathfrak{p}$ be a finite prime of F which is unramified in the extension B/F, but which divides the level $\mathfrak{f}$. Then the Euler $\mathfrak{p}$-factor of $L(f,s,B/F)$ is:

i) $\mathfrak{p}$ inertial: $(\mathbb{N}\mathfrak{f}\vartheta_{\mathfrak{p}})^{2s+1}\,\mathfrak{y}(\mathfrak{f},\psi)\,(1-\alpha_1\mathbb{N}\mathfrak{p}^3X^3)^{-1}$.

ii) $\mathfrak{p}$ partially split: $(\mathbb{N}\mathfrak{f}\vartheta_{\mathfrak{p}})^{2s+1}\mathfrak{y}(\mathfrak{f},\psi)\times$

$$\times\,(1+\alpha_1\alpha_2\mathbb{N}\mathfrak{p}^{-3}X^{-3})(1-\alpha_1{}^2\mathbb{N}\mathfrak{p}^{-2}X^2)^{-1}(1-\alpha_2{}^2\mathbb{N}\mathfrak{p}^{-4}X^4)^{-1}.$$

iii) $\mathfrak{p}$ completely split: $(\mathbb{N}\mathfrak{f}\vartheta_{\mathfrak{p}})^{2s+1}\mathfrak{y}(\mathfrak{f},\psi')\,(1+\alpha_1\alpha_2\alpha_3\mathbb{N}\mathfrak{p}^{-3}X^{-3})\times$

$$\times\,(1-\alpha_1{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_2{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_3\mathbb{N}\mathfrak{p}^2X^2)^{-1}.$$

Here

$$\mathfrak{y}(\mathfrak{f},\psi)=\int\tau_{\mathfrak{p}}(x)\,\psi(x)\,dx,$$

where $\chi=\psi\circ\mathrm{Norm}$.

(Proof in section 4).

(1.11) Theorem. Suppose that $\mathfrak{p}$ is a real prime of F, with three real primes $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ of B lying over it. Suppose that the local Whittaker functions at $\mathfrak{P}_i$ are all of the form of (1.2.4), i.e.,

$$W_{\mathfrak{P}_i}(\tilde{y},1)=\|\tilde{y}\|_{\mathfrak{P}_i}{}^{\kappa/2}\exp(-2\pi\|\tilde{y}\|_{\mathfrak{P}_i})$$

where κ depends only on $\mathfrak{p}$. Then the Euler $\mathfrak{p}$-factor given by the integral of (1.7) is

$$(-1)^{\kappa/2}\,2^{6-4s-3\kappa}\pi^{3-s-3\kappa/2}\times$$

$$\times\,[\Gamma(s-1+\kappa/2)^3\Gamma(s-2+3\kappa/2)]/[\Gamma(s+\kappa/2)\Gamma(2s+\kappa-2)].$$

(Proof in section 5).

(1.12) Theorem. Now further suppose that f is a holomorphic eigencuspform of level one with trivial central character, of "weight" $(2\kappa, \ldots, 2\kappa)$, $\kappa \in \mathbb{Z}$. Let $\langle , \rangle$ be the Petersson inner product on such cuspforms. Then

$$L^{\#}(f,\kappa-2,B/F) = \zeta_F(2\kappa)^{-1}\, \zeta_F(4\kappa-2)\, \langle f,f\rangle^{-1}\, L(f,\kappa-2,B/F)$$

is an algebraic number, and under an automorphism σ of $\mathbb{C}$,

$$L^{\#}(f,\kappa-2.B/F)^{\sigma} = L^{\#}(f^{\sigma},\kappa-2,B/F),$$

where in the right-hand side of the latter expression σ acts on f as a global section of a line bundle on the associated Shimura variety. (I.e., if the rings of integers of F and B have class-number one, this is the action on the Fourier coefficients of f). By the result of (1.11),

$$L^{\#}(f,\kappa-2,B/F) = (\text{rational}) \times \pi^{d(3-3\kappa)}\, \zeta_F(2\kappa)^{-1}\, \zeta_F(4\kappa-2)^{-1} \times$$

$$\times \prod_{\wp<\infty} L(f,\kappa-2,B/F)_{\wp}$$

(Proof in section 6).

2. Coset Computations

The first main technical point in the whole calculation is determination of nice coset representatives for

$$P'(F) \backslash G'(F) / \iota(G(F)).$$

As we will see below, this double coset space has finitely-many elements (of some conceptual significance), only one of which makes a 'cuspidal' contribution in the end.

(2.1) Notation. Assume that $\dim_F B=3$. Let $\mathrm{End}_F B$ be the ring of F-vectorspace endomophisms of B. Let $\sigma_1, \sigma_2, \sigma_3$ be the 3 distinct F-algebra homomorphisms of B into a fixed algebraic closure $\overline{F}$ of F, and put $tr=\sigma_1+\sigma_2+\sigma_3$. Define $e \in \mathrm{End}_F B$ by

$$e(\beta) = tr(\beta) \in F \subset B.$$

We define an F-subtorus T_1 of T by

$$T_1(R) = T(R) \cap GL(2,R) \subset G(R)$$

for an F-algebra R. Define an F-subgroup U_0 of U by

$$U_0(R) = \{u(x) \in U(R):\ x \in R \otimes B,\ tr(x) = 0\}$$

where we take the R-linear extension of the trace $tr: B \to F$, and $u(x)$ is as in (1.1). Also, let

$$U_1(R) = \{u(x) \in U(R):\ x \in R\}.$$

Say that an element θ of G' is <u>degenerate</u> if the isotropy group of θ (in $G(\mathbb{A})$)

contains the adelic points of the unipotent radical of a parabolic subgroup in some F-simple factor of G. Say that θ is <u>nondegenerate</u> otherwise. We say that a double coset $P'(F)\theta G(F)$ is <u>degenerate</u> if it has a degenerate representative θ; otherwise say the double coset is <u>nondegenerate</u>.

The following proposition makes essential use of the fact that B is a cubic (and not higher degree) extension of F.

(2.2) Proposition. <u>The double coset space</u>

$$P'(F) \setminus G'(F) / G(F)$$

<u>has a unique nondegenerate</u> (in the sense of (2.1)) <u>double coset with nondegenerate representative</u>

$$\theta = \begin{pmatrix} 1 & 3^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} 1 & -3^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+e/3 & -e/9 \\ e & 1-e/3 \end{pmatrix} \qquad \in G'(F) \subset GL(2, \mathrm{End}_F B),$$

<u>where</u> $e \in \mathrm{End}_F B$ <u>is as in</u> (2.1). <u>The isotropy group</u> $\Theta(F)$ <u>of</u> θ <u>is</u>

$$\Theta(F) = G(F) \cap \theta^{-1} P'(F) \theta = T_1(F) U_0(F) = U_0(F) T_1(F),$$

<u>using notation of</u> (1.1). <u>Further, with</u> P'' <u>as in</u> (1.5),

$$G(\mathbb{A}) \cap \theta^{-1}P''\theta \supset U_0(\mathbb{A}).$$

Proof. We will consider G and G' to be subgroups of $GL(2,End_F B)$, as in (1.3). Let I be the identity map in $End_F B$, and 0 the zero map. We have a non-degenerate bilinear form $\langle\ ,\ \rangle$ on B given by

$$\langle x,y \rangle = tr(xy),$$

where tr is as in (2.1). Let $g \to g^T$ be the corresponding transpose map on $End_F B$. Consider an element

$$\begin{pmatrix} I & 0 \\ s & I \end{pmatrix} \in G'.$$

It is not difficult to check that such an element lies in G' if and only if $s^T = s$, and that this element lies in G if and only if $s \in B \subset End_F B$. As in the classical situation with symplectic groups, we have a natural identification

$$P'(F)\backslash G'(F) \approx Aut_F B \backslash X,$$

where

$$X = \{\ x=(x_1,x_2) \in (End_F B)^2 : x_1 x_2^{\ T} = x_2 x_1^{\ T}, \text{ and } (x_1,x_2) \text{ is of 'full rank', i.e., } x_1 + x_2 : B \oplus B \to B \text{ is onto}\}.$$

It is elementary linear algebra to verify that for given $x \in X$, there is

$$u = \begin{pmatrix} I & \lambda \\ 0 & I \end{pmatrix}$$

in G(F) (i.e., with $\lambda \in B$) so that $xu = (y_1,y_2)$ with $y_2 \in Aut_F B$. Then by left multiplication by a suitable element g of $GL(1,End_F B)$ we can normalize to

$$g{\times}u = (s\ ,\ I),$$

with $s^T = s$.

Now put

$$B^\perp = \{s \in \mathrm{End}_F B : s^T = s,\ \mathrm{trace}(\beta s) = 0 \text{ for all } \beta \in B\},$$

where here trace is the operator trace. (One checks easily that, for $\beta \in B$, $\beta^T = \beta$). We have $\dim_F B^\perp = 3$ (as F-vectorspace). There is a natural action of $B^\times$ on $B^\perp$, given as follows. For α in $B^\times$, s in $B^\perp$, β in B, define

$$s^\alpha(\beta) = \alpha s(\alpha\beta).$$

Note that

$$\alpha\,(s\ ,\ I) \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = (s^\alpha, I),$$

where the α on the left is viewed as lying in $\mathrm{Aut}_F B$. Further, for λ in $F^\times$,

$$(s\ ,\ I) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} = (\lambda s\ ,\ I).$$

Thus, the vector space action of $F^\times$ on $B^\perp$ and the action of $B^\times$ on $B^\perp$ are both induced by the actions of $\mathrm{Aut}_F B$ and $G(F)$ on elements (s,I) of X.

First, we determine the orbits of $(F^\times)\times(B^\times)$ on $B^\perp$. Over an algebraic closure $\overline{F}$ of F, one readily verifies that this action of $B^\times$ on the $\overline{F}$-vectorspace $B^\perp \otimes \overline{F}$ has characters $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$ (rational over $\overline{F}$).

In the case $B = F\times F\times F$, all the characters are already rational over F, and we have a decomposition (over F)

$$B^\perp = V(1,2) \oplus V(2,3) \oplus V(1,3),$$

where $V(i,j)$ is the $\sigma_i\sigma_j$-representation space of $B^\times=(F^\times)\times(F^\times)\times(F^\times)$. Then it is elementary to see that there are precisely 8 orbits, with representatives of the form

$$\varepsilon_{12} \oplus \varepsilon_{23} \oplus \varepsilon_{13},$$

where each ε_{ij} is either 0 or a fixed non-zero element of $V(i,j)$. With a choice of coordinates on B according to these subrepresentations (so that the elements s are represented by symmetric matrices over F), we have (possibly redundant) representatives

$$s = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix}$$

where each ε_{ij} is either 0 or 1.

In the case $B=F\times M$, with M a quadratic field extension of F, the representation space $\sigma_1\sigma_2\oplus\sigma_2\sigma_3\oplus\sigma_1\sigma_3$ of $B^\times$ decomposes over F into F-irreducible summands

$$V_1 = \sigma_1\otimes(\sigma_2\oplus\sigma_3),$$

$$V_2 = \sigma_2\sigma_3,$$

where we take σ_2 and σ_3 to be the imbeddings of M into $\overline{F}$. One directly determines that there are just 4 $(B^\times)\times(F^\times)$-orbits, with representatives

$$\varepsilon_1 \oplus \varepsilon_2,$$

where ε_i is either 0 or a fixed non-zero element of V_i. In coordinates, we can take (possibly redundant) representatives

$$s = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_1^{T} & \varepsilon_2 \end{pmatrix}$$

where $\varepsilon_1 \in \mathrm{Hom}(M,F)$ is either 0 or the trace τ from M to F, $\varepsilon_1^{T} \in \mathrm{Hom}_F(F,M)$ is correspondingly either 0 or the natural (field) imbedding, and $\varepsilon_2 \in \mathrm{Hom}_F(M,M)$ is either 0 or $(\tau^{T} \circ \tau - 1)$.

When B is a field, the the representation of $B^{\times}$ on $B^{\perp}$ is clearly irreducible over F. We create an B−vectorspace structure ψ on $B^{\perp}$ as follows. For $\beta \in B^{\times}$, $s \in B^{\perp}$, define

$$\varphi(\beta)s = \beta^{-1}s\beta^{-1}\mathbb{N}(\beta),$$

where $\mathbb{N}$ is the norm from B to F, and the action of $\mathbb{N}\beta$ is by the F−vectorspace action. Put $\varphi(0)s=0$. It is easy to check that φ gives B^{T} the structure of a one−dimensional B−vectorspace, since as a $B^{\times}$−representation space it is just

$$\sigma_1 \oplus \sigma_2 \oplus \sigma_3.$$

Hence, there are just 2 orbits, represented by 0 and any non−zero element of $B^{\perp}$.

Next we complete the computation of the nondegenerate double coset representatives, at the same time determining the corresponding isotropy groups. For each possible double coset representative η, it will be shown that either η is in the same double coset as θ, or else η is degenerate.

For B a field, the above indicates that we have at most representatives

$$\eta_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \eta_1 = \begin{pmatrix} I & 0 \\ s & I \end{pmatrix}$$

where s is any non-zero element of $B^{\perp}$. For that matter, we can right-multiply by suitable elements of G(F) to make s be any element of $\mathrm{End}_F B$ with non-zero projection to $B^{\perp}$, and $s^{\top}=s$. We may as well take s=e, which can be shown to have the required properties. And η_1 is visibly in the same double coset as θ. One sees directly that the isotropy group of η_0 is the F-parabolic subgroup P of G, so η_0 is degenerate. (We will compute the isotropy group of θ (with arbitrary non-zero m) in a uniform manner below).

When B=F×M, with M a quadratic field extension of F, by the above discussion we have at most four representatives

$$\eta_s = \begin{pmatrix} I & 0 \\ s & I \end{pmatrix}$$

with s as above. Clearly the case s=0 gives a degenerate representative. Likewise, the case

$$s = \begin{pmatrix} 0 & 0 \\ 0 & (\tau^{\top}\circ\tau-1) \end{pmatrix}$$

with τ as above clearly gives an η_s which commutes with $SL(F)\times\{1\}\subset SL(2,B)$ (imbedding by $\alpha\rightarrow\alpha\times 1$). Thus, this representative is degenerate. On the other hand, for

$$e = \begin{pmatrix} 0 & \tau \\ \tau^T & (\tau^T \circ \tau - 1) \end{pmatrix}$$

we have s+1=e, so that the corresponding η_s lies in the same double coset as does θ, as in the case that B is a field. What remains to be shown is that for

$$s = \begin{pmatrix} 0 & \tau \\ \tau^T & 0 \end{pmatrix}$$

the corresponding representative η_s lies in the same double coset as does θ (so that this η_s is redundant with the previously-mentioned representatives). We note that

$$\begin{pmatrix} 1_F & 0 \\ \tau^T & 1 \end{pmatrix} \quad (s\ \ I) \begin{pmatrix} 1_F & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1_F & 0 & 1_F & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1_F & 0 & 1_F & 0 \\ 0 & 1 & 0 & 0 \\ 1_F & 0 & 0_F & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is just

$$\begin{pmatrix} 0 & \tau & 1_F & 0 \\ \tau^T & (\tau^T \circ \tau - 1) & 0 & 1 \end{pmatrix}$$

again, where the unlabelled 1's represent the identity in $Hom_F M$. This shows that this η_s is redundant. Thus, in the case B=F×M we have shown that θ represents the only nondegenerate double coset.

The case $B=F\times F\times F$ is treated similarly.

Now we compute the isotropy group $\Theta(F)$ attached to θ, in a manner which does not depend on the precise nature of B. Take

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G(F).$$

Then the condition $\theta g\theta^{-1}\in P'$ is just that

$$h=$$

$$=ea+c-3^{-1}ec-3^{-1}eae-3^{-1}ce+9^{-1}ece-ebe-de+3^{-1}ede=0\in \mathrm{End}_F B.$$

By considering the requirement $h(\beta)=0$ for $\beta\in B$ with $tr(\beta)=0$, we find that $c=0$. Then from $h(\beta)=0$ for $\beta\in F\subset B$, we find that $tr(b)=0$. Again considering $h(\beta)=0$ for $tr(\beta)=0$, one finds that $a\in F$ (actually, $F^{\times}$). Since $g\in G(F)$, this implies that $d\in F^{\times}$. Then it is easy to check that with these restrictions the element h of $\mathrm{End}_F B$ is the zero endomorphism, so that g lies in $\Theta(F)$.

The last assertion of the proposition follows similarly.

This finishes the proof of the proposition.///

3. The Euler Factorization

Now we obtain the expression (1.7) for the integral of the adapted Eisenstein series against an eigencuspform as an "Euler product" of local integrals. Explicit Euler factors in the examples of (1.2) will be computed in the next section.

Proof of (1.7). Altogether, this is just an 'unwinding' argument. Let $\{\eta\}$ be a collection of representatives for

$$P'(F)\backslash G'(F)/G(F),$$

let $\Theta(\eta)$ be the isotropy group of η, and $X(\eta)=Z(\mathbb{A})\Theta(\eta)\backslash G(\mathbb{A})$. Then, by the usual unwinding, the integral of (1.7) is

$$\Sigma_\eta \int_{X(\eta)} f(g)\, \overline{\nu}(\eta g;\chi',\rho',\overline{s})\, dg.$$

Now if η is 'degenerate' in the sense of section 2, then $\overline{\nu}(\eta g;\chi',\rho',\overline{s})$ is left invariant (in g) under the adelic points $V(\mathbb{A})$ of the unipotent radical V of an F–parabolic in G. That is, integrating over $V(F)\backslash V(\mathbb{A})$ (and with u(x) as in (1.1)),

$$\int \overline{\nu}(\eta u(x) g;\chi',\rho',\overline{s})\, \tau(\xi x)\, dx = 0$$

for $\xi\in B^\times$. Since f is a 'cuspform', the integral over $X(\eta)$ must vanish. Therefore, the only contribution to the integral is from the single non–degenerate representative θ of (2.2).

Now $\nu(\theta g;\chi',\rho',s)$ is left $U_0(\mathbb{A})$-invariant, left $T_1(F)$-invariant, and left $(Z(\mathbb{A}),\chi')$-equivariant, with T_1 as in (1.3). Hence, integrating over $U_0(F)\backslash U(\mathbb{A})$,

$$\int \overline{\tau}(\xi x)\ \nu(\theta u(x)g;\chi',\rho',\overline{s})\ dx\ =\ 0$$

for $\xi\in B-F^\times$, and this integral is finite for $\xi\in F^\times$ for real part of s sufficiently large. We may integrate over the smaller coset space

$$Z(\mathbb{A})U_0(F)T_1(F)\backslash G(\mathbb{A})/K\mathbb{K}(\mathfrak{f}).$$

Since f is supported on $P(\mathbb{A})K\mathbb{K}(\mathfrak{f})$, we may in fact integrate over the representatives

$$u(x)\ \delta(y\tilde{y},\tilde{y}^{-1}) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y\tilde{y} & 0 \\ 0 & \tilde{y}^{-1} \end{pmatrix}$$

with $x\in(\mathbb{A}\otimes B)/B$, $y\in\mathbb{A}^\times/F^\times$, $\tilde{y}\in(\mathbb{A}\otimes B)^\times/\mathbb{A}^\times$. Thus, in the notation of (1.1), (using the $U_0(\mathbb{A})$-invariance of ν, and using (2.2))

$$\int f(g)\ \overline{E}(g;\chi',\rho',\overline{s})\ dg =$$

$$= \sum_\xi \int W(\xi y\tilde{y},\tilde{y}^{-1})\ \tau(\xi x)\ \overline{\nu}(\theta u(x)\delta(y\tilde{y},\tilde{y}^{-1});\chi',\rho',\overline{s})\ \|y^3N(\tilde{y}^2)\|^{-1}\ dx\ d^\cdot y\ d^\cdot\tilde{y},$$

where $x\in\mathbb{A}$, $y\in\mathbb{A}^\times/F^\times$, $\tilde{y}\in(\mathbb{A}\otimes B)^\times/\mathbb{A}^\times$, $\xi\in F^\times$, and $d^\cdot y$ is the usual normalization of multiplicative Haar measure. (We need the factor of $\|y^3N(\tilde{y}^2)\|^{-1}$ for the Haar measure on P). For each ξ, we can replace x by $x\xi^{-1}$ and y by $y\xi^{-1}$ to put this into the form (integrated over $x\in\mathbb{A}$, $y\in\mathbb{A}^\times$, and over

$\tilde{y}\in(\mathbb{A}\otimes B)^{\times}/\mathbb{A}^{\times}$):

$$\int W(y\tilde{y},\tilde{y}^{-1})\ \tau(x)\ \overline{\nu}(\theta u(x)\delta(y\tilde{y},\tilde{y}^{-1});\chi',\rho',\overline{s})\ \|y^3N(\tilde{y}^2)\|^{-1}\ dx\ d^{\times}y\ d^{\times}\tilde{y}.$$

Note that $\tau(x)$ is the additive character on $\mathbb{A}\otimes B$, $x\in\mathbb{A}$, so $\tau(x)=\tau_0(3x)$, where τ_0 is the additive character on $\mathbb{A}$, normalized as in (1.1). In this integral we replace x by $3^{-1}x$, noting that the idele norm of 3 is 1, by the product formula. By construction, ν is a product over primes, by definition τ_0 is such a product, and by hypothesis W is also. Thus, we obtain the assertion of (1.7).////

4. Explicit Euler Factors

Now we will complete the proofs of (1.9), and (1.10). Again we point out that these calculations apply to "newforms" with characters for groups analogous to $\Gamma_0(N)$, over arbitrary number fields.

(4.1) Let χ and ρ be of the type mentioned in (1.2). Suppose that χ and ρ are adaptable over F, with χ', ρ', ψ as in (1.4). Again, this means that $\chi=\psi\circ\mathrm{Norm}$, χ restricted to $\mathbb{A}^\times$ is χ', etc. Now we will obtain some preliminary expressions for the indicated local integrals, for the explicit local Whittaker functions of (1.2), with the assumptions of (1.2) on ρ. This will involve an explicit expression for

$$(4.1.1) \qquad \varphi_{\wp}(x,y,\tilde{y}) = \overline{\nu}_{\wp}(\theta u(x/3)\delta(y\tilde{y},\tilde{y}^{-1});\chi',\rho',\overline{s})$$

with $x\in F_{\wp}$, $y\in F_{\wp}^{\times}$, $\tilde{y}\in(B\otimes F_{\wp})^{\times}$, and for the local Fourier transform (in x) of this function. Note that we have replaced $x\in F_{\wp}$ by $x/3$, as in the calculation of the general formal Euler factorization in the previous section. In the notation of (1.2), for primes $\mathfrak{P}_1$, $\mathfrak{P}_2$, ... lying over a (finite) prime $\wp$ of F, let $\alpha_i=\alpha(\mathfrak{P}_i)$, $\beta_i=\beta(\mathfrak{P}_i)$. (If $\wp|\mathfrak{f}$ then we take $\beta(\mathfrak{P}_i)=0$). For $\wp$ not dividing $\mathfrak{f}$, let Φ_i be the group of permutations of $\{\alpha_i,\beta_i\}$, and $\Phi=\Phi_1\times\Phi_2\times\ldots$. Let $r=\prod_i \alpha_i$, and

$$\Delta=\prod_i (\alpha_i-\beta_i).$$

For $\sigma\in\Phi$, $\wp$ not dividing $\mathfrak{f}$, put $r^\sigma=\prod_i \sigma(\alpha_i)$, $\Delta^\sigma=\prod_i(\alpha_i{}^\sigma-\beta_i{}^\sigma)$, etc. We note that since $\mathfrak{f}$ is an ideal of the integers of F, all the primes $\mathfrak{P}$ lying over a finite prime $\wp$ of F have the same sort of explicit 'local Whittaker function': either they are all of the form (1.2.2), or of the form (1.2.3). Let $\sigma_1,\sigma_2,\sigma_3$ be the non-trivial $F_\wp$-algebra homomorphisms of $B\otimes_F F_\wp$ into $F_\wp$, and let $N(\beta)=\sigma_1(\beta)\sigma_2(\beta)\sigma_3(\beta)$ be the local norm. For a prime $\wp$ of F, $x\in F_\wp$, note that

$$\prod_{\mathfrak{P}} \tau_{\mathfrak{P}}(x) = \tau_\wp(\mathrm{trace}(x)) = \tau_\wp(3x),$$

where $\tau_\wp$ is the "canonical" character on $\mathbb{A}$, as in (1.1).

(4.2) Lemma. Fix a finite prime $\wp$ of F, which is unramified in B/F. Take $x\in F_\wp$, $y\in F_\wp{}^\times$, $\tilde{y}=(y_1,y_2,y_3)\in(B\otimes F_\wp)^\times$, φ as in (3.2.1). Normalize $\tilde{y}$ so that $\mathrm{ord}_{\mathfrak{P}}(\tilde{y})\geq 0$ for all $\mathfrak{P}$ lying over $\wp$, and so that $\mathrm{ord}_{\mathfrak{P}}(\tilde{y})=0$ for some $\mathfrak{P}$ lying over $\wp$. Then for $\wp$ not dividing $\mathfrak{f}$ we have

$$\varphi_\wp(x,y,\tilde{y}) =$$
$$= \psi(\gamma^{-1}N(\tilde{y}))\, \|N(\tilde{y})^2y^3/\gamma^2\|_\wp{}^s,$$

where γ is the greatest common divisor of y and x. For $\wp$ dividing the level $\mathfrak{f}$,

$$\varphi_\wp(x,y,\tilde{y}) = \begin{cases} \psi(\gamma^{-1}N(\tilde{y}))\, \|N(\tilde{y})^2y^3/\gamma^2\|_\wp{}^s & (y\in x\mathfrak{f}\mathcal{O}_\wp) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Recall that the representation $\mu_{\wp}$ of (1.5) is trivial on the subgroup $P''(F_{\wp})$ of $P'(F_{\wp})$ consisting of elements

$$\begin{pmatrix} a^T & b \\ 0 & a^{-1} \end{pmatrix}$$

where $\det(a)=1$. Also, for $\wp$ dividing $\mathfrak{f}$, by hypothesis ρ_0 is trivial on the subgroup C of $G'(\vartheta_{\wp})$ consisting of elements of the form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ modulo } \mathfrak{f}\vartheta_{\wp}.$$

If $\wp$ does not divide $\mathfrak{f}$, then put $C=G'(\vartheta_{\wp})$. Therefore, if we find $p\in P''(F_{\wp})$, k in C so that

$$p\theta \begin{pmatrix} y\tilde{y} & x\tilde{y}^{-1} \\ 0 & \tilde{y}^{-1} \end{pmatrix} k = \begin{pmatrix} * & * \\ 0 & w \end{pmatrix},$$

then

$$\varphi_{\wp}(\theta \begin{pmatrix} y\tilde{y} & x\tilde{y}^{-1} \\ 0 & \tilde{y}^{-1} \end{pmatrix}) = \psi(\det(w)^{-1}) \, \|y^3 \det(w)^{-2}\|_{\wp}{}^{s}.$$

Thus, we need only look at the lower half of these matrices, left modulo

$$H = \{F_{\wp}\text{-automorphisms of } B\otimes F_{\wp} \text{ with determinant } 1\},$$

and right modulo C.

The lower half of our product is

$$(e \quad 1-e/3) \begin{pmatrix} y\tilde{y} & x\tilde{y}^{-1} \\ 0 & \tilde{y}^{-1} \end{pmatrix} = (ey\tilde{y} \quad ex\tilde{y}^{-1}+(1-e/3)\tilde{y}^{-1}).$$

Except in the case that $B\otimes F_{\wp}$ is a non-Galois cubic extension of $F_{\wp}$, it is not hard to see that one may choose an $\mathcal{O}_{\wp}$–basis e_1, e_2, e_3 on $\mathcal{O}\otimes\mathcal{O}_{\wp}$ so that in these coordinates

$$e = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In particular:

i) We take the basis (1,0,0), (0,1,0), (0,0,1) for $\wp$ completely split;

ii) For $B\otimes F_{\wp}=F_{\wp}\times M$, with M an unramified quadratic field extension of $F_{\wp}$, let ω_1, ω_2 be an $\mathcal{O}_{\wp}$–basis of the integers $\tilde{\mathcal{O}}$ of M so that the α_i are permuted by Galois, and $\omega_1+\omega_2=1$. Then take the basis (1,0), $(0,\omega_1)$, $(0,\omega_2)$ of $\mathcal{O}\otimes\mathcal{O}_{\wp}$.

iii) For $B\otimes F_{\wp}$ an unramified Galois field extension, we can take an $\mathcal{O}_{\wp}$–basis for $\mathcal{O}\otimes\mathcal{O}_{\wp}$ of the form ω_1, ω_2, ω_3, where the ω_i are permuted among themselves by Galois, and $\omega_1+\omega_2+\omega_3=1$.

We will treat these cases first. Note that the ω_i which appear must be units. Now take $\tilde{y}$ normalized as in the statement of the Lemma. That is, in the respective cases mentioned above,

i) $\tilde{y}=(\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)\in\mathcal{O}_{\wp}\times\mathcal{O}_{\wp}\times\mathcal{O}_{\wp}$ with $\min\{\mathrm{ord}_{\wp}(\tilde{y}_i)\}=0$;

ii) $\tilde{y}=(\tilde{y}_1,\tilde{y}_2)\in\mathcal{O}_{\wp}\times\tilde{\mathcal{O}}$, with either (case (iia)) $y_1=1$ or (case (iib)) $y_2=1$;

iii) $\tilde{y}=1$.

Then in cases (i), (iia), and (iii) we have

$$(ey\tilde{y} \quad ex\tilde{y}^{-1}+(1-e/3)\tilde{y}^{-1}) =$$

$$= \begin{pmatrix} yy_1 & yy_2 & yy_3 & xy_1^{-1}+2y_1^{-1}/3 & xy_2^{-1}-y_2^{-1}/3 & xy_3^{-1}-y_3^{-1}/3 \\ yy_1 & yy_2 & yy_3 & xy_1^{-1}-y_1^{-1}/3 & xy_2^{-1}+2y_2^{-1}/3 & xy_3^{-1}-y_3^{-1}/3 \\ yy_1 & yy_2 & yy_3 & xy_1^{-1}-y_1^{-1}/3 & xy_2^{-1}-y_2^{-1}/3 & xy_3^{-1}+2y_3^{-1}/3 \end{pmatrix}$$

where in case (i) some $y_i=1$, in case (iia) at least two of the y_i are 1, and in case (iii) all $y_i=1$. Without loss of generality, we can <u>renumber</u> the y_j so that $y_1=1$. By acting on the left by H, one easily transforms this to

$$\begin{pmatrix} y & yy_2 & yy_3 & 3x & 0 & 0 \\ 0 & 0 & 0 & -1 & y_2^{-1} & 0 \\ 0 & 0 & 0 & -1 & 0 & y_3^{-1} \end{pmatrix}.$$

Since y_2 and y_3 are in $\vartheta_{\wp}$, we can act on the right by

$$\begin{pmatrix} A^{T-1} & 0 \\ 0 & A \end{pmatrix} \in C$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ y_2 & 1 & 0 \\ y_3 & 0 & 1 \end{pmatrix}$$

to transform this to

$$\begin{pmatrix} y & 0 & 0 & 3x & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & y_3^{-1} \end{pmatrix}.$$

From the preliminary remarks in the proof of this lemma, upon replacing x by $3^{-1}x$, this yields the result in the cases (i), (iia), and (iii) when $B\otimes F_\wp$ is not a non-Galois cubic field extension of $F_\wp$.

In case (iib), we have $\tilde{y}_1=1$, $\tilde{y}_2\in M^\times\cap\tilde{\mathfrak{o}}$. Let $\eta=\tilde{y}_2$. Temporarily, let τ be the trace from M to $F_\wp$, and δ the diagonal imbedding of $F_\wp$ into M. Then

$$(e \quad 1-e/3)\begin{pmatrix} y\tilde{y} & x\tilde{y}^{-1} \\ 0 & \tilde{y}^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} y & \tau y\eta & x+2/3 & \tau(x-1/3)\eta^{-1} \\ \delta y & \delta\tau y\eta & \delta(x-1/3) & [(x-1/3)\delta\tau+1]\eta^{-1} \end{pmatrix}$$

in the $F_\wp\times M$ coordinates. The same type of reduction as before puts this into the form

$$\begin{pmatrix} y & 0 & x(1+\tau\delta) & 0 \\ 0 & 0 & 0 & \eta^{-1} \end{pmatrix} = \begin{pmatrix} y & 0 & 3x & 0 \\ 0 & 0 & 0 & \eta^{-1} \end{pmatrix}.$$

Thus, we have the same result, upon replacing x by $3^{-1}x$.

When $B\otimes F_\wp=M$ is an unramified non-Galois cubic extension of $F_\wp$, we can still take an $\mathfrak{o}_\wp$-basis $1,\omega,\omega'$ for $\mathcal{O}\otimes\mathfrak{o}_\wp$ so that trace$(\omega)=1$, trace$(\omega')=0$. Then, in these coordinates,

$$e = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, using $\tilde{y}=1$, we consider (in a similar manner to the other cases)

$$\begin{pmatrix} 3y & y & 0 & 3x & x-1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the left action of H, as before, we adjust this to

$$\begin{pmatrix} 3y & y & 0 & 3x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the same result is obtained in this case, as well, after making the substitution of $3^{-1}x$ for x.////

(4.3) Lemma. *With notation and normalizations as above,*

$$\int \tau_{\wp}(x)\, \varphi_{\wp}(x,y,\tilde{y})\, dx =$$
$$= \psi(N(\tilde{y}))\, \|y^3 N(\tilde{y})^2\|_{\wp}^{s}\, (1-\mathbb{N}\wp^{-2s}\psi(\pi)) \times$$
$$\times\, (1-[\mathbb{N}\wp^{2s-1}\psi(\pi^{-1})]^{m+1})\, (1-\mathbb{N}\wp^{2s-1}\psi(\pi^{-1}))^{-1},$$

for $\wp$ *not dividing* $\mathfrak{f}$, *where* $\mathrm{ord}_{\wp}(y)=m\geq 0$, *and the integral is over* $F_{\wp}$. *For* $\wp$ *dividing* $\mathfrak{f}$,

$$\int \tau_{\wp}(x)\, \varphi_{\wp}(x,y,\tilde{y})\, dx =$$

$$= \psi(N(\hat{y}))\, \|y^{s}N(\hat{y})^{2}\|^{s}\, (\mathbb{N}\mathfrak{f}\vartheta_{\mathfrak{p}})^{2s+1}\, \mathfrak{g}(\mathfrak{f},\psi),$$

<u>where we have the character sum</u>

$$\mathfrak{g}(\mathfrak{f},\psi) = \int \tau_{\mathfrak{p}}(x)\, \psi(x)\, dx,$$

<u>integrating over</u> $x \in F_{\mathfrak{p}}$ <u>so that</u> $\mathrm{ord}_{\mathfrak{p}}(x) = -\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})$.

Proof. Let π be a local parameter at $\mathfrak{p}$. For $\mathfrak{p}$ not dividing $\mathfrak{f}$, the indicated integral is $\psi(N(\hat{y}))\|y^{s}N(\hat{y})^{2}\|^{s}$ times

$$\int \psi(y)^{-1}\|y\|_{\mathfrak{p}}^{-2s}dx + \int \psi(x)^{-1}\|x\|_{\mathfrak{p}}^{-2s}dx + \int \psi(x)^{-1}\tau_{\mathfrak{p}}(x)\|x\|_{\mathfrak{p}}^{-2s}dx,$$

where the first integral is over $\{x : \mathrm{ord}_{\mathfrak{p}}x \geq \mathrm{ord}_{\mathfrak{p}}y\}$, the second is over $\{x : \mathrm{ord}_{\mathfrak{p}}x = -1\}$, and the third is over $\{x : \mathrm{ord}_{\mathfrak{p}}y \geq \mathrm{ord}_{\mathfrak{p}}x \geq 0\}$. These integrals yield the sum

$$(\mathbb{N}\mathfrak{p})^{-m}\mathbb{N}\mathfrak{p}^{2ms}\psi(\pi^{-m}) - \mathbb{N}\mathfrak{p}^{-2s}\psi(\pi) +$$

$$+ \sum_{0 \leq k < m} \mathbb{N}\mathfrak{p}^{k(2s-1)}\psi(\pi^{-k})(1-\mathbb{N}\mathfrak{p}^{-1}) =$$

$$= [\mathbb{N}\mathfrak{p}^{2s-1}\psi(\pi)^{-1}]^{m} - \mathbb{N}\mathfrak{p}^{-2s}\psi(\pi) +$$

$$+ (1-[\mathbb{N}\mathfrak{p}^{2s-1}\psi(\pi^{-1})]^{m})(1-\mathbb{N}\mathfrak{p}^{-1})(1-\mathbb{N}\mathfrak{p}^{2s-1}\psi(\pi^{-1}))^{-1},$$

which simplifies to the desired expression.

For $\mathfrak{p}$ dividing $\mathfrak{f}$, one may readily verify that the only part of the integral which gives a non-zero contribution is the integral over

$$\{x \in F_{\mathfrak{p}} : \mathrm{ord}_{\mathfrak{p}}(x) = -\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})\},$$

by the usual elementary arguments regarding vanishing of character sums. Then one obtains the indicated result immediately.////

(4.4) Lemma. Let $\wp$ be unramified in B/F. For primes $\mathfrak{P}_i$ lying over $\wp$, $\tilde{y}\in\mathcal{O}\otimes F_\wp$ as in (4.2), put

$$m(i) = \mathrm{ord}_{\mathfrak{P}_i}(\tilde{y}).$$

For $\wp$ not dividing $\mathfrak{f}$, integrating over $x\in F_\wp$, $y\in F_\wp^\times$, we have

$$\iint \tau_\wp(x)\, W_\wp(y\tilde{y},\tilde{y}^{-1})\, \varphi_\wp(x,y,\tilde{y})\, \|y\|_\wp^{-1}\, dx\, d^\times y =$$
$$= (1-\mathbb{N}\wp^{-2s}\psi(\pi)) \times \|\mathbb{N}\tilde{y}\|_\wp^{2s} \times$$
$$\times \textstyle\sum_{\sigma\in\Phi} (r\Delta^{-1})^\sigma \prod_i (\alpha_i^\sigma)^{2m(i)} (1-r^\sigma\mathbb{N}\wp^{3-3s})^{-1}(1-r^\sigma\mathbb{N}\wp^{2-s}\psi(\pi^{-1}))^{-1}.$$

For $\wp$ dividing $\mathfrak{f}$, this integral is equal to

$$(\mathbb{N}\mathfrak{f}\vartheta_\wp)^{2s+1}\,\eta(\mathfrak{f},\psi)\|N(\tilde{y})\|^{2s}\,(1-r\,\mathbb{N}\wp^{3-3s})^{-1}\prod_i \alpha_i^{m(i)}.$$

Proof. Let π be a local parameter at $\wp$. We have

$$W_\wp(y\tilde{y},\tilde{y}^{-1}) = \chi(\tilde{y}^{-1})\, W_\wp(y\tilde{y}^2,1) = \psi(N\tilde{y})^{-1}W_\wp(y\tilde{y}^2,1).$$

Then, by (4.3), for $\wp$ not dividing $\mathfrak{f}$, the integral over y is

(4.4.1) $$\psi(N\tilde{y})^{-1}\psi(N\tilde{y})\|N(\tilde{y})\|_\wp^{2s}(1-\mathbb{N}\wp^{-2s}\psi(\pi))(1-\mathbb{N}\wp^{2s-1}\psi(\pi^{-1}))^{-1} =$$
$$= \|N(\tilde{y})\|_\wp^{2s}(1-\mathbb{N}\wp^{-2s}\psi(\pi))(1-\mathbb{N}\wp^{2s-1}\psi(\pi^{-1}))^{-1}$$

times the sum

$$\textstyle\sum_{\sigma\in\Phi}(r\Delta^{-1}\prod_i\alpha_i^{m(i)})^\sigma \left[\sum_{m\geq 0}\mathbb{N}\wp^{-3ms+3m}(r^\sigma)^m(1-[\mathbb{N}\wp^{2s-1}\psi(\pi^{-1})]^{m+1})\right],$$

where we use the explicit form of each $W_\wp$ as in (1.2.2). For fixed $\sigma\in\Phi$, the inner sum is

$$(1-r^\sigma\mathbb{N}\wp^{3-3s})^{-1} - \mathbb{N}\wp^{2s-1}\psi(\pi^{-1})(1-r^\sigma\mathbb{N}\wp^{2-s}\psi(\pi^{-1}))^{-1} =$$
$$= [1-\mathbb{N}\wp^{2s-1}\psi(\pi^{-1})][1-r^\sigma\mathbb{N}\wp^{3-3s}]^{-1}[1-r^\sigma\mathbb{N}\wp^{2-s}\psi(\pi^{-1})]^{-1}.$$

The first factor cancels with a factor in (4.4.1), giving the asserted result.

For $\mathfrak{p}$ dividing $\mathfrak{f}$, the argument starting from (4.3) is even simpler. We use the explicit form (1.2.3) for the local Whittaker functions, and the fact that adaptability again entails $\chi(\tilde{y}^{-1})\psi(N(\tilde{y}))=1$. ////

The following three propositions complete our preliminary examination of the (finite-prime) $\mathfrak{p}$-factors $\tilde{L}(f,s,B/F)_{\mathfrak{p}}$ for our integral (notation as in (1.7)), in the case that the local Whittaker functions are as in (1.2). This completes the proof of (1.10). We keep the notation of the previous part of this section.

(4.5) Proposition. Let $\mathfrak{p}$ be completely split in B/F. Let $X=\mathbb{N}\mathfrak{p}^{-s}$, and let π_i be a local parameter at $\mathfrak{P}_i$, in $B\otimes F_{\mathfrak{p}}$, π a local parameter at $\mathfrak{p}$. Then in the present case the local integral

$$\int W_{\mathfrak{p}}(y\tilde{y},\tilde{y}^{-1})\ \tau_{\mathfrak{p}}(x)\ \varphi_{\mathfrak{p}}(x,y,\tilde{y})\ \|y^3N(\tilde{y}^2)\|_{\mathfrak{p}}^{-1}\ dx\ d^{\cdot}y\ d^{\cdot}\tilde{y}$$

is, for $\mathfrak{p}$ not dividing $\mathfrak{f}$,

$$(1-\psi(\pi)X^2)\ \sum_{\sigma\in\Phi}\Big[(\alpha_1\alpha_2\alpha_3/\Delta)\ (1+\alpha_1\alpha_2\alpha_3\mathbb{N}\mathfrak{p}^3X^3)\times$$

$$\times(1-\alpha_1{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_2{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_3{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_1\alpha_2\alpha_3\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}\Big]^{\sigma}.$$

For $\mathfrak{p}$ dividing $\mathfrak{f}$, this local integral is

$$(\mathbb{N}\mathfrak{f}\vartheta_{\mathfrak{p}})^{2s+1}\eta(\mathfrak{f},\psi')\ (1+\alpha_1\alpha_2\alpha_3\mathbb{N}\mathfrak{p}^3X^3)\ \times$$

$$\times\ (1-\alpha_1{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_2{}^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_3\mathbb{N}\mathfrak{p}^2X^2)^{-1}.$$

Proof. This follows very easily from the expression of (4.3) for the integral over x and y, observing the normalization of $\tilde{y}$ as in (4.2), and recalling the factor of $\|N(\tilde{y}^2)\|_{\wp}^{-1}$. The integral is the difference of two geometric series. Upon cancelling the common factors $(1-\alpha_1\alpha_2\alpha_3 N\wp^3 X^3)^{\sigma}$ in the numerator and denominator of the summed geometric series, we obtain the indicated result.///

(4.6) Proposition. Let $\wp$ be inertial in B/F. Put $X = N\wp^{-s}$. Then for $\wp$ not dividing $\mathfrak{f}$, the local integral is

$$\int W_{\wp}(y\tilde{y},\tilde{y}^{-1})\ \tau_{\wp}(x)\ \varphi_{\wp}(x,y,\tilde{y})\ \|y^3 N(\tilde{y}^2)\|_{\wp}^{-1}\ dx\ d^{\cdot}y\ d^{\cdot}\tilde{y} =$$

$$= (1-\psi(\pi)X^2)\sum_{\sigma\in\Phi}(\alpha_1\Delta^{-1})^{\sigma}\ (1-\alpha_1^{\ \sigma}N\wp^3X^3)^{-1}\ (1-\alpha_1^{\ \sigma}\psi(\pi^{-1})N\wp^2X)^{-1}.$$

For $\wp$ dividing $\mathfrak{f}$ this local integral is

$$(N\mathfrak{f}\vartheta_{\wp})^{2s+1}\ \eta(\mathfrak{f},\psi)\ (1-\alpha_1 N\wp^3X^3)^{-1}.$$

Proof. This is straightforward, following similar observations as in the proof of (4.5).///

(4.7) Proposition. Let $\wp\mathcal{O} = \mathfrak{P}_1\mathfrak{P}_2$ where the residue class extension degree of $\mathfrak{P}_2$ is two. Let $X = N\wp^{-s}$, and let π_1, π_2, π be local parameters at $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\wp$, respectively. For $\wp$ not dividing $\mathfrak{f}$, the local integral is

$$\int W_{\wp}(y\tilde{y},\tilde{y}^{-1})\ \tau_{\wp}(x)\ \varphi_{\wp}(x,y,\tilde{y})\ \|y^3N(\tilde{y}^2)\|_{\wp}^{-1}\ dx\ d^{\cdot}y\ d^{\cdot}\tilde{y}\ =$$

$$=\ (1-\psi(\pi)X^2)\sum_{\sigma\in\Phi}\Big[\alpha_1\alpha_2\Delta^{-1}(1+\alpha_1\alpha_2N\wp^3X^3)\ \times$$

$$\times\ (1-\alpha_1^{\ 2}N\wp^2X^2)^{-1}(1-\alpha_2^{\ 2}N\wp^4X^4)^{-1}(1-\alpha_1\alpha_2\psi(\pi^{-1})N\wp^2X)^{-1}\Big]^{\sigma}.$$

For $\mathfrak{p}$ dividing $\mathfrak{f}$, this integral is

$$(\mathbb{N}\mathfrak{f}\mathfrak{d}_{\mathfrak{p}})^{2s+1}\eta(\mathfrak{f},\psi)(1+\alpha_1\alpha_2\mathbb{N}\mathfrak{p}^3X^3)(1-\alpha_1^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_2^2\mathbb{N}\mathfrak{p}^4X^4)^{-1}.$$

Proof. Similar to the two previous.///

(4.8) Proof of (1.9), $\mathfrak{p}$ inertial. For $\mathfrak{p}$ inertial in B/F, from (4.6), the Euler $\mathfrak{p}$-factor is $(1-\psi(\pi)X^2)$ times the expression

$$\alpha_1(1-\alpha_1\mathbb{N}\mathfrak{p}^3X^3)^{-1}(1-\alpha_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}(\alpha_1-\beta_1)^{-1} +$$
$$+\beta_1(1-\beta_1\mathbb{N}\mathfrak{p}^3X^3)^{-1}(1-\beta_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}(\beta_1-\alpha_1)^{-1} =$$

$$(1-\alpha_1\mathbb{N}\mathfrak{p}^3X^3)^{-1}(1-\beta_1\mathbb{N}\mathfrak{p}^3X^3)^{-1}(1-\alpha_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}(1-\beta_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}(\alpha_1-\beta_1)^{-1}\times$$
$$\times\left[\alpha_1(1-\beta_1\mathbb{N}\mathfrak{p}^3X^3)(1-\beta_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)-\beta_1(1-\alpha_1\mathbb{N}\mathfrak{p}^3X^3)(1-\alpha_1\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)\right].$$

The expression inside the square brackets simplifies to

$$(\alpha_1-\beta_1)\times(1-\mathbb{N}\mathfrak{p}^2\psi(\pi^{-1})X^4),$$

so that this simplifies to the expression of (1.9), using $\alpha_1\beta_1=\mathbb{N}\mathfrak{P}_1^{-1}=\mathbb{N}\mathfrak{p}^{-3}$.///

(4.9) Proof of (1.9), $\mathfrak{p}$ partially split. Now take $\mathfrak{p}\mathcal{O}=\mathfrak{P}_1\mathfrak{P}_2$, with the residue class field extension degree of $\mathfrak{P}_2$ being 2. The Euler $\mathfrak{p}$-factor is $(1-\psi(\pi)X^2)$ times

$$\Sigma_{\sigma\in\Phi}[(\alpha_1\alpha_2/\Delta)(1+\alpha_1\alpha_2\mathbb{N}\mathfrak{p}^3X^3)(1-\alpha_1\alpha_2\psi(\pi^{-1})\mathbb{N}\mathfrak{p}^2X)^{-1}\times$$
$$\times(1-\alpha_1^2\mathbb{N}\mathfrak{p}^2X^2)^{-1}(1-\alpha_3^2\mathbb{N}\mathfrak{p}^4X^4)^{-1}]^{\sigma}.$$

This is a rational function $P(X)/Q(X)$ of X, where the numerator P and the denominator Q are polynomials in X and we take

$$Q(X) = (1-\alpha_1{}^2 N\mathfrak{p}^2X^2)(1-\beta_1{}^2 N\mathfrak{p}^2X^2)(1-\alpha_2{}^2 N\mathfrak{p}^4X^4)(1-\beta_2{}^2 N\mathfrak{p}^4X^4) \times$$
$$\times \prod_{s\in\Phi} (1-\alpha_1\alpha_2\psi(\pi)^{-1}N\mathfrak{p}^2X)^\sigma.$$

We want to show that $(1-(\alpha_2{}^\sigma)^2 N\mathfrak{p}^4X^4)$ divides the numerator (for each $\sigma\in\Phi$), and that $(1-\psi(\pi^{-1})N\mathfrak{p}^2X^4)$ divides the numerator. For notational ease, replace $\alpha_1{}^\sigma$ by $\alpha_1{}^\sigma(\psi(\pi)/N\mathfrak{p})^{1/2}$, $\alpha_2{}^\sigma$ by $\alpha_2{}^\sigma\psi(\pi)/N\mathfrak{p}$, and X by $X/(\psi(\pi)N\mathfrak{p})^{1/2}$. Then ψ and $N\mathfrak{p}$ disappear from this expression, and $\alpha_i\beta_i=1$.

After this change of variables, the above expression becomes

$$\sum_{\sigma\in\Phi} [(\alpha_1\alpha_2/\Delta)(1+\alpha_1\alpha_2X^3)(1-\alpha_1\alpha_2X)^{-1} \times$$
$$\times (1-\alpha_1{}^2X^2)^{-1}(1-\alpha_3{}^2X^4)^{-1}]^\sigma,$$

and we want to show that $(1-X^4)$, $(1-\alpha_2{}^2X^4)$, and $(1-\beta_2{}^2X^4)$ divide the numerator P' of this rational expression, where we take denominator

$$Q'(X) = (1-\alpha_1{}^2X^2)(1-\beta_1{}^2X^2)(1-\alpha_2{}^2X^4)(1-\beta_2{}^2X^4) \times$$
$$\times \prod_{s\in\Phi} (1-\alpha_1\alpha_2X)^\sigma.$$

First, substitute $X=\pm1$ into this expression. It becomes

$$\sum_\sigma [(\alpha_1\alpha_2/\Delta)(1\pm\alpha_1\alpha_2)(1\pm(-\alpha_1\alpha_2))^{-1}(1-\alpha_1{}^2)^{-1}(1-\alpha_2{}^2)^{-1}]^\sigma=$$
$$=\Delta^{-2}(\alpha_1+\beta_1)^{-1}(\alpha_2+\beta_2)^{-1}\sum_\sigma [(1\pm\alpha_1\alpha_2)(1\pm(-\alpha_1\alpha_2))^{-1}]^\sigma.$$

It is easy to check that the sum is zero. Next, substitute $X=\pm i$. The above expression becomes

$$\Sigma_\sigma\,[[\alpha_1\alpha_2/\Delta](1\pm(-i\alpha_1\alpha_2))(1\pm(-i\alpha_1\alpha_2))^{-1}(1+\alpha_1^2)^{-1}(1+\alpha_2^2)^{-1}]^\sigma=$$

$$=(\alpha_1+\beta_1)^{-1}(\alpha_2+\beta_2)^{-1}\Sigma_\sigma\,(\Delta^{-1})^\sigma=0.$$

To finish the proof in this case, by formal symmetry it suffices to show that $(1-\alpha_2^2X^4)$ divides the numerator P'. Since this polynomial occurs in the denominators of 2 terms in the sum, it suffices to show that $(1-\alpha_2^2X^4)$ divides

$$(\alpha_1/(\alpha_1-\beta_1))(1+\alpha_1\alpha_2X^3)/(1-\alpha_1X^2)(1-\alpha_1\alpha_2X)+$$
$$+(\beta_1/(\beta_1-\alpha_1))(1+\beta_1\alpha_2X^3)/(1-\beta_1X^2)(1-\beta_1\alpha_2X).$$

Let $\gamma=\pm\alpha_2^{1/2}$. Substituting $X=\pm\alpha_2^{1/2}=\gamma$, this is

$$\alpha_1(1\pm\alpha_1/\gamma)/(1-\alpha_1^2/\gamma^2)(1\pm(-\alpha_1\gamma))(\alpha_1-\beta_1)+$$
$$+\beta_1(1\pm\beta_1/\gamma)/(1-\beta_1^2/\gamma^2)(1\pm(-\beta_1\gamma))(\beta_1-\alpha_1)=$$
$$=[\gamma/(\alpha_1-\beta_1)]\times[(-\gamma\pm\alpha_1)^{-1}(-\beta_1\pm\gamma)^{-1}-(-\gamma\pm\beta_1)^{-1}(-\alpha_1\pm\gamma)^{-1}]=0.$$

Substituting $X=i\gamma$, this is

$$\alpha_1(1\pm(-i\alpha_1/\gamma))/(1+\alpha_1^2/\gamma^2)(1\pm(-i\alpha_1\gamma))(\alpha_1-\beta_1)+$$
$$+\beta_1(1\pm(-i\beta_1/\gamma))/(1+\beta_1^2/\gamma^2)(1\pm(-i\beta_1\gamma))(\beta_1-\alpha_1)=$$
$$=[i\gamma/(\alpha_1-\beta_1)]\times[(\gamma\pm i\alpha_1)^{-1}(i\beta_1\pm\gamma)^{-1}-(\gamma\pm i\beta_1)^{-1}(i\alpha_1\pm\gamma)^{-1}]=0.$$

By degree considerations, these must be all the factors of P'. From this we have the assertion of (1.9) in the case that $\mathfrak{p}$ is "partially split".////

(4.10) Proof of (1.9), $\mathfrak{p}$ completely split. Now take $\mathfrak{p}$ completely split in B/F, not dividing $\mathfrak{f}$. The Euler $\mathfrak{p}$-factor is $(1-\psi(\pi)X^2)$ times

$$\sum_{\sigma\in\Phi} [(\alpha_1\alpha_2\alpha_3/\Delta)(1+\alpha_1\alpha_2\alpha_3 \mathbb{N}\wp^3X^3)(1-\alpha_1\alpha_2\alpha_3\psi(\pi^{-1})\mathbb{N}\wp^2X)^{-1} \times$$
$$\times (1-\alpha_1{}^2\mathbb{N}\wp^2X^2)^{-1} (1-\alpha_2{}^2\mathbb{N}\wp^2X^2)^{-1}(1-\alpha_3{}^2\mathbb{N}\wp^2X^2)^{-1}]^\sigma.$$

This is a rational function $P(X)/Q(X)$ of X, where the numerator P and the denominator Q are polynomials in X, and

$$Q(X) =$$
$$= \prod_{\sigma\in\Phi} (1-\alpha_1\alpha_2\alpha_3\psi(\pi^{-1})\mathbb{N}\wp^2X)^\sigma \times \prod_i (1-\alpha_i{}^2\mathbb{N}\wp^2X^2)(1-\beta_i{}^2\mathbb{N}\wp^2X^2).$$

We want to show that $(1-(\alpha_i{}^\sigma)^2\mathbb{N}\wp^2X^2)$ divides the numerator P (for each $\sigma\in\Phi$, and for each i), and that $(1-\psi(\pi^{-1})\mathbb{N}\wp^2X^4)$ divides the numerator. As in the previous case, replace each $\alpha_i{}^\sigma$ by $\alpha_i{}^\sigma(\psi(\pi)/\mathbb{N}\wp)^{1/2}$, and X by $X/(\psi(\pi)\mathbb{N}\wp^3)^{1/2}$, so that $\alpha_i\beta_i=1$, and ψ and $\mathbb{N}\wp$ disappear from the simplified rational expression P'/Q'.

First, we show that $(1-X^4)$ divides the numerator P' of this rational expression. For a rational expression R in $\{\alpha_i, \beta_i\}$, let T[R] denote the "trace"

$$T[R] = \sum_{\sigma\in\Phi} R^\sigma.$$

Substituting $X=\pm1$, the expression above becomes

$$T\Big[\alpha_1\alpha_2\alpha_3(1\pm\alpha_1\alpha_2\alpha_3)/\Delta(1-\alpha_1{}^2)(1-\alpha_2{}^2)(1-\alpha_3{}^2)(1\pm(-\alpha_1\alpha_2\alpha_3))\Big] =$$
$$= \Delta^{-2}\, T[-(1\pm\alpha_1\alpha_2\alpha_3)/(1\pm(-\alpha_1\alpha_2\alpha_3))] = 0.$$

Substituting $X=\pm i$, this expression becomes

$$T\Big[\alpha_1\alpha_2\alpha_3(1\pm(-i\alpha_1\alpha_2\alpha_3))/\Delta(1+\alpha_1{}^2)(1+\alpha_2{}^2)(1+\alpha_3{}^2)(1\pm(-i\alpha_1\alpha_2\alpha_3))\Big] =$$
$$= ((\alpha_1+\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3))^{-1}\, T[\Delta] = 0.$$

By formal symmetry, to show that the 6 factors $(1-(\alpha_1^\sigma)^2X^2)$ divide the numerator, it suffices to show that $(1-\alpha_1^2X^2)$ divides the rational expression

$$\sum_{\sigma\in\Psi}[(\alpha_2\alpha_3/\Delta)(1+\alpha_1\alpha_2\alpha_3X^3)(1-\alpha_2^2X^2)^{-1}(1-\alpha_3^2X^2)^{-1}\times$$
$$\times(1-\alpha_1\alpha_2\alpha_3X)^{-1}]^\sigma,$$

where $\Psi=\Phi_2\times\Phi_3$, with Φ_i as in (4.1). For a rational expression R in $\{\alpha_i,\beta_i\}$, let

$$S[R]=\sum_{\sigma\in\Psi}\operatorname{sgn}(\sigma)\,R^\sigma,$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ. Note that for all $\sigma\in\Phi$, $\Delta^\sigma=\operatorname{sgn}(\sigma)\Delta$. By elementary algebra one sees that, to show that $(1-\alpha_1^2X^2)$ divides the above expression, it suffices to show that

$$0=S[\alpha_2\alpha_3(1\pm\alpha_2\alpha_3/\alpha_1^2)/(1-\alpha_2^2/\alpha_1^2)(1-\alpha_3^2/\alpha_1^2)(1\pm(-\alpha_2\alpha_3))]=$$
$$=\alpha_1^{-2}\,S[\alpha_2\alpha_3(\alpha_1^2\pm\alpha_2\alpha_3)/(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(1\pm(-\alpha_2\alpha_3))]=$$

$$=\alpha_1^{-2}\Big[(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\beta_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\beta_3^2)\prod_{\sigma\in\Psi}(1\pm(-\alpha_2\alpha_3)^\sigma)\Big]^{-1}\times$$
$$\times\,S[\alpha_2\alpha_3(\alpha_1^2\pm\alpha_2\alpha_3)(\alpha_1^2-\beta_2^2)(\alpha_1^2-\beta_3^2)(1\pm(-\beta_2\alpha_3))(1\pm(-\alpha_2\beta_3))(1\pm(-\beta_2\beta_3))].$$

The argument of S in the last line is a polynomial of the form $A_3\alpha_1^6+A_2\alpha_1^4+A_1\alpha_1^2+A_0$, in α_1. To show that this value of S is zero we must show that each $S[A_i]=0$.

We have

$$S[A_0] = S[-\alpha_2\alpha_3(\pm\alpha_2\alpha_3)(-\beta_2{}^2)(-\beta_3{}^2)(1\pm(-\beta_2\alpha_3))(1\pm(-\alpha_2\beta_3)) \times$$
$$\times\ (1\pm(-\beta_2\beta_3))] =$$
$$= -S[\pm 2-\beta_2\alpha_3-\alpha_2\beta_3-2\beta_2\beta_3\pm\beta_2{}^2\pm\beta_3{}^2] = 0.$$

Next,

$$S[A_3] = S[\alpha_2\alpha_3(1\pm(-\beta_2\alpha_3))(1\pm(-\alpha_2\beta_3))(1\pm(-\beta_2\beta_3))] =$$
$$= -S[-\alpha_2\alpha_3\pm\alpha_3{}^2\pm\alpha_2{}^2\pm 1-\alpha_2\alpha_3-\beta_2\alpha_3-\alpha_2\beta_3\pm 1] = 0.$$

Third,

$$S[A_1] =$$
$$= S[\alpha_2\alpha_3(-\beta_2{}^2\beta_3{}^2\pm\beta_2\alpha_3\pm\alpha_2\beta_3)(-2\pm 2\beta_2\beta_3\pm\alpha_2\beta_3\pm\beta_2\alpha_3-\beta_2{}^2-\beta_3{}^2)] =$$
$$= S[\beta_2\beta_3(2+\beta_2{}^2+\beta_3{}^2)+(\alpha_2{}^2+\alpha_3{}^2)(2\beta_2\beta_3+\beta_2\alpha_3+\alpha_2\beta_3)]$$
$$\pm\ S[-(\alpha_2{}^2+\alpha_3{}^2)(\beta_2{}^3+\beta_3{}^2+2)-(\beta_2\beta_3(2\beta_2\beta_3+\beta_2\alpha_3+\alpha_2\beta_3)] =$$
$$= 2(\alpha_2+\beta_2+\alpha_2{}^3+\beta_2{}^3)S[\beta_3] + (\alpha_3{}^3+\beta_3{}^3)S[\beta_2] + 2(\alpha_2+\beta_2)S[\alpha_3]$$
$$\pm\ S[-2+2\alpha_2{}^2+2\alpha_3{}^2+\beta_2{}^2+\beta_3{}^2] \pm S[-\beta_2{}^2\alpha_3{}^2-\alpha_2{}^2\beta_3{}^2-2\beta_2{}^2\beta_3{}^2] = 0.$$

(In the latter grouping of terms, its is very easy to see that each indicated evaluation of S is zero). Finally,

$$S[A_2] =$$
$$= -S[\alpha_2\alpha_3(\pm\alpha_2\alpha_3-\beta_2{}^3-\beta_3{}^2)(-2\pm(2\beta_2\beta_3+\beta_2\alpha_3+\alpha_2\beta_3)-\beta_2{}^2-\beta_3{}^2)] =$$
$$= S[(\beta_2\alpha_3+\alpha_2\beta_3)(2+\beta_2{}^2+\beta_3{}^2)+\alpha_2{}^2\alpha_3{}^2(2\beta_2\beta_3+\beta_2\alpha_3+\alpha_2\beta_3)]$$
$$\pm\ S[\alpha_2{}^2\alpha_3{}^2(2+\beta_2{}^2+\beta_3{}^2)+(\beta_2\alpha_3+\alpha_2\beta_3)(2\beta_2\beta_3+\beta_2\alpha_3+\alpha_2\beta_3)] =$$
$$= 2(\alpha_2+\beta_2)(\alpha_3+\beta_3)S[1] + (\alpha_2{}^3+\beta_2{}^3)S[\alpha_3] + (\alpha_3{}^3+\beta_3{}^3)S[\alpha_2]$$
$$\pm\ S[2+\alpha_3{}^2+\alpha_2{}^2+2\beta_2{}^2+2\beta_3{}^2] \pm S[2\alpha_2{}^2\alpha_3{}^2+\beta_2{}^2\alpha_3{}^2+\alpha_2{}^2\beta_3{}^2] =$$
$$= 0.$$

Again, the last grouping of terms allows easy verification that this sum of values of S is indeed zero.

Altogether, we have checked that the numerator P' has the asserted factors. By degree considerations, these must be all factors of the numerator. Thus, we obtain the completely split case of (1.9).///

Remarks. In the previous calculations of the explicit Euler factors, there does not seem to be any obvious reason why the simplifications should take place as they do, though presumably there is some general purely algebraic identity of which these are special cases.

6. The Archimedean Integral (holomorphic case)

Now we consider the Euler factors at infinite primes in the case of holomorphic automorphic forms.

Proof (of 1.11). Let $\wp$ be a real prime of F, with $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ real primes of B lying over $\wp$, and suppose that the local Whittaker functions are of the form of (1.2.4), i.e.,

$$W_{\mathfrak{P}_i}(\tilde{y},1) = |\tilde{y}|^{\alpha} \exp(-2\pi|\tilde{y}|),$$

where $\alpha = \kappa/2$ depends only on $\wp$, and not on $\mathfrak{P}_i$. (I.e., the "weight" is κ). We will explicitly compute the integral indicated in (1.7), as was done in the previous section for finite primes. We are supposing that the character χ is trivial on $F_\wp^\times$.

First, we may choose representatives

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{y} & 0\\ 0 & \tilde{y}^{-1}\end{pmatrix}$$

for $T_1(F_\wp)\backslash P(F_\wp)$ (as indicated in (1.7)) so that $(x,y,\tilde{y})$ satisfy

$$\tilde{y}=(y_1,y_2,y_3)\in(B\otimes F_\wp)^\times \approx (F_\wp^\times)\times(F_\wp^\times)\times(F_\wp^\times),$$

$$y=1,$$

$$x\in F_\wp.$$

Then we have

$$W_{\wp}(\tilde{y},\tilde{y}^{-1}) = \prod_i W_{\wp_i}(y_i^{2},1) = |y_1y_2y_3|^{2\kappa} \exp(-2\pi(y_1^{2}+y_2^{2}+y_3^{2})),$$

since these local Whittaker functions are invariant under the center of $GL(2,B\otimes F_{\wp})$.

As in the previous section (see (4.1)), we need an explicit expression for the function

$$\varphi_{\wp}(x,\tilde{y}) = \bar{\nu}_{\wp}(\theta u(x/3)\delta(\tilde{y},\tilde{y}^{-1});\chi',\rho',\bar{s}),$$

for its Fourier transform, and for the integral in $\tilde{y}$. Now for $g\in G'(F_{\wp})$ of determinant 1, with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\nu_{\wp}(g;\chi',\rho',\bar{s}) = |\det(ci+d)|^{-2s}\det(ci+d)^{-\alpha}\det(-ic+d)^{\alpha},$$

by the usual sort of calculation (see (1.5)).

Let $g=\theta u(x/3)\delta(\tilde{y},\tilde{y}^{-1})$. As in the proof of (4.2), we can left multiply g by elements of P'' to normalize the lower half $(c\ d)$ of g to the form

$$\begin{pmatrix} y_1 & y_2 & y_3 & xy_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & -y_1^{-1} & y_2^{-1} & 0 \\ 0 & 0 & 0 & -y_1^{-1} & 0 & y_3^{-1} \end{pmatrix}.$$

Then

$$\det(ci+d) = (y_1y_2y_3)^{-1}\,[x+i(y_1^{2}+y_2^{2}+y_3^{2})].$$

Thus,

$\varphi_{\wp}(x,\tilde{y}) =$

$= (y_1y_2y_3)^{2s}[x-i(y_1{}^2+y_2{}^2+y_3{}^2)]^{-s-\alpha}[x+i(y_1{}^2+y_2{}^2+y_3{}^2)]^{-s+\alpha},$

using the obvious appropriate choice of branches. Thus, altogether, the integral we wish to evaluate, which will give the Euler $\wp$-factor in this case, is

$$\int (y_1y_2y_3)^{2s+\kappa}\exp(-2\pi(y_1{}^2+y_2{}^2+y_3{}^2))[x-i(y_1{}^2+y_2{}^2+y_3{}^2)]^{-s-\kappa/2} \times$$
$$\times [x+i(y_1{}^2+y_2{}^2+y_3{}^2)]^{-s+\kappa/2} \exp(2\pi ix) \|N(\tilde{y}^2)\|_{\wp}^{-1} d^\cdot y_1 d^\cdot y_2 d^\cdot y_3 dx,$$

where $x\in\mathbb{R}$ with usual Lebesgue measure, $y_i\in\mathbb{R}^\times$, and $d^\cdot y_i = dy_i/\|y_i\|$. Replacing $y_i{}^2$ by $y_i>0$ for each i, this may be written as

$$(-1)^\kappa\int (y_1y_2y_3)^{s+\alpha-1}\exp(-2\pi(y_1+y_2+y_3))[ix+(y_1+y_2+y_3)]^{-s-\alpha} \times$$
$$\times [-ix+(y_1+y_2+y_3)]^{-s+\alpha} \exp(2\pi ix)\, d^\cdot y_1 d^\cdot y_2 d^\cdot y_3 dx.$$

This may be evaluated by standard methods, as follows.

From the identity (for $Re(s)>0$)

$$\int_{0<\eta} \exp(-2\pi\eta(y+ix))\, \eta^s\, d^\cdot\eta = (2\pi)^{-s}\Gamma(s)(y+ix)^{-s}$$

we can compute the Fourier transforms (by Fourier inversion, for $Re(s)$ sufficiently large for given $\kappa=2\alpha$):

$$\int_{\mathbb{R}} \exp(-2\pi ix\eta)\, [ix+y]^{-s-\alpha}dx =$$
$$= \begin{cases} (2\pi)^{s+\alpha}\, \Gamma(s+\alpha)^{-1} \exp(2\pi y\eta)(-\eta)^{s+\alpha-1} & (\eta<0) \\ 0 & (\eta>0); \end{cases}$$

$$\int_{\mathbb{R}} \exp(-2\pi ix\eta)\, [-ix+y]^{-s+\alpha}dx =$$
$$= \begin{cases} (2\pi)^{s-\alpha}\, \Gamma(s-\alpha)^{-1} \exp(-2\pi y\eta)\eta^{s-\alpha-1} & (\eta>0) \\ 0 & (\eta<0). \end{cases}$$

By the identity $(f_1 f_2)\hat{} = f_1 * f_2$ (indicating convolution), we have

$$\int_{\mathbb{R}} \exp(2\pi i x)\,[ix+(y_1+y_2+y_3)]^{-s-\alpha}[-ix+(y_1+y_2+y_3)]^{-s+\alpha} dx =$$

$$= (2\pi)^{2s}\Gamma(s+\alpha)^{-1}\Gamma(s-\alpha)^{-1} \times$$

$$\times \int_{0<\eta} \exp(-2\pi(1+2\eta)(y_1+y_2+y_3))\, \eta^{s-\alpha-1}(1+\eta)^{s+\alpha-1}\, d\eta.$$

Therefore, the above integral is (integrating from 0 to ∞ in each variable)

$$\int (y_1y_2y_3)^{s+\alpha-1}\exp(-4\pi(1+\eta)(y_1+y_2+y_3))\eta^{s-\alpha-1}(1+\eta)^{s+\alpha-1}d\eta\, d^{\cdot}y_1\, d^{\cdot}y_2\, d^{\cdot}y_3$$
$$= (2\pi)^{2s}\Gamma(s+\alpha)^{-1}\Gamma(s-\alpha)^{-1}(4\pi)^{-3s-3\alpha+3}\Gamma(s+\alpha-1)^3 \times$$
$$\times \int_{0<\eta} (1+\eta)^{-2s-2\alpha+2}\, \eta^{s-\alpha-1}\, d\eta.$$

It is well-known that

$$\int_{0<\eta} \eta^a(1+\eta)^{-b}d\eta = \Gamma(a+1)\Gamma(b-a-1)/\Gamma(b).$$

Hence, the integral over η above is

$$\Gamma(s-\alpha)\Gamma(s+3\alpha-2)/\Gamma(2s+2\alpha-2).$$

Therefore, altogether, this local integral is

$$(-1)2^{6-4s-6\alpha}\pi^{3-s-3\alpha} \times$$

$$\times\ \Gamma(s+\alpha-1)^3\Gamma(s+3\alpha-2)/\Gamma(s+\alpha)\Gamma(2s+2\alpha-2).$$

This finishes the proof of (1.11).///

7. Special Values (holomorphic case)

Now we can directly demonstrate our simple case of the special value result, as a corollary of the integral representation, and of the arithmetic of Eisenstein series and Hilbert modular forms. (The basic results regarding the latter are to be found in [Sh2]).

Proof of (1.12). Now we restrict our attention to the case that F is totally real of degree d over $\mathbb{Q}$, B is a product of totally real number fields, and the automorphic form f is (an adelic) holomorphic eigencuspform of level one of "weight" (2κ, ..., 2κ) with trivial central character and trivial finite-prime right representation. (Again, the most general case of holomorphic eigenforms requires considerable further effort, and involves several rather subtle issues not directly connected with the present considerations). In our present adelic setting, it is most economical to use the terminology and results of [H1] and [H2] regarding the arithmetic of automorphic forms, though we are only using a very special case of the results of these papers. Also see [T] regarding some arithmetic properties of certain level-one Eisenstein series. If it were the case that the rings of integers of F and B had class number one, then the more classical notions of arithmeticity via literal Fourier coefficients would suffice for part of this argument (as in [G3]).

Let

$$X = Z(\mathbb{A})G(F)\backslash G(\mathbb{A})/\mathbb{K}K,$$

$$X' = Z'(\mathbb{A})G'(F)\backslash G'(\mathbb{A})/\mathbb{K}'K'$$

be the two (level-one pieces of) the associated Shimura varieties. Denote by $\iota: X \to X'$ the algebraic morphism from X to X' induced by the imbedding of groups. Both X and X' have models over $\mathbb{Q}$, and this can be arranged so that ι is defined over $\mathbb{Q}$. Let $E(g,2\kappa)$ denote the Eisenstein series of (1.5) in this case, with $s=\kappa$, trivial ρ_0, $\mathfrak{f}=1$, and weight 2κ (as in (1.2)). It is known (see [H1]), that (with suitable choice of the $\mathbb{Q}$-structure on X') this Eisenstein series is "$\mathbb{Q}$-arithmetic", i.e., is a section of a line bundle on X' defined over $\mathbb{Q}$. Further, $E\circ\iota$ is a global section of a line bundle ω on X defined over $\mathbb{Q}$. As in the classical treatment in [Sh2], the space of "weight $\tilde{\kappa}=(2\kappa, \ldots, 2\kappa)$" cuspforms on X is a $\mathbb{Q}$-subspace of the space of global sections of ω (and this latter space of global sections is defined over $\mathbb{Q}$, as well).

Let $\{\varphi_i\}$ be an orthogonal basis for the space of cuspforms of weight $\tilde{\kappa}$ on X, consisting of eigenfunctions for the Hecke algebra. The action of the Hecke algebra on the space of weight $\tilde{\kappa}$ cuspforms is defined over $\mathbb{Q}$, semi-simple, closed under adjoints, and is commutative, as in the most classical case. Also as in the most classical case, these eigenfunctions are global sections of ω defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Let η be the projection (with respect to the action of the Hecke algebra) of $E\circ\iota$ to the space of cuspforms of weight $\tilde{\kappa}$. By the previous remarks, η is still a global section of ω defined over $\mathbb{Q}$. Therefore, putting $\bar{\eta}=\sum_i c_i\varphi_i$, we must have $c_i=\langle\varphi_i,\eta\rangle\langle\varphi_i,\varphi_i\rangle^{-1}$, and

$c_i \in \bar{\mathbb{Q}}$. Further, letting $\varphi \longrightarrow \varphi^\sigma$ denote the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on sections of ω, we have

$$\sum_i c_i \varphi_i = \bar{\eta} = \bar{\eta}^\sigma = \sum_i c_i^\sigma \varphi_i^\sigma.$$

Since (in this case) the Hecke algebra commutes with the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the φ_i^σ are again (orthogonal) eigenfunctions (and are cuspforms). Thus,

$$(\langle \varphi_i, \eta \rangle \langle \varphi_i, \varphi_i \rangle^{-1})^\sigma = c_i^\sigma = \langle \varphi_i^\sigma, \eta \rangle \langle \varphi_i^\sigma, \varphi_i^\sigma \rangle^{-1} \in \bar{\mathbb{Q}}.$$

This finishes the demonstration of our simple example of the special-value phenomenon. The explicit calculation of the archimedean factor gives the second statement of the theorem (1.12). ///

Remarks. In the more general situation, one must refer to somewhat more delicate rationality properties of the spaces of newforms with characters, and the conjugation result involves conjugation of the characters, and Eisenstein series as well. Further, to obtain the <u>other</u> expected special values, one needs the much more delicate arithmetic properties of the Eisenstein series at other specializations of the parameter s, for example, as in [Sh3], [Sh4].

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