(*March 1, 2005*)

Artin L-functions

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Let **o** be the ring of algebraic integers in a number field k, **O** the ring of integers in a finite Galois extension K of k, with Galois group G. For a prime P in **O** lying over a prime p in **o**, the **decomposition (sub-) group** $G_P \subset G$ is the subgroup stabilizing (*not* necessarily pointwise fixing) P.

The fixed field $L = K^{G_P}$ of G_P has the property that it is the largest subfield of K (containing k) such that P is the only prime of **O** lying over $Q = P \cap L$. The residue fields are related by $\mathbf{o}/p = \mathbf{O}'/Q$, where $\mathbf{O}'$ is the ring of algebraic integers in L.

Then G_P acts on the residue field $\mathbf{O}/P$, and in fact *surjects* to the Galois group of $\mathbf{O}/P$ over $\mathbf{o}/p$. The kernel I_P is called the **inertia** subgroup, which is trivial if P is unramified over p, so the inertia subgroup is trivial for *almost all* p.

Let $\mathbf{o}/p$ have q elements. Then the Galois group of $\mathbf{O}/P$ over $\mathbf{o}/p$ is generated by the Frobenius automorphism $\alpha \to \alpha^q$. Let Φ_P be the inverse image of $\alpha \to \alpha^q$ in the decomposition group G_P. There are other notations as well, such as $\Phi_P = (P, K/k)$.

For P *ramified* over p, we only have an I_P-*coset* rather than an *element*, and more complicated considerations are necessary. We won't worry about this, since at worst only finitely many primes are ramified.

Since the Galois group of K/k is *transitive* on primes P lying over p, all the Frobenius elements Φ_P for P over p are *conjugate*. Thus, attached to the prime p *downstairs* is a *conjugacy class* of Frobenius elements in $\mathrm{Gal}(K/k)$.

When the Galois group is *abelian*, the conjugacy class of Frobenius elements Φ_P for primes P over p necessarily consists of a single element, called the **Artin** symbol.

We will associate to a finite-dimensional representation ρ of $\mathrm{Gal}(K/k)$ a Dirichlet series with Euler product, the **Artin L-function**, as follows. To conform with standard usage, now use v to denote a (finite) place of **o**, p_v the associated prime ideal in **o**, q_v the residue field cardinality $\mathbf{o}/p_v$, and Φ_v the conjugacy class of Frobenius elements attached to p_v, for v unramified in the extension K/k. Let S be the finite set of (finite) places ramified in K/k. Define the Artin L-function

$$L(s,\rho) = \prod_{v \notin S} \frac{1}{\det(1_\rho - q_v^{-s}\Phi_v)}$$

The indicated determinant is indeed well-defined since does only depend upon the conjugacy class.

Artin conjectured in the 1930's that for ρ irreducible and not the trivial representation the L-function is *entire*.

For abelian $\mathrm{Gal}(K/k)$ **classfield theory** proves that these L-functions are among the L-functions attached to Hecke characters, and Hecke (and Iwasawa and Tate) proved that Hecke L-functions have analytic continuations, proving Artin's conjecture in this case. In the abelian Galois group case Artin L-functions are called **abelian L-functions**.

For non-abelian $\mathrm{Gal}(K/k)$, R. Brauer proved that there is a *meromorphic* continuation by showing that these L-functions are quotients of products of *abelian* L-functions attached to intermediate fields, by proving that all irreducibles ρ of the Galois group can be expressed as **Z**-linear combinations of induced representations of one-dimensional representations on subgroups. This does *not* prove the entire-ness, however.

In the 1960's R. Langlands offered a new viewpoint on Artin's conjecture, namely that for an n-dimensional irreducible ρ the Artin L-function should be equal to an L-function associated to a *cuspform* (or cuspidal *automorphic representation*) on $GL(n)$, whose analytic continuation had been proven just about then, by Jacquet-Piatetski-Shapiro-Shalika, and also by Jacquet-Godement. That is, Langlands changed the issue to assertion that an L-function coming from Galois theory (the Artin L-function) should be equal to an analytically defined L-function (the automorphic one).

Except for the abelian case and two-dimensional examples, very little has been proven.