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Archimedean Zeta Integrals for Unitary Groups

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We prove that certain archimedean integrals arising in global zeta integrals involving holomorphic discrete series on unitary groups are predictable powers of π times rational or algebraic numbers. In some cases we can compute the integral exactly in terms of values of gamma functions, and it is plausible that the value in the most general case is given by the corresponding expression. Non-vanishing of the algebraic factor is readily demonstrated via the explicit expression.

Regarding analytical aspects of such integrals, whether archimedean or p-adic, a recent systematic treatment is [Lapid Rallis 2005] in the Rallis conference volume. [1] In particular, the results of Lapid and Rallis allow us to focus on the arithmetic aspects of the special values of the integrals. This is implicit in (4.4)(iv) in [Harris 2006]. [2]

- Context
- Qualitative theorem
- Quantitative theorem

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1. *Context*

We are interested in evaluating archimedean local integrals arising from restricting Siegel-type Eisenstein series from a maximally rationally split unitary group $U(n,n)$ to a product $U(\Phi)\times U(-\Phi)$ of smaller unitary groups, and integrating against $f\otimes f^\vee$ for a holomorphic cuspform on $U(\Phi)$, where $f^\vee$ is the complex conjugated function, on $U(-\Phi)$. In the (rational) double coset space

$$\text{(Siegel parabolic in } U(n,n)) \backslash U(n,n) \;/\; (U(\Phi)\times U(-\Phi))$$

there just one double coset whose contribution to the integral (against a cuspform) is non-zero, and the corresponding global integral *unwinds* to an integral which can be viewed as a product of local integral operators. We arrive at the following situation.

From a global situation in which a unitary group is defined via a totally real number field E_o and a CM extension E, we fix throughout this discussion a real imbedding of E_o and a complex imbedding of E, and identify these fields with their images. Then G becomes a unitary group which over E conjugate (by the inertia theorem for hermitian forms) over E to the unitary group of the hermitian form

$$H=\begin{pmatrix} -1_p & 0\\ 0 & 1_q\end{pmatrix}$$

with $p\geq q>0$. We use this copy of the group, with the side effect that all rationality claims can be made only over E, not $\mathbb{Q}$, for example. Make the usual choice

$$K=\{\begin{pmatrix} * & 0\\ 0 & *\end{pmatrix}\in G\}\approx U(p)\times U(q)$$

of maximal compact subgroup of G. Let $\mathfrak{g}$ be the real Lie algebra of G (literally, over E_o, etc.), with complexification $\mathfrak{g}_{\mathbb{C}}$, and as usual

$$\mathfrak{p}_+=\{\begin{pmatrix} 0 & *\\ 0 & 0\end{pmatrix}\in\mathfrak{g}_{\mathbb{C}}\}\qquad \mathfrak{p}_-=\{\begin{pmatrix} 0 & 0\\ * & 0\end{pmatrix}\in\mathfrak{g}_{\mathbb{C}}\}$$

[1] E. Lapid, S. Rallis, *On the local factors of representations of classical groups*, in *Automorphic representations, L-functions and applications: progress and prospects*, de Gruyter, 2005.

[2] M. Harris, *A simple proof of rationality of Siegel-Weil Eisenstein series*, this volume.

The Harish-Chandra decomposition is

$$G \subset N_+ \cdot K_{\mathbb{C}} \cdot N_- \subset G_{\mathbb{C}}$$

with $N_\pm = \exp \mathfrak{p}_\pm$. With minimal parabolic

$$B_K = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \text{ and } B \text{ lower-triangular}\}$$

in $K_{\mathbb{C}}$, the group

$$B = B_K \cdot N_- = \{\text{lower triangular } g \in G_{\mathbb{C}} \approx GL(p+q, \mathbb{C})\}$$

is a minimal parabolic in $G_{\mathbb{C}}$. Further, there is an open subset Ω of N_+ such that

$$G \cdot B = G \cdot K_{\mathbb{C}} \cdot N_- = \Omega \cdot K_{\mathbb{C}} \cdot N_-$$

The irreducible τ of K extends to a holomorphic irreducible of $K_{\mathbb{C}}$, and this extension has extreme weight $\xi = \xi_\tau$, a one-dimensional representation of the minimal parabolic B_K in $B_{\mathbb{C}}$. Extend ξ to a one-dimensional representation of B by making it trivial on N_-. Harish-Chandra described the dual $\pi_\tau^\vee$ of the holomorphic discrete series π_τ with extreme K-type τ as a collection of functions φ on G which extend holomorphically to an open subset of the complexification of G:

$$\pi_\tau^\vee = \{\text{holomorphic } \varphi \text{ on } GB : \varphi \in L^2(G),\ \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x) \text{ for } x \in GB,\ b \in B\}$$

with the *left* regular representation

$$L_h\varphi(g) = \varphi(h^{-1}g)$$

Since π_τ is in the discrete series, there is a single copy of $\pi_\tau \otimes \pi_\tau^\vee$ inside $L^2(G)$, with the biregular representation of $G \times G$ on $L^2(G)$ (where the first factor of G acts by the right regular representation and the second factor acts by the left regular representation). In particular, the copy of $\tau \otimes \pi_\tau^\vee$ in $L^2(G)$ (where τ is the extreme K-type in π_τ) is

$$\tau \otimes \pi_\tau^\vee = \{\text{holomorphic } \varphi \text{ on } GB : \varphi \in L^2(G),\ \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x) \text{ for } x \in GB, b \in B\}$$

since the right N_--invariance certainly implies right annihilation by $\mathfrak{p}_-$. Then the copy $\tau \otimes \tau^\vee$ of the tensor product of the extreme K-types in $\pi_\tau \otimes \pi_\tau^\vee$ consists of functions which, further, are left annihilated by $\mathfrak{p}_+$, namely

$$\tau \otimes \tau^\vee = \left\{\text{holomorphic } \varphi \text{ on } G \cdot B : \begin{matrix} \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x) \text{ for } x \in GB, b \in B, \\ \varphi \text{ right annihilated by } \mathfrak{p}_- \text{ and left annihilated by } \mathfrak{p}_+ \end{matrix}\right\}$$

Since these functions φ are holomorphic on the non-empty open subset GB of $G_{\mathbb{C}}$, the left annihilation by $\mathfrak{p}_+$ implies that φ extends to a left N_+ invariant and right N_- invariant holomorphic function on

$$N_+ \cdot K_{\mathbb{C}} \cdot N_-$$

Thus,

$$\tau \otimes \tau^\vee = \left\{\text{holomorphic } \varphi \text{ on } N_+ \cdot K_{\mathbb{C}} \cdot N_- : \begin{matrix} \varphi(xb) = \xi(b)^{-1} \cdot \varphi(x) \text{ for } x \in GB, b \in B, \\ \varphi \text{ left } N_- \text{ invariant} \end{matrix}\right\}$$

Thus, any function φ in this copy of $\tau \otimes \tau^\vee$ extends to $N_- K_{\mathbb{C}} N_+$, is completely determined by its values on $K_{\mathbb{C}}$, and on $K_{\mathbb{C}}$ the function φ is holomorphic and lies in the (holomorphic extension of the) unique copy of $\tau \otimes \tau^\vee$ inside $L^2(K)$. The latter consists of (holomorphic extensions of) coefficient functions $c_{u,v}$ for $u, v \in \tau$, where as usual the matrix coefficient function is

$$c_{u,v}(\theta) = \langle \tau(\theta)u, v\rangle$$

where $\langle , \rangle$ is a fixed K-invariant hermitian inner product on τ. We can assume that for g in the complexification $K_{\mathbb{C}}$

$$\tau(g^*) = \tau(g)^*$$

where $g \longrightarrow g^*$ is the involution on $K_{\mathbb{C}}$ which fixes the real points K of $K_{\mathbb{C}}$.

The integral of interest is of the form

$$Tf(g) = \int_G f(h)\,\overline{\eta(g^{-1}h)}\,dh$$

with f and η as follows. It is important to understand that this integral may not be absolutely convergent, but is defined by analytic continuation, which is known to exist, by the result of Lapid and Rallis cited above. We use an analytic continuation (below) convenient for appraisal of rationality properties.

Anticipating insertion of a convergence factor below, rewrite the integral (replacing h by gh) as

$$Tf(g) = \int_G f(gh)\,\overline{\eta(h)}\,dh$$

For us, f is a *bounded*[3] function on G, is right-annihilated by $\mathfrak{p}_-$, has right K-type τ, and under the right regular representation generates (a *copy* of) the holomorphic discrete series representation π_τ of G with extreme K-type τ. [4]

The function η is in $L^2(G)$, is annihilated on the left by $\mathfrak{p}_+$, is annihilated on the right by $\mathfrak{p}_-$, and has right K-type τ. For lowest K-type τ of sufficiently high extreme weight, the universal $(\mathfrak{g}, K)$-module generated by a vector v_τ of K-type τ and annihilated by $\mathfrak{p}_-$ is *irreducible.* [5] Thus, η generates the holomorphic discrete series $\tau \otimes \tau^\vee \subset L^2(G)$ and is a finite sum of functions made from holomorphic extensions of coefficient function of τ, namely functions of the form

$$\eta_{x,y}(n_+\theta n_-) = c_{x,y}(\theta) = \langle \tau(\theta)x, y\rangle$$

with $x, y \in \tau$.

The way that η arises in practice allows some control over its choice inside the extreme K-type $\tau\otimes\tau^\vee$ inside $\pi\otimes\pi^\vee$. Since $\tau\otimes\tau^\vee$ occurs just once inside $\pi\otimes\pi^\vee$, the space of K-conjugation invariant functions in $\tau\otimes\tau^\vee$ is one-dimensional. Take K-conjugation invariant η defined by

$$\eta(n_+\theta n_-) = \dim\tau\cdot\operatorname{tr}\tau(k) = \dim\tau\cdot\sum_i \langle\tau(\theta)x_i, x_i\rangle \in \tau\otimes\tau^\vee \subset \pi\otimes\pi^\vee \subset L^2(G)$$

where $\{x_i\}$ is an orthonormal basis for τ. Note that the expression for η does not depend upon the choice of orthonormal basis, but does present the holomorphic extension to $K_{\mathbb{C}}$ of the character $\operatorname{tr}\tau$ as a function in $\tau\otimes\tau^\vee \subset \pi\otimes\pi^\vee$. Recall that

$$\int_K \operatorname{tr}\tau(k)\cdot k\cdot v\,dk = \frac{v}{\dim\tau}$$

[3] This boundedness is a crude but sufficient description of the asymptotic behavior of f. In fact, f comes from a holomorphic cuspform, so may have somewhat better decay properties than mere boundedness.

[4] Thus, while f generates a unitary representation, this neither implies nor assumes that f is in $L^2(G)$. Indeed, in the case of interest, f is definitely *not* square-integrable.

[5] This has been known at least since H. Rossi, M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions, and the application to the holomorphic discrete series of a semisimple Lie group*, J. of Func. An. **13** (1973), pp. 324-389, and *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1976), pp. 1-59. Also, a direct proof of this irreducibility follows easily from consideration of eigenvalues of the Casimir operator on generalized Verma modules.

for a vector v in any copy of τ, from Schur orthogonality relations.

Normalize a measure on G as follows. Use a Cartan decomposition

$$G = C \cdot K \approx C \times K$$

where

$$C = \{g \in G = U(p,q) : g = g^* \text{ is positive-definite hermitian}\}$$

Give K a Haar measure with total mass 1. Parametrize C by

$$D_{p,q} \ni z \longrightarrow h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2}z^* & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

where $D_{p,q}$ is the classical domain

$$D_{p,q} = \{z = p\text{-by-}q \text{ complex matrix} : 1_p - zz^* \text{ is positive definite }\}$$

The unitary group $U(p,q)$ acts on $D_{p,q}$ by generalized linear fractional transformations, and has invariant measure on $D_{p,q}$ which we normalize to

$$d^*z = \frac{dz}{\det(1_q - z^*z)^{p+q}} = \frac{dz}{\det(1_p - zz^*)^{p+q}}$$

where dz is the product of usual additive Haar measures.

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2. *Qualitative theorem*

In the situation described above,

Theorem: With f of right K-type τ in a holomorphic discrete series representation $\pi = \pi_\tau$ with lowest K-type τ, and with η as above,

$$(Tf)(g) = \int_G f(gh)\,\overline{\eta(h)}\,dh = \pi^{pq} \cdot (E\text{-rational number}) \cdot f(g)$$

Proof: For computational purposes, we would *like* f to be in $L^2(G)$, allowing use of the Harish-Chandra decomposition as for η. However, f unlikely to be in $L^2(G)$, despite the fact that it generates a (unitary) representation isomorphic to a discrete series representation. Still, *if* $\eta \in L^1(G)$, then the map $f \longrightarrow Tf$ would be a continuous endomorphism of the representation space generated by f, justifying computation of the integral in any preferred model for π_τ. Despite the fact that, for lowest K-type τ *too low*, η is not in $L^1(G)$, η can be modified by a *convergence factor* parametrized by $s \in \mathbb{C}$ (normalized to be trivial at $s = 0$). This convergence factor is *bounded* for $\mathrm{Re}(s) \geq 0$, is 1 at $s = 0$, and makes the modified η integrable for $\mathrm{Re}(s)$ sufficiently large. The resulting integral has an analytic continuation back to the point $s = 0$, and is evaluated there. [6] The convergence factor is left and right K-invariant, so the K-conjugation invariance of the original η is not disturbed.

A Cartan decomposition $h = h_z k$ and Harish-Chandra decomposition $h_z = n_z^+ \theta_z n_z^-$ are

$$h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2}z^* & (1_q - z^*z)^{-1/2} \end{pmatrix} = \begin{pmatrix} 1_p & z \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1_q & 0 \\ z^* & 1_q \end{pmatrix}$$

[6] The modified integral with complex parameter s is *not* the same as the integral against a section of degenerate principal series on $U(n,n)$ parametrized as usual by s. That is, we do *not* compute the archimedean zeta integral as a function of the *usual* s. Rather, we compute the *value* of the zeta integral at a special point, via analytic continuation.

Thus,

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

Thus, replace $\overline{\eta}$ by

$$\xi(g) = \xi_s(g) = \overline{\eta(g)} \cdot \det(1_p - zz^*)^s \quad \text{(for Cartan decomposition } g = h_z \cdot k)$$

where $s \in \mathbb{C}$, $\mathrm{Re}(s) \geq 0$. Thus, $\xi_{s=0} = \overline{\eta}$. For $\mathrm{Re}(s)$ sufficiently large (positive), $\xi \in L^1(G)$. (This is clear just below.) The integral with the convergence factor inserted becomes

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh$$

For s such that $\xi = \xi_s$ is in $L^1(G)$, the integral for $T_s f$ lies in the representation space V generated by f, so for $\mathrm{Re}(s)$ sufficiently large the function $s \longrightarrow T_s f$ is a holomorphic V-valued function of s. Since η was K-conjugation invariant, and since the convergence factor is left and right K-invariant, ξ is K-conjugation invariant for all s. Thus, at least for $\mathrm{Re}(s)$ large, the map $f \longrightarrow T_s f$ maps V to V, commutes with K, and stabilizes K-isotypes in V. The representation V contains only a single copy of the extreme K-type τ, so $f \longrightarrow T_s f$ is a *scalar* map on the copy of τ inside π. Thus, there is a constant Ω_s depending only upon τ (and depending holomorphically upon s) such that

$$T_s f = \Omega_s \cdot f$$

For $\mathrm{Re}(s)$ sufficiently positive so that we can use a different model, we do change the model of the representation to compute Ω_s.

There is *pointwise* (in g) equality

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh$$

of holomorphic $\mathbb{C}$-valued functions, so the same *pointwise* identity

$$T_s f(g) = \int_G f(gh)\, \xi_s(h)\, dh = \Omega_s \cdot f(g)$$

holds for the analytic continuation. Since the far right-hand side is simply a scalar multiple of f, this ruse gives the desired result even for η not in $L^1(G)$.

To compute the constant Ω_s, it suffices to compute a single pointwise value

$$T_s f(1) = \int_G f(h)\, \xi_s(h)\, dh = \Omega_s \cdot f(1)$$

For $\mathrm{Re}(s)$ sufficiently large such that $\xi_s \in L^1(G)$, we are entitled to take a different model. Take f to be in

$$\tau \otimes \tau^\vee \subset \pi \otimes \pi^\vee \subset L^2(G)$$

Then use the Harish-Chandra decomposition and express f as a finite sum of functions of the special form

$$f(h) = f(h_z \cdot k) = f_{u,v}(h_z \cdot k) = f_{u,v}(n_+ \theta_z n_- k) = f_{u,v}(n_+\, \theta_z k\, k^{-1} n_z^- k) = f_{u,v}(\theta_z k)$$

$$= \langle \tau(\theta_z k) u, v \rangle = \langle \tau(k) u, \tau(\theta_z) v \rangle$$

where $h = h_z \cdot k$ is the Cartan decomposition, $h_z = n_+ \cdot \theta_z \cdot n_-$ is the Harish-Chandra decomposition, $u, v \in \tau$, and

$$\tau(\theta_z)^* = \tau(\theta_z^*) = \tau(\theta_z)$$

(This uses the holomorphic extension of τ to $K_{\mathbb{C}}$.) Then

$$\Omega_s \cdot f(1) = T_s f(1) = \int_G f(h)\, \xi(h)\, dh = \int_C \int_K f_{u,v}(h_z k)\, \xi(h_z k)\, dk\, d^* z$$

Recall that ξ is

$$\xi(h_z k) = \overline{\eta}(n_z^+ \theta_z n_z^- k) \cdot \det(1_p - zz^*)^s = \dim\tau \cdot \overline{\operatorname{tr}\tau(\theta_z k)} \cdot \det(1_p - zz^*)^s$$

with the special choice of η. Thus, the integral is

$$Tf(1) = \dim\tau \int_C \int_K \langle \tau(k)u,\, \tau(\theta_z)v\rangle\, \overline{\operatorname{tr}\tau(\theta_z k)}\, \det(1_p - zz^*)^s\, dk\, d^* z$$

Let $\{x_i\}$ be an orthonormal basis for τ and take the special choice of η as earlier, namely

$$\eta(n_+ \theta n_-) = \dim\tau \cdot \operatorname{tr}\tau(\theta) = \dim\tau \cdot \sum_i \langle \tau(\theta)x_i, x_i\rangle$$

Then, unsurprisingly, part of the integral is computed via Schur's orthogonality relations, namely, [7]

$$\sum_i \int_K \langle \tau(k)u,\, \tau(\theta_z)v\rangle\, \overline{\langle \tau(\theta_z k)x_i, x_i\rangle}\, dk = \sum_i \int_K \langle \tau(k)u,\, \tau(\theta_z)v\rangle\, \overline{\langle \tau(k)x_i, \tau(\theta_z)x_i\rangle}\, dk$$

$$= \sum_i \frac{\langle u, x_i\rangle}{\dim\tau} \langle \tau(\theta_z)x_i, \tau(\theta_z)v\rangle = \sum_i \frac{\langle u, x_i\rangle}{\dim\tau} \langle x_i, \tau(\theta_z^2)v\rangle = \frac{\langle u, \tau(\theta_z^2)v\rangle}{\dim\tau}$$

where K has total measure 1 and use $\tau(\theta_z)^* = \tau(\theta_z^*) = \tau(\theta_x)$. Thus, the factor $\dim\tau$ cancels, as planned, and

$$\Omega_s \cdot f(1) = T_s f(1) = \int_C \langle u, \tau(\theta_z^2)v\rangle\, (1_p - zz^*)^s\, d^* z = \langle u,\, \left(\int_C \tau(\theta_z^2)\, \det(1_p - zz^*)^s\, d^* z\right)\, v\rangle = \langle u,\, Sv\rangle$$

where $S = S_s$ is the endomorphism (anticipated to be a scalar)

$$S = S_s = S_s(\tau) = \int_C \tau(\theta_z^2)\, \det(1_p - zz^*)^s\, d^* z$$

where, as above,

$$\theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^* z)^{-1} \end{pmatrix}$$

The irreducible τ of $K \approx U(p) \times U(q)$ necessarily factors as a tensor product

$$\tau \approx \tau_1 \otimes \tau_2$$

with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$. Thus, the endomorphism S of the finite-dimensional vector space τ is

$$S = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^* z)\, \det(1_p - zz^*)^s\, d^* z \in \operatorname{End}_{\mathbb{C}}(\tau)$$

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[7] Since we use the holomorphic extension of τ to $K_{\mathbb{C}}$, there is a minor hazard of a needless false assertion about the application of Schur orthogonality. The explicit argument makes clear that this invocation is legitimate.

We claim that the endomorphism S of the finite-dimensional complex vector space τ is *scalar*. Indeed, the map $z \longrightarrow \alpha z \beta^*$ with $\alpha \in U(p)$, $\beta \in U(q)$ is an automorphism of $D_{p,q}$ that leaves the invariant measure unchanged. Applying this change of variables inside the integral defining S

$$S = \int_{D_{p,q}} \tau_1(\alpha(1_p - zz^*)\alpha^*) \otimes \tau_2^{-1}(\beta(1_q - z^*z)\beta^*) \, \det(1_p - zz^*)^s \, d^*z = [\tau_1(\alpha) \otimes \tau_2(\beta)] \cdot S \cdot [\tau_1(\alpha) \otimes \tau_2(\beta)]^{-1}$$

Thus, S commutes with $\tau(k)$ for $k \in K$. By Schur's lemma, since τ is irreducible as a representation of K (apart from its behavior on $K_{\mathbb{C}}$), the endomorphism S of τ is *scalar*, as claimed.

Thus, with scalar S and $f = f_{u,v}$,

$$\Omega_s \cdot f(1) = S \cdot \langle u, v \rangle = S \cdot f(1)$$

Thus, in fact, as an operator on τ,

$$\Omega_s = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z) \, \det(1_p - zz^*)^s \, d^*z$$

Let $z = \alpha r \beta$ with $\alpha \in U(p)$, $\beta \in U(q)$, and where r is diagonal (even if rectangular)

$$r = \text{ p-by-q } = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_q \\ 0 & \dots & 0 \end{pmatrix}$$

with diagonal entries $-1 < r_i < 1$. Let

$$\Delta(r) = \prod_{1 \le i < j \le q} \left(r_i^2 - r_j^2\right)^2$$

Then, with a constant C determined subsequently, the integral of a function φ on the domain $D_{p,q}$ is [8]

$$\int_{D_{p,q}} \varphi(z) \, \frac{dz}{\det(1_q - z^*z)^{p+q}} = C \cdot \int_{U(p)\times U(q)} \int_{(-1,1)^q} \varphi(\alpha r \beta) \, d\alpha \, d\beta \, \frac{\Delta(r) \, dr}{\det(1_q - r^*r)^{p+q}}$$

where $U(p)$ and $U(q)$ have total measure 1 and dr is the product of usual (additive) Haar measures on intervals $(-1, +1)$. Thus,

$$S = C \cdot \int_{U(p)\times U(q)} [\tau_1(\alpha) \otimes \tau_2(\beta)] \circ I \circ [\tau_1(\alpha) \otimes \tau_2(\beta)]^{-1} \, d\alpha \, d\beta$$

where the inner operator is

$$I = \int_{(-1,1)^q} \tau_1(1_p - rr^*) \otimes \tau_2^{-1}(1_q - r^*r) \frac{\det(1_q - r^*r)^s \, \Delta(r) \, dr}{\det(1_q - r^*r)^{p+q-s}}$$

The outer integration is exactly the projection

$$\mathrm{End}_{\mathbb{C}}(\tau) \longrightarrow \mathrm{End}_{U(p)\times U(q)}(\tau)$$

[8] This formula can be construed as a variant of Weyl's integration formula. It can be derived directly by considering the exponential map to the symmetric space.

where $\mathrm{End}_{\mathbb{C}}(\tau)$ has the natural $K = U(p) \times U(q)$ structure

$$k \cdot \varphi = \tau(k) \circ \varphi \circ \tau(k)^{-1}$$

for $k \in K$ and $\varphi \in \mathrm{End}_{\mathbb{C}}(\tau)$.

Give $\mathrm{End}_{\mathbb{C}}(\tau)$ a E-rational structure compatible with whatever E-rational structure

$$\mathfrak{k}_E = \mathfrak{gl}(p) \otimes \mathfrak{gl}(q)$$

we have on the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}(p, \mathbb{C}) \otimes \mathfrak{gl}(q, \mathbb{C})$ of $K = U(p) \times U(q)$ under the action of $\mathfrak{k}_E$. We will directly compute that the inner integral for S is in

$$\left(\frac{2^s\,\Gamma(s)^2}{\Gamma(2s)}\right)^q \cdot E(s) \qquad\qquad (E(s) = \text{rational functions in } s, \text{ coefficients in } E\ )$$

That is, up to the indicated leading factor, the inner integral is a rational function of s with coefficients in E. Then we show that the projection P to K-invariants is an element in the center $Z(\mathfrak{k}_E)$ of the enveloping algebra $U = U(\mathfrak{k}_E)$ of $\mathfrak{k}_E$. Thus, apart from the normalizing constant C, up to the leading factor the whole integral for S is again in $E(s)$. Then we determine the constant C by a different computation of S with a special choice of $\tau = \tau_1 \otimes \tau_2$.

Any finite-dimensional representation of $U(n)$ gives a representation of the complexified Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of $U(n)$ with a highest weight. In the context in which these integrals arise, one often does not have a $\mathbb{Q}$-rational structure, but only over a CM-field or totally real subfield of such. Consider the *E-rational* Lie algebra $\mathfrak{gl}(n, E)$. Following Dixmier, [9] construct the universal enveloping algebra $U(\mathfrak{gl}(n, E))$ over E, and the Poincaré-Birkhoff-Witt theorem holds. Let $\mathfrak{n}$ be the strictly upper-triangular subalgebra, and $\mathfrak{a}$ the diagonal subalgebra. For a E-rationally-valued character $\lambda : \mathfrak{a} \longrightarrow E$ of $\mathfrak{a}$ (following Dixmier) we have the Verma module over E

$$M_\lambda = U/\left(U \cdot \mathfrak{n} \,\oplus\, \sum_\alpha U \cdot (\alpha - (\lambda - \rho)\alpha)\right)$$

as the quotient of U by the left ideal generated by $\mathfrak{n}$ and the differences $\alpha - (\lambda - \rho)\alpha$ for $\alpha \in \mathfrak{a}$, with ρ the usual half-sum of positive roots (*positive* roots being the roots of $\mathfrak{a}$ in $\mathfrak{n}$). As over $\mathbb{C}$, there is a unique maximal proper submodule N_λ, namely, the sum of all submodules not containing the $\lambda - \rho$ weight space. As over $\mathbb{C}$, the E-irreducible quotient M_λ/N_λ is finite-dimensional for λ *integral* and *dominant*. Further, this construction commutes with extension of scalars to $\mathbb{C}$, so $(M_\lambda/N_\lambda) \otimes \mathbb{C}$ is still irreducible.

Since the highest weights $\lambda - \rho$ for finite-dimensional irreducibles are *integral* (and dominant), they certainly take E-rational values on the E-rational diagonal subalgebra $\mathfrak{a}_E$. Therefore, any finite-dimensional irreducible complex representation τ of $U(n)$ has an E-rational structure

$$\tau = (M_\lambda/N_\lambda) \otimes_E \mathbb{C}$$

for some λ, with *E-rational* Verma module M_λ as above. Let τ_E be such an E-rational form of τ, and $\tau_E^\vee$ a E-rational form of its dual. An irreducible τ of $U(p) \times U(q)$ factors as $\tau \approx \tau_1 \otimes \tau_2$ with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$, so we can choose E-rational structures $\tau_{i,E}$ on τ_i and put

$$\tau_E = \tau_{1,E} \otimes_E \tau_{2,E}$$

[9] J. Dixmier, *Enveloping Algebras*, (English translation 1995, A.M.S.) North-Holland, 1977, shows that essentially all the usual algebraic constructions depend only upon the underlying field being of characteristic 0. Specifically, behavior of the enveloping algebra, Verma modules, and expression of finite-dimensional irreducibles as quotients of Verma modules proceeds over arbitrary fields of characteristic 0, with the obvious trivial modifications, in the fashion usually carried out over $\mathbb{C}$.

We have the usual natural isomorphism

$$\mathrm{End}_E(\tau_E) \approx \tau_E \otimes \tau_E^\vee$$

Let P be the projection to the trivial $K = U(p) \times U(q)$ subrepresentation inside $\mathrm{End}_E(\tau_E) \otimes_E \mathbb{C}$. (The *absolute* irreducibility and Schur's lemma together assure that this subspace is exactly one-dimensional, and consists of scalar operators.) We want to show that

$$P\left(\mathrm{End}_E(\tau_E)\right) \subset E \cdot \mathrm{id}_\tau \subset \mathrm{End}_\mathbb{C}(\tau_\mathbb{C})$$

that is, that E-rational $\mathbb{C}$-endomorphisms project to E-rational (scalar) K-endomorphisms.

Following Dixmier, the Harish-Chandra isomorphism and its proof hold for the E-rational Lie algebra $\mathfrak{k}_E = \mathfrak{gl}(p, E) \otimes \mathfrak{gl}(q, E)$. In particular, the center Z_E of $U(\mathfrak{k}_E)$ distinguishes finite-dimensional irreducible $\mathfrak{k}_E$-modules with distinct E-rational highest weights. That is, given finite-dimensional irreducibles V and V' with E-rational highest weights $\lambda = \lambda'$, there is $z \in Z_E$ such that $z(\lambda) \neq z(\lambda')$. In particular, given $\lambda \neq 0$, there is z_λ in $U(\mathfrak{k}_E)$ such that $z_\lambda(\lambda) = 0$ and $z_\lambda(0) = 1$. Then, with Λ the *finite* collection of λ's indexing the irreducibles in $\mathrm{End}_\mathbb{C}(\tau) = \mathrm{End}_E(\tau_E) \otimes_E \mathbb{C}$,

$$P = \prod_{\lambda \in \Lambda} z_\lambda$$

projects endomorphisms to the K-invariant subspace. Thus, projection to K-endomorphisms preserves E-rationality.

The diagonal subgroups in $GL(p, \mathbb{C})$ and $GL(q, \mathbb{C})$ act on the weight spaces in τ. The inner integral in the description of S preserves each weight space and acts on it by a scalar. Noting the identity

$$(t^2 - u^2) = (t^2 - 1) - (u^2 - 1)$$

one can see that each such scalar is expressible as a sum with E-rational coefficients of products of integrals of the form (with $n \in \mathbb{Z}$)

$$\int_{-1}^1 (1-t^2)^n \, \frac{(1-t^2)^s \, dt}{(1-t^2)^{p+q}} = \int_0^1 [4t(1-t)]^n \, \frac{2\,[4t(1-t)]^s \, dt}{[4t(1-t)]^{p+q}}$$

$$= 2^{2n+1-p-q+s} \int_0^1 t^{n-p-q} \, (1-t)^{n-p-q+s} \, dt$$

$$= 2^{2n+1-p-q+s} \, \frac{\Gamma(n-p-q+1+s)\,\Gamma(n-p-q+1+s)}{\Gamma(2n-2p-2q+2+2s)} \in \frac{2^s \Gamma(s)^2}{\Gamma(2s)} \cdot E(s)$$

(replacing t by $2t-1$ in the first expression). Thus, the inner integral acts by scalars on all weight spaces, and these scalars (up to the leading factor) are in $E(s)$. Thus, when $s = 0$, this endomorphism is in $\mathrm{End}_E(\tau_E)$. We have shown that projection to $U(p) \times U(q)$-endomorphisms preserves E-rationality, so

$$S_{s=0} = C \cdot (E\text{-rational scalar endomorphism of } \tau)$$

To determine the normalization constant C, it suffices to compute the endomorphism

$$S = S(\tau) = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z) \, d^*z$$

for any particular choice of $\tau = \tau_1 \otimes \tau_2$. The following is the simplest version of the fuller qualitative computation done in the following section.

Let

$$\tau_1(x) = (\det x)^m \quad (\text{on } GL(p,\mathbb{C})) \qquad \text{and} \qquad \tau_2(y) = (\det y)^{-n} \quad (\text{on } GL(q,\mathbb{C}))$$

with m, n sufficient large to assure that we are in the L^1 range. For positive integer ℓ, define the cone

$$C_\ell = \{\text{positive-definite hermitian } \ell\text{-by-}\ell \text{ complex matrices}\}$$

and associated gamma function

$$\Gamma_\ell(s) = \int_{C_\ell} e^{-\operatorname{tr} x}\, (\det x)^s \frac{dx}{(\det x)^\ell} \qquad\qquad (\text{for } \operatorname{Re}(s) > \ell - 1)$$

where dx denotes the product of the usual measures on $\mathbb{R}$ for the diagonal components of x, and the usual measure on $\mathbb{C} \approx \mathbb{R}^2$ for the off-diagonal components. It is a classical fact [10] and is straightforward to determine directly that

$$\Gamma_\ell(s) = \pi^{\ell(\ell-1)/2} \prod_{i=1}^{\ell} \Gamma(s-i+1)$$

Let $x^{1/2}$, $y^{1/2}$ be the unique positive-definite hermitian square roots of $x \in C_p$ and $y \in C_q$. Then (with arguments to Γ_p and Γ_q that are intelligible only with some hindsight)

$$\det(x)^{m+n}\, \det(y)^{m+n} \cdot S$$

$$= (\det x)^n\, (\det y)^m \left[\det(x^{1/2})^m\, \det(y^{1/2})^n\right] \cdot S \cdot \left[\det(x^{1/2})^m \cdot \det(y^{1/2})^n\right]$$

Multiply both sides by $e^{-\operatorname{tr} x - \operatorname{tr} y}$ and integrate over $C_p \times C_q$ against the invariant measures $(\det x)^{-p} dx$ and $(\det y)^{-q} dy$ to obtain

$$\Gamma_p(m+n)\, \Gamma_q(m+n) \cdot S$$

$$= \int_{C_p} \int_{C_q} \int_{D_{p,q}} e^{-\operatorname{tr} x - \operatorname{tr} y}\, \det(x - x^{1/2} z z^* x^{1/2})^n\, \det(y - y^{1/2} z^* z y^{1/2})^m \frac{(\det x)^{n-p}\, (\det y)^{m-q}\, dx\, dy\, dz}{\det(1_p - zz^*)^{p+q}}$$

Replacing $z \in D_{p,q}$ by $x^{-1/2} z y^{-1/2}$ converts the integral over $C_p \times C_q \times D_{p,q}$ into an integral over

$$Z = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \in C_{p+q}$$

Replacing z by $x^{-1/2} z y^{-1/2}$ replaces the additive Haar measure measure dz by

$$(\det x)^{-q}\, (\det y)^{-p}\, dz$$

The normalizing factor $\det(1_p - zz^*)^{-(p+q)}$ associated with dz can be rewritten as

$$\det(1_p - zz^*)^{-(p+q)} = \det(1_p - zz^*)^{-q}\, \det(1_q - z^* z)^{-p}$$

and then under the replacement of z by $x^{-1/2} z y^{-1/2}$ becomes

$$\det(1_p - x^{-1/2} z y^{-1} z^* x^{-1/2})^{-q}\, \det(1_q - y^{-1/2} z^* x^{-1} z y^{-1/2})^{-p}$$

[10] Gamma functions attached to cones appeared in C.L. Siegel *Über die analytische Theorie der quadratischen Formen*, Ann. Math. **36** (1935), pp. 527-606, C.L. Siegel *Über die Zetafunktionene indefinier quadratischer Former*, Math. Z. **43** (1938), pp. 682-708, H. Maaß *Siegel's modular forms and Dirichlet series*, SLN 216, Berlin, 1971. See also the application in G. Shimura *Confluent hypergeometric functions on tube domains*, Math. Ann. **260** (1982), pp. 269-302. In any case, the indicated identity follows readily by an induction, and is elementary.

$$= (\det x)^q \det(x - zy^{-1}z^*)^{-q} \, (\det y)^p \det(y - zx^{-1}z^*)^{-p}$$

Putting this together, the integral becomes

$$\int_{C_{p+q}} e^{-\mathrm{tr}\, Z} \, \det(x - zy^{-1}z^*)^m \, \det(y - z^*x^{-1}z)^n \, \frac{(\det x)^{n-p} \, (\det y)^{m-q} \, dx \, dy \, dz}{\det(x - zy^{-1}z^*)^q \det(y - z^*x^{-1}z)^p}$$

The identity

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} 1 & zy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}$$

shows that

$$\det Z = \det(x - zy^{-1}z^*) \cdot \det y = \det(y - z^*x^{-1}z) \cdot \det x$$

Thus, altogether,

$$\Gamma_p(m+n)\Gamma_q(m+n) \cdot S = \int_{C_{p+q}} e^{-\mathrm{tr}\, Z} \, (\det Z)^{m+n} \, \frac{dZ}{(\det Z)^{p+q}} = \Gamma_{p+q}(m+n)$$

Thus, for this particular choice of τ,

$$S = \frac{\Gamma_{p+q}(m+n)}{\Gamma_p(m+n-p)\,\Gamma_q(m+n-q)}$$

$$= \frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i+1)}{\prod_{i=0}^{p-1} \Gamma(m+n-i+1) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-i+1)} \cdot \frac{\pi^{(p+q)(p+q-1)/2}}{\pi^{p(p-1)/2} \cdot \pi^{q(q-1)/2}}$$

The net exponent of π is

$$(p+q)(p+q-1)/2 - p(p-1)/2 - q(q-1)/2 = pq$$

as anticipated. Thus, *indirectly*, we have shown that

$$C = \pi^{pq} \cdot (E\text{-rational number})$$

where necessarily that E-rational number is independent of m, n, being a normalization of a measure. Then, for *arbitrary* τ, evaluating at $s = 0$,

$$S = \pi^{pq} \cdot (E\text{-rational scalar endomorphism of } \tau)$$

This proves the *qualitative* assertion formulated above. ///

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3. *Quantitative theorem*

The simple case

$$\tau(k_1 \times k_2) = (\det k_1)^m \, (\det k_2)^{-n} \qquad (\text{with } m \geq p,\ n \geq q,\ k_1 \in U(p) \text{ and } k_2 \in U(q))$$

used in the last section to determine a normalization gives an explicit form of the dependence upon the representation τ, namely that (up to a uniform non-zero E-rational number) in that special case

$$Tf = \pi^{pq} \cdot \frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i)}{\prod_{i=0}^{p-1} \Gamma(m+n-p-i) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-q-i)} \cdot f$$

This computation can be extended, as we do in this section, at least to the case that only *one* of the two τ_i is one-dimensional.

As discussed in the context-setting section, any function φ in the copy of $\tau \otimes \tau^\vee$ inside $\pi_\tau \otimes \pi_\tau^\vee$ is completely determined by its values on $K_\mathbb{C}$, and on $K_\mathbb{C}$ the function φ is holomorphic and lies in the (holomorphic extension of the) unique copy of $\tau \otimes \tau^\vee$ inside $L^2(K)$. The latter consists of (holomorphic extensions of) (matrix) coefficient functions

$$c_{u,v}(\theta) = \langle \tau(\theta)u, v\rangle \qquad\qquad (u, v \in \tau)$$

where $\langle,\rangle$ is a fixed K-invariant hermitian inner product on τ. We can assume that for g in the complexification $K_\mathbb{C}$

$$\tau(g^*) = \tau(g)^*$$

where $g \longrightarrow g^*$ is the involution on $K_\mathbb{C}$ which fixes the real points K of $K_\mathbb{C}$. Thus, we may take f of the form (in the $N_+K_\mathbb{C}N_-$ coordinates)

$$f(g) = f_{u,v}(g) = f_{u,v}(n_+\theta n_-) = c_{u,v}(\theta)$$

As recalled above, for extreme K-type τ with sufficiently high extreme weight the universal $(\mathfrak{g}, K)$-module generated by a vector v_τ of K-type τ and annihilated by $\mathfrak{p}_-$ is irreducible. Thus, the annihilation of η by $\mathfrak{p}_-$ and the specification of K-type of η imply that η generates the holomorphic discrete series $\tau \otimes \tau^\vee$, and η is a finite sum

$$\eta_{\mu,\nu}(n_+\theta n_-) = c_{\mu,\nu}(\theta) = \langle \tau(\theta)\mu, \nu\rangle$$

with $\mu, \nu \in \tau$. Use the same normalization of measure on G as earlier.

Let τ_1 have extreme weights $(\kappa_1, \ldots, \kappa_p$, and let

$$\kappa_{p+1} = \ldots = \kappa_{p+q} = \kappa_p$$

Let τ_2 have extreme weights $(\lambda_{p+1}, \ldots, \lambda_{p+q}$ (note the funny indexing!), and let

$$\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda_{p+1}$$

Note also that in practice these λ_i will most often be negative integers. For example the scalar example of the previous section is recovered by taking $\lambda_i = -n$, with n as in in the previous.

A similar computation as the special case of the previous section will yield:

Theorem: For $\tau = \tau_1 \otimes \tau_2$ with *either* τ_1 *or* τ_2 *one-dimensional,*

$$(Tf)(1) = \int_G f_{u,v}(h)\,\overline{\eta_{\mu,\nu}(h)}\,dh = \pi^{pq} \cdot \langle u, \mu\rangle \cdot \langle v, \nu\rangle \cdot \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p+q-i) - \lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_i - (p-i) - \lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) - \lambda_i)}$$

Proof: As in the proof of the previous theorem, use a Cartan decomposition $h = h_z k$ and the Harish-Chandra decomposition $h_z = n_z^+ \theta_z n_z^-$

$$h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2} z^* & (1_q - z^*z)^{-1/2} \end{pmatrix} = \begin{pmatrix} 1_p & z \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1_q & 0 \\ z^* & 1_q \end{pmatrix}$$

so

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

so the integral is

$$(Tf)(1) = \int_G f_{u,v}(h)\,\overline{\eta_{\mu,\nu}(h)}\,dh = \int_C \int_K f_{u,v}(h_z k)\,\overline{\eta_{\mu,\nu}(h_z k)}\,dk\,dz^*$$

The special form of $f_{u,v}$ gives

$$f_{u,v}(h_z k) = f_{u,v}(n_z^+ \theta_z n_z^- k) = f_{u,v}(\theta_z k \cdot k^{-1} n_z^- k) = f_{u,v}(\theta_z k) = \langle \tau(\theta_z k)u, v\rangle$$

since $\theta_z k \in K_{\mathbb{C}}$. Similarly for $\eta_{\mu,\nu}$. As in the previous proof, we insert a convergence factor $\det(1_p - zz^*)^s$, so the integral is

$$\int_C \int_K \langle \tau(\theta_z k)u, v\rangle\, \overline{\langle \tau(\theta_z k)\mu, \nu\rangle}\, \det(1_p - zz^*)^s \, dk\, d^*z$$
$$= \int_C \int_K \langle \tau(k)u, \tau(\theta_z)^* v\rangle\, \overline{\langle \tau(k)\mu, \tau(\theta_z)^* \nu\rangle}\, \det(1_p - zz^*)^s \, dk\, d^*z$$

The Schur inner product relations compute the integral over K, leaving

$$(Tf)(1) = \frac{\langle u, \mu\rangle}{\dim \tau} \cdot \int_C \langle \tau(\theta_z)^* \nu, \tau(\theta_z)^* v\rangle\, \det(1_p - zz^*)^s \, d^*z$$

Rearrange this slightly, to

$$\frac{\langle u, \mu\rangle}{\dim \tau} \cdot \int_C \langle \nu, \tau(\theta_z)\tau(\theta_z)^* v\rangle\, \det(1_p - zz^*)^s \, d^*z$$
$$= \frac{\langle u, \mu\rangle}{\dim \tau} \cdot \int_C \langle \nu, \tau(\theta_z^2)\, v\rangle\, \det(1_p - zz^*)^s \, d^*z = \frac{\langle u, \mu\rangle}{\dim \tau} \cdot \langle \nu, \left(\int_C \tau(\theta_z^2)\, \det(1_p - zz^*)^s \, d^*z \right) v\rangle$$

since $\tau(g^*) = \tau(g)^*$ for g in $K_{\mathbb{C}}$, and since $\theta_z^* = \theta_z$. Thus, we need to compute the endomorphism

$$S = \int_C \tau(\theta_z^2)\, \det(1_p - zz^*)^s \, d^*z$$

As above,

$$\theta_z = \begin{pmatrix} (1_p - zz^*)^{1/2} & 0 \\ 0 & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

so

$$\theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^*z)^{-1} \end{pmatrix}$$

The irreducible τ of $K \approx U(p) \times U(q)$ necessarily factors as an external tensor product

$$\tau \approx \tau_1 \otimes \tau_2$$

with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$. Thus, the endomorphism S of the finite-dimensional vector space τ is

$$S = S_s = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^*z)\, \det(1_p - zz^*)^s \, d^*z \in \mathrm{End}_{\mathbb{C}}(\tau)$$

As noted in the proof of the qualitative result above, the endomorphism S of the finite-dimensional complex vector space τ is *scalar*, since a change of variables in the defining integral shows that it commutes with $\tau(k)$ for all $k \in K$, and we invoke Schur's lemma.

Even though we seem forced eventually to make the restrictive hypothesis that τ_2 (or τ_1) is *scalar*, we set up the general form of the computation. Again, let τ_1 have extreme weight $(\kappa_1, \ldots, \kappa_p)$ and τ_2 have extreme weight $(\lambda_{p+1}, \ldots, \lambda_{p+q})$, where our convention is that the extreme weight vectors $v_1 \in \tau_1$ and $v_2 \in \tau_2$ satisfy

$$\tau_1 \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ * & & m_p \end{pmatrix} v_1 = m_1^{\kappa_1} \ldots m_p^{\kappa_p} \cdot v_1 \qquad \tau_2 \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ * & & m_q \end{pmatrix} v_2 = m_1^{\lambda_{p+1}} \ldots m_q^{\lambda_{p+q}} \cdot v_2$$

We introduce a notion of gamma functions somewhat more general than that earlier (still falling into a family treated long ago by Siegel and others). Again, for positive integer n, let

$$C_n = \{\text{positive-definite hermitian } n\text{-by-}n \text{ complex matrices}\}$$

For an irreducible representation σ of $U(n)$, extend σ to a holomorphic representation of $GL(n,\mathbb{C})$, and for complex s define

$$\Gamma_n(\sigma,s) = \int_{C_n} e^{-\mathrm{tr}\, x}\, \sigma(x)\, (\det x)^s \frac{dx}{(\det x)^n} \in \mathrm{End}_{\mathbb{C}}(\sigma)$$

where dx denotes the product of the usual measures on $\mathbb{R}$ for the diagonal components of x, and the usual measure on $\mathbb{C}$ for the off-diagonal components.

Proposition: The endomorphism-valued function $\Gamma_n(\sigma,s)$ is *scalar*, and can be evaluated in terms of the extreme weight of σ, namely, with extreme weight

$$(\sigma_1,\ldots,\sigma_n) \quad \text{with } \sigma_1 \le \ldots \le \sigma_n$$

we have

$$\Gamma_n(\sigma,s) = \pi^{n(n-1)/2} \prod_{i=1}^{n} \Gamma(\sigma_i - (n-i) + s)$$

Proof: For $\alpha \in U(n)$, replacing x by $\alpha x \alpha^*$ in the integral for $\Gamma_n(\sigma,s)$ yields

$$\Gamma_n(\sigma,s) = \int_{C_n} e^{-\mathrm{tr}\,(\alpha x \alpha^*)}\, \sigma(\alpha x \alpha^*)\, (\det x)^s\, d^*x$$

$$= \sigma(\alpha)\cdot \int_{C_n} e^{-\mathrm{tr}\,(\alpha^* \alpha x)}\, \sigma(x)\, (\det x)^s\, d^*x \cdot \sigma(\alpha)^{-1} = \sigma(\alpha)\cdot \Gamma_n(\sigma,s)\cdot \sigma(\alpha)^{-1}$$

Thus, $\Gamma_n(\sigma,s)$ commutes with $\sigma(\alpha)$ for all $\alpha \in U(n)$. By Schur's lemma, $\Gamma_n(\sigma,s)$ is scalar. Replacing x by $y^{1/2} x y^{1/2}$ and using the fact that $\Gamma_n(\sigma,s)$ is scalar gives the second integral formula.

To compute the scalar, it suffices to compute the effect of the operator on an extreme weight vector for σ. Let P be the group of lower-triangular matrices in $GL(n,\mathbb{C})$ with positive real diagonal entries. This subgroup is still transitive on the cone C_n with the action $g(x) = g^* x g$, so

$$\Gamma_n(\sigma,s) = 2^n \cdot \int_P e^{-\mathrm{tr}\, p^* p}\, \sigma(p^* p)\, (\det p^* p)^s\, dp$$

with suitable right Haar measure dp. Let $\langle,\rangle$ be a K-invariant inner product on σ, which we may suppose satisfies

$$\langle \sigma(g) u, w\rangle = \langle u, \sigma(g^*) w\rangle$$

for all $g \in GL(n,\mathbb{C})$: that is, the Hilbert space adjoint $\sigma(g)^*$ of $\sigma(g)$ is $\sigma(g^*)$, where g^* is conjugate transpose. Then with an extreme weight vector v for σ

$$\langle \Gamma_n(\sigma) v, v\rangle = \int_{C_n} e^{-\mathrm{tr}\, p^* p}\, \langle \sigma(p)^* \sigma(p) v, v\rangle\, dp = \int_{C_n} e^{-\mathrm{tr}\, p^* p}\, \langle \sigma(p) v, \sigma(p) v\rangle\, dp$$

$$= \int_{C_n} e^{-\mathrm{tr}\, p^* p}\, |m_1|^{2\sigma_1} \ldots |m_n|^{2\sigma_n}\, dp \cdot \langle v, v\rangle$$

where

$$p = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ u_{ij} & & 1 \end{pmatrix} \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}$$

The latter integral is

$$\int_{m_i>0}\int_{u_{ij}\in\mathbb{C}} e^{-\left[m_1^2(1+|u_{21}|^2+\ldots+|u_{n1}|^2)+m_2^2(1+|u_{32}|^2+\ldots+|u_{n2}|^2)+\ldots+m_n^2\right]}\, m_1^{2(\sigma_1+s)}\ldots m_n^{2(\sigma_n+s)}\, dp$$

Replacing each u_{ij} by $u_{ij}\sqrt{\pi}/m_j$ changes the measure by $\pi\, m_j^{-2(n-j)}$ (since $u_{ij}\in\mathbb{C}$), and each integral over u_{ij} becomes (with usual Haar measure on $\mathbb{C}\approx\mathbb{R}^2$)

$$\int_{\mathbb{C}} e^{-\pi|u|^2}\, du = 1$$

Thus, the integral is

$$\pi^{n(n-1)/2}\int_{m_1>0}\cdots\int_{m_n>0} e^{-(m_1^2+\ldots+m_n^2)}\, m_1^{2(\sigma_1-(n-1)+s)}\, m_2^{2(\sigma_2-(n-2)+s)}\ldots m_n^{2(\sigma_n+s)}\frac{dm_1}{m_1}\cdots\frac{dm_n}{m_n}$$

$$= \pi^{n(n-1)/2}\cdot 2^{-n}\,\Gamma(\sigma_1-(n-1)+s)\Gamma(\sigma_2-(n-2)+s)\ldots\Gamma(\sigma_{n-1}-1+s)\Gamma(\sigma_n+s)$$

after replacing each m_i by $\sqrt{m_i}$. With respect to the coordinates u_{ij} and m_i above, versus the coordinate $x\in C_n$, the Jacobian determinant of the map $p\longrightarrow p^*p$ is 2^n, so the powers of 2 go away. ///

Similarly,

Proposition: The operator-valued gamma function defined by

$$\Gamma_n(\sigma^*,s)=\Gamma_n(\sigma^{*,-1},s)=\int_{C_n} e^{-\mathrm{tr}\,x}\,\sigma^{-1}(x)\,(\det x)^s\frac{dx}{(\det x)^n}$$

where σ has extreme weights σ_i, is *scalar*, given by

$$\Gamma_n(\sigma^*,s)=\pi^{n(n-1)/2}\prod_{i=1}^n\Gamma(-\sigma_i-(n-i)+s)$$

Proof: This is a trivial variation on the previous proof. ///

Now evaluate S. Since S is scalar, using the unique positive-definite hermitian square roots of $x\in C_p$ and $y\in C_q$,

$$\tau_1(x)\otimes\tau_2^{-1}(y)\cdot S=\left[\tau_1(x^{1/2})\otimes\tau_2^{-1}(y^{1/2})\right]\cdot\left[\tau_1(x^{1/2})\otimes\tau_2^{-1}(y^{1/2})\right]\cdot S$$

$$=\left[\tau_1(x^{1/2})\otimes\tau_2^{-1}(y^{1/2})\right]\cdot S\cdot\left[\tau_1(x^{1/2})\otimes\tau_2^{-1}(y^{1/2})\right]$$

Multiply both sides by

$$e^{-\mathrm{tr}\,x-\mathrm{tr}\,y}\,(\det x)^{s_1+s}\,(\det y)^{s_2+s}$$

with s_1 and s_2 to be determined later, and integrate over $C_p\times C_q$ against the invariant measure to obtain (keeping in mind that $\tau_2(h^*)=\tau_2(h)^*$)

$$\Gamma_p(\tau_1,s_1+s)\otimes\Gamma_q(\tau_2^{*,-1},s_2+s)\cdot S=$$

$$\int_{C_p\times C_q\times D_{p,q}} e^{-\mathrm{tr}\,x-\mathrm{tr}\,y}\tau_1(x-x^{1/2}zz^*x^{1/2})\otimes\tau_2^{-1}(y-y^{1/2}z^*zy^{1/2})\frac{dz}{\det(1_p-zz^*)^{p+q-s}}\frac{\det x^{s_1+s}dx}{\det x^p}\frac{\det y^{s_2+s}dy}{\det y^q}$$

Observe that we must take the transpose-conjugate of τ_2^{-1} in order to have a (anti-holomorphic) representation in the gamma function, though in fact we have

$$\tau_2^*(y)=\tau_2(y^*)=\tau_2(y)\qquad\text{(for } y=y^*)$$

Replacing $z \in D_{p,q}$ by $x^{-1/2} z y^{-1/2}$ converts the integral over $C_p \times C_q \times D_{p,q}$ into an integral over

$$Z = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \in C_{p+q}$$

A change of measure by $(\det x)^{-q} (\det y)^{-p}$ comes out. (The exponents are *not* divided by 2, despite the square roots, since the z variable is *complex*, and each complex coordinate has two real coordinates.) In fact, we want to break the $\det(1_p - zz^*)^{p+q-s}$ into two pieces,

$$\det(1_p - zz^*)^{p+q-s} = \det(1_p - zz^*)^{p-\frac{s}{2}} \cdot \det(1_q - z^* z)^{q-\frac{s}{2}}$$

We also use the identity

$$\begin{bmatrix} x & z \\ z^* & y \end{bmatrix} = \begin{bmatrix} 1 & zy^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^* x^{-1} & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y - z^* x^{-1} z \end{bmatrix} \begin{bmatrix} 1 & x^{-1}z \\ 0 & 1 \end{bmatrix}$$

which implies that

$$\det Z = \det(x - zy^{-1}z^*) \cdot \det y = \det(y - z^* x^{-1} z) \cdot \det x$$

Thus,

$$\det(1_p - zz^*)^{p+q-s} \, (\det x)^{p-s_1-s} \, (\det y)^{q-s_2-s} = (\det Z)^{p+q-s} \, (\det x)^{-s_1} \, (\det y)^{-s_2}$$

and the integral becomes

$$\Gamma_p(\tau_1, s_1 + s) \otimes \Gamma_q(\tau_2^{*,-1}, s_2 + s) \cdot S$$

$$= \int_{C_{p+q}} e^{-\mathrm{tr}\, Z} \, \tau_1(x - zy^{-1}z^*) \otimes \tau_2^{-1}(y - z^* x^{-1} z) (\det x)^{s_1} \, (\det y)^{s_2} \, \frac{(\det Z)^s \, dZ}{(\det Z)^{p+q}}$$

Since the left-hand side of the latter equality is a product of scalar operators, the right-hand side is scalar.

Let $\tilde{\tau}_1$ be the irreducible representation of $GL(p+q, \mathbb{C})$ with extreme weight vector $\tilde{v}_1$ with weight

$$\tilde{\tau}_1 \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_{p+q} \end{pmatrix} \tilde{v}_1 = t_1^{\kappa_1} \ldots t_{p+q}^{\kappa_{p+q}} \cdot \tilde{v}_1$$

where we take

$$\kappa_{p+1} = \kappa_{p+2} = \ldots = \kappa_{p+q} = \kappa_p$$

The restriction of $\tilde{\tau}_1$ to

$$\{ \begin{pmatrix} A & 0 \\ 0 & 1_q \end{pmatrix} : A \in GL(p, \mathbb{C}) \} \approx GL(p, \mathbb{C})$$

has (among its several extreme weight vectors) the vector $\tilde{v}_1$ as extreme weight vector with

$$\tilde{\tau}_1 \begin{pmatrix} A & 0 \\ 0 & 1_q \end{pmatrix} \cdot \tilde{v}_1 = t_1^{\kappa_1} \ldots t_p^{\kappa_p} \cdot \tilde{v}_1$$

for lower-triangular A with diagonal entries t_i. Thus, a copy of τ_1 containing $\tilde{v}_1$ lies inside $\tilde{\tau}_1$. Then

$$\langle \tilde{\tau}_1 \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \tilde{v}_1, \tilde{v}_1 \rangle = \langle \tilde{\tau}_1 \left[\begin{pmatrix} 1 & zy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix} \right] \tilde{v}_1, \tilde{v}_1 \rangle$$

$$= \langle \tilde{\tau}_1 \left[\begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y^{-1}z^* & 1 \end{pmatrix} \right] \tilde{v}_1, \; \tilde{\tau}_1 \begin{pmatrix} 1 & y^{-1}z^* \\ 0 & 1 \end{pmatrix} \tilde{v}_1 \rangle$$

$$= \langle \tilde{\tau}_1 \begin{pmatrix} x - zy^{-1}z^* & 0 \\ 0 & y \end{pmatrix} \tilde{v}_1, \tilde{v}_1 \rangle = \langle \tau_1(x - zy^{-1}z^*) \, (\det y)^{\kappa_p} \cdot \tilde{v}_1, \tilde{v}_1 \rangle$$

The other identity

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}$$

has the roles of upper-triangular and lower-triangular reversed, and also τ_2 appears as τ_2^{-1}. Let $\tilde{\tau}_2$ be the irreducible representation of $GL(p+q,\mathbb{C})$ with extreme weight vector $\tilde{v}_2$ with weight

$$\tilde{\tau}_2 \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_{p+q} \end{pmatrix} \tilde{v}_2 = t_1^{\lambda_1} \ldots t_{p+q}^{\lambda_{p+q}} \cdot \tilde{v}_2$$

where we take

$$\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda_{p+1}$$

Then for *lower*-triangular D in $GL(q,\mathbb{C})$

$$\tilde{\tau}_2 \begin{pmatrix} 1_p & 0 \\ 0 & D \end{pmatrix} \cdot \tilde{v}_2 = t_{p+1}^{\lambda_{p+1}} \ldots t_{p+q}^{\lambda_{p+q}} \cdot \tilde{v}_2$$

Thus, a copy of τ_2 containing $\tilde{v}_2$ lies inside the restriction of $\tilde{\tau}_2$ to $GL(q,\mathbb{C})$. Then

$$\langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \tilde{v}_2, \tilde{v}_2 \rangle = \langle \tilde{\tau}_2^{-1} \left[\begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix} \right] \tilde{v}_2, \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \tilde{\tau}_2^{-1} \begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \tilde{v}_2,\ \tilde{\tau}_2^{*,-1} \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix} \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \tilde{v}_2, \tilde{\tau}_2^{-1} \begin{pmatrix} 1 & 0 \\ z^*x^{-1} & 1 \end{pmatrix} \tilde{v}_2 \rangle$$

$$= \langle \tilde{\tau}_2^{-1} \begin{pmatrix} x & 0 \\ 0 & y - z^*x^{-1}z \end{pmatrix} \tilde{v}_2,\ \tilde{v}_2 \rangle = \langle \tau_2^{-1}(y - z^*x^{-1}z)\,(\det x)^{-\lambda_p}\, \tilde{v}_2, \tilde{v}_2 \rangle$$

From these computations at last we see that it is wise to take

$$s_1 = -\lambda_p \qquad s_2 = \kappa_p$$

We do so. Combining these two computations,

$$\langle (\tilde{\tau}_1 \otimes \tilde{\tau}_2^{-1}) \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} (\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

$$= (\det x)^{-\lambda_p} \cdot (\det y)^{\kappa_p} \cdot \langle [\tau_1(x - zy^{-1}z^*) \otimes \tau_2^{-1}(y - z^*x^{-1}z)]\,(\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

Multiplying both sides of the latter equality by $e^{-\operatorname{tr} Z}\,(\det Z)^s$ and integrating over $Z \in C_{p+q}$ with respect to the measure $dZ/(\det Z)^{p+q}$ gives

$$\langle \Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)\,(\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

$$= \langle \int_{C_{p+q}} e^{-\operatorname{tr} Z}\,(\det x)^{-\lambda_p} (\det y)^{\kappa_p} \tau_1(x - zy^{-1}z^*) \otimes \tau_2^{-1}(y - z^*x^{-1}z)\,\frac{(\det Z)^s\,dZ}{(\det Z)^{p+q}}\,(\tilde{v}_1 \otimes \tilde{v}_2), \tilde{v}_1 \otimes \tilde{v}_2 \rangle$$

$$= \langle \Gamma_p(\tau_1, s - \lambda_p) \otimes \Gamma_q(\tau_2, s + \kappa_p) \cdot S \cdot (\tilde{v}_1 \otimes \tilde{v}_2),\ \tilde{v}_1 \otimes \tilde{v}_2 \rangle \rangle$$

From above, we know that S is scalar, and also that $\Gamma_p(\tau_1, -\lambda_p + s)$ and $\Gamma_q(\tau_2^{*,-1}, \kappa_p + s)$ are scalar. *If* $\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)$ were scalar (*which is not at all evident!*), then we could write

$$S = S_s = \frac{\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)}{\Gamma_p(\tau_1, -\lambda_p + s)\,\Gamma_q(\tau_2^{*,-1}, \kappa_p + s)}$$

However, again, it is not at all clear that $\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)$ is scalar, certainly not by the arguments used above, since the representation $\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}$ is neither purely holomorphic nor purely antiholomorphic, unlike what was correctly invoked above. Nevertheless, in fact, we would need *less* to finish this general computation, since we are concerned with evaluation just of a *single* inner product.

Leaving the general question aside for now, to reduce to the purely holomorphic or purely anti-holomorphic situation, it suffices to take one of τ_1 or τ_2 one-dimensional, so that anything not purely holomorphic or purely antiholomorphic can be subsumed in the power-of-determinant. Taking τ_2 to be scalar, we have

$$\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda_{p+1} = \ldots = \lambda_{p+q}$$

And with τ_2 scalar, the Γ_{p+q} gamma function *is scalar*, namely

$$\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{-1}, s) = \Gamma_{p+q}(\tilde{\tau}_1, s - \lambda_p) = \pi^{(p+q)(p+q-1)/2} \prod_{i=1}^{p+q} \Gamma(\kappa_i - (p+q-i) + (s - \lambda_p))$$

Thus, in this case, we *can conclude*

$$S = \frac{\Gamma_{p+q}(\tilde{\tau}_1 \otimes \tilde{\tau}_2^{*,-1}, s)}{\Gamma_p(\tau_1, -\lambda_p + s)\ \Gamma_q(\tau_2^{*,-1}, \kappa_p + s)} = \frac{\Gamma_{p+q}(\tilde{\tau}_1, s - \lambda_p)}{\Gamma_p(\tau_1, -\lambda_p + s)\ \Gamma_q(\kappa_p - \lambda_p + s)}$$

Expanding this into ordinary gammas, as in the special case done earlier, the net power of π is π^{pq}, and

$$S = \pi^{pq} \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p+q-i) + s - \lambda_p)}{\prod_{i=1}^{p} \Gamma(\kappa_i - (p-i) + s - \lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) + s - \lambda_p)}$$

Similarly, if τ_1 is scalar, then, thanks to the odd indexing scheme, we have

$$S = \pi^{pq} \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_p - (p+q-i) + s - \lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_p - (p-i) + s - \lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) + s - \lambda_i)}$$

Thus, we can write a common expression applicable to both cases, namely

$$S = \pi^{pq} \frac{\prod_{i=1}^{p+q} \Gamma(\kappa_i - (p+q-i) + s - \lambda_i)}{\prod_{i=1}^{p} \Gamma(\kappa_i - (p-i) + s - \lambda_p)\ \prod_{i=1}^{p+q} \Gamma(\kappa_p - (q-i) + s - \lambda_i)}$$

When evaluated at $s = 0$ this gives the expression asserted in the theorem. ///
