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Global automorphic Sobolev spaces

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The goal is legitimization of term-wise differentiation of L^2 spectral expansions, so that computations producing a classical outcome are correct. We are fond of L^2 expansions because they are what Plancherel gives.

Typically, L^2 expansions are not continuous, much less differentiable, so the issue cannot be *proving* classical differentiability, which does *not* hold.

To say that L^2 spectral expansions are term-wise differentiable in a *distributional* sense is often valid, but too weak, since it is difficult to return from the large world of distributions to the smaller world of L^2 functions.

Further, already for Fourier transforms on $\mathbb{R}^n$, the integral expressing Fourier inversion is *not* a superposition of L^2 functions, since the exponentials are not in $L^2(\mathbb{R}^n)$.

Notions of L^2 Sobolev spaces are a balance of the simplicity of Hilbert space structures with extensions of notions of differentiability, insofar as solving elliptic partial differential equations of sufficiently high degree can move back to L^2. That is, Sobolev spaces are within *finite distance* of L^2, in terms of basic processes of analysis.

Especially with respect to *invariant* operators such as Casimir operators, especially in delicate situations such as *automorphic forms*, Plancherel theorems most naturally yield corollaries about L^2-differentiation, not about classical pointwise differentiation.

[1.1] Spherical automorphic spectral expansion

For simplicity, take $X = \Gamma\backslash G/K$ where $\Gamma = SL_2(\mathbb{Z}[i])$ and $G = SL_2(\mathbb{C})$ and $K = SU(2)$. Let Δ be the G-invariant Laplacian on G/K descended from Casimir on G. Functions f in $L^2(\Gamma\backslash G/K)$ decompose in an L^2 sense

$$f \;=\; \sum_F \langle f, F\rangle \cdot F \;+\; \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} \;+\; \int_{-\infty}^{\infty} \langle f, E_{\frac{1}{2}+it}\rangle \cdot E_{\frac{1}{2}+it}\, dt$$

where F runs over an orthonormal basis of *cuspforms*. Plancherel is

$$\|f\|^2 \;=\; \sum_F |\langle f, F\rangle|^2 \;+\; \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \;+\; \int_0^{\infty} |\langle f, E_{\frac{1}{2}+it}\rangle|^2\, dt$$

Note that Plancherel does not need nor assert anything about pointwise values of cuspforms or Eisenstein series.

[1.2] Spectral parametrization

Let

$$\Xi \;=\; \{\text{orthonormal basis of cuspforms}\} \;\cup\; \{1\} \;\cup\; \tfrac{1}{2}+i[0,\infty)$$

where the half-line parametrizes Eisenstein series $E_{\frac{1}{2}+it}$. The **spectral measure** on Ξ gives each cuspform and the constant function 1 point-mass measure, and gives the half-line $\frac{1}{2} + i[0,\infty)$ a constant multiple of Lebesgue measure. Let

$$\Phi_\xi \;=\; \begin{cases} F & (\text{for } \xi = F) \\ 1/\langle 1,1\rangle^{1/2} & (\text{for } \xi = 1) \\ E_s & (\text{for } \xi = s = \frac{1}{2}+it) \end{cases} \qquad (\text{for } \xi \in \Xi)$$

Then the spectral decomposition can be rewritten as

$$f \;=\; \int_\Xi \langle f, \Phi_\xi\rangle \cdot \Phi_\xi \, d\xi$$

and Plancherel is

$$|f|^2 \;=\; \int_\Xi |\langle f, \Phi_\xi\rangle|^2\,d\xi$$

The integrals $\langle f, \Phi_\xi\rangle$ are the **spectral components**, but the implied integrals do not converge for all f in L^2, especially integrals against Eisenstein series. Nevertheless, Plancherel asserts that the literal integrals on *test functions extend* to an isometry

$$\mathscr{F} \;:\; L^2(X) \;\longrightarrow\; L^2(\Xi)$$

The essential point is that on test functions $\mathscr{F}$ *is* given by literal integrals against the preferred functions: cuspforms, constants, and Eisenstein series.

[1.3] Spectral expansions of derivatives

For test functions, integration by parts is immediately legitimate to compute spectral projections. Let λ_ξ be the eigenvalue of Δ on Φ_ξ. Note that these eigenvalues are *real*. Then

$$\langle \Delta f, \Phi_\xi\rangle \;=\; \int_X f\;\Delta\overline{\Phi}_\xi \;=\; \lambda_\xi\cdot\int_X f\;\overline{\Phi}_\xi \qquad (\text{for } f\in C_c^\infty(X))$$

Thus, for test functions,

$$\Delta f \;=\; \int_\Xi \langle \Delta f, \Phi_\xi\rangle\,\Phi_\xi\,d\xi \;=\; \int_\Xi \lambda_\xi\cdot\langle f,\Phi_\xi\rangle\,\Phi_\xi\,d\xi$$

That is, the differential operator Δ differentiates *term-wise*, in the sense of moving inside the integration over Ξ giving the *spectral synthesis*. More succinctly,

$$\Delta f \;=\; \mathscr{F}^{-1}\mathscr{F}\Delta f \;=\; \mathscr{F}^{-1}\lambda_\xi\mathscr{F} f$$

[1.4] Small and smaller Sobolev spaces

To discuss general *differentiability* of functions on $X=\Gamma\backslash G/K$ it is sufficient for us to use the right action of the Lie algebra $\mathfrak{g}$ of G on $\Gamma\backslash G$. Most of these derivatives will no longer be right K-invariant, but that harms nothing. Indeed, up to a constant, the measure and integral on $\Gamma\backslash G/K$ are simply that of $\Gamma\backslash G$ restricted to right K-invariant functions. So it is consistent with the above to write

$$|f|^2 \;=\; \int_{\Gamma\backslash G}|f|^2 \qquad \langle f,\varphi\rangle \;=\; \int_{\Gamma\backslash G} f\,\overline{\varphi} \qquad (\text{for } f,\varphi\in C_c^\infty(\Gamma\backslash G))$$

Let $U\mathfrak{g}^{\le\ell}$ be the finite-dimensional subspace of elements of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ consisting of elements of degree $\le \ell$. This is stable under K. Each $\alpha\in U\mathfrak{g}^{\le\ell}$ gives a seminorm ν_α on $C_c^\infty(\Gamma\backslash G)$ by

$$\nu_\alpha(f) \;=\; |\alpha f|^2 \qquad (\text{right action of } U\mathfrak{g})$$

Sups of finite linear combinations of these seminorms give the same topology as seminorms given by *bounded subsets* B of $U\mathfrak{g}^{\le\ell}$, by

$$\nu_B(f) \;=\; \sup_{\beta\in B}\nu_\beta(f)$$

For $\ell\ge 0$, a *smaller* Sobolev space $H^\ell_o(\Gamma\backslash G)$ is the completion of *test functions* $C_c^\infty(\Gamma\backslash G)$ in the topology given by all ν_α with $\alpha\in U\mathfrak{g}^{\le\ell}$, or, equivalently, by all ν_B with B bounded subsets of $U\mathfrak{g}^{\le\ell}$. The Sobolev space of interest is

$$H^\ell_o(X) \;=\; H^\ell_o(\Gamma\backslash G)^K \;=\; \{\text{right } K\text{-fixed elements of } H^\ell_o(\Gamma\backslash G)\}$$

Since $U\mathfrak{g}^{\le\ell}$ is finite-dimensional, the topology on $H^\ell_o(\Gamma\backslash G)$ *can be* given by finitely-many Hilbert space norms, so is *isomorphic to* a Hilbert space. However, at this point there is no *canonical* Hilbert space structure:

only the *topology* is canonical. Nevertheless, the collection of *bounded sets* in a topological vector space sense depends only upon the *topology*, and the *strong dual* $H_o^{\ell *}$ is canonically defined as a topological vector space. Then the fact that the limit has a finite cofinal subset implies that the strong dual is a Hilbert space, invoking Riesz-Fischer on the limitands.

The *small* Sobolev space $H^\ell(\Gamma\backslash G)$ is the completion of the space of *smooth* functions having finite ν_α-norm for all $\alpha \in U\mathfrak{g}^{\leq \ell}$. And $H^\ell(X)$ is the right K-invariant elements of $H^\ell(\Gamma\backslash G)$. These also have the canonical topologies.

From Urysohn's lemma, the group G acts continuously on $L^2(\Gamma\backslash G)$. Thus, since the Adjoint action of G stabilizes $U\mathfrak{g}^{\leq \ell}$, the group G acts continuously on $H^\ell(\Gamma\backslash G)$.

The differential operator Δ on test functions gives a map

$$\Delta \;:\; H_o^\ell \cap C_c^\infty(\Gamma\backslash G) \;\longrightarrow\; H_o^{\ell-2} \cap C_c^\infty(\Gamma\backslash G)$$

continuous by design. Thus, Δ extends by continuity to a continuous linear map

$$\widetilde{\Delta}_\ell \;:\; H_o^\ell(\Gamma\backslash G) \;\longrightarrow\; H_o^{\ell-2}(\Gamma\backslash G) \qquad\qquad (\text{for } \ell \geq 2)$$

Similarly, Δ extends by continuity to

$$\widetilde{\Delta}_\ell \;:\; H^\ell(\Gamma\backslash G) \;\longrightarrow\; H^{\ell-2}(\Gamma\backslash G) \qquad\qquad (\text{for } \ell \geq 2)$$

These extensions are L^2 **differentiation** by Δ on $\Gamma\backslash G$.

Since Δ commutes with K, the L^2 differentiation by Δ descends to $H_o^\ell(X) \to H_o^{\ell-2}(X)$ and $H^\ell(X) \to H^{\ell-2}(X)$.

[1.4.1] Remark: We will prove just below that these two definitions yield the same completion, that is, $H_o^\ell(\Gamma\backslash G) = H^\ell(\Gamma\backslash G)$.

[1.4.2] Remark: We will prove below that, in fact, the semi-norms attached to polynomials in Δ suffice to give the topology on Sobolev spaces, returning us to the straightforward Hilbert space setting.

[1.5] Smooth cut-offs To prove that test functions are dense in H^ℓ, smooth cut-off functions are required. Some mild specifics about the geometry of $\Gamma\backslash G$ are necessary. Use Iwasawa decomposition coordinates $z = (x, y)$ on G/K, from

$$n_x \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad\qquad m_y \;=\; \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \qquad\qquad (\text{with } x \in \mathbb{C} \text{ and } y > 0)$$

With $N = \{n_x\}$ and $M = \{m_y\}$, the Iwasawa decomposition $G = NMK$ shows that NM maps surjectively to G/K. By reduction theory, for sufficiently small $t > 0$ the Siegel set

$$\mathfrak{S}_t \;=\; \{z \in G/K \;:\; y \geq t\} \;\subset\; \mathfrak{H}$$

covers the quotient $\Gamma\backslash G/K$. Thus, the inverse image of this in G covers $\Gamma\backslash G$. Fix a smooth function η_o on $\mathbb{R}$ such that $0 \leq \eta_o \leq 1$ everywhere, η_o is identically 1 on $(-\infty, 0]$, and η_o is identically 0 on $[1, \infty)$. Let $q : \mathfrak{S}_t \to \Gamma\backslash G/K$ be the quotient map $G/K \to \Gamma\backslash G/K$ *restricted* to $\mathfrak{S}_t$. Define a smooth cut-off function on $\Gamma\backslash G/K$ for large $R > 0$, by defining it on a Siegel set $\mathfrak{S}_t$, by

$$\eta(z) \;=\; \eta_R(z) \;=\; \eta_o\big(\log y - R) \qquad\qquad (\text{for } z = n_x m_y K \in \mathfrak{S}_t)$$

The obvious issue is well-definedness.

Reduction theory assures existence of sufficiently large $t_1 > t$ such that $\gamma \cdot z = z'$ with $\gamma \in \Gamma$, $z \in \mathfrak{S}_t$ and $z' \in \mathfrak{S}_{t_1}$ implies $z \in \mathfrak{S}_{t_1}$. Thus, for $R > t_1$, since $\eta = 1$ on $\mathfrak{S}_t - \mathfrak{S}_{t_1}$, it is certainly *well defined* on $q\big(\mathfrak{S}_t - \mathfrak{S}_{t_1}\big)$.

Meanwhile, note that $\mathfrak{S}_{t_1}$ is stable under

$$\Gamma_\infty \;=\; \{\gamma \in \Gamma \;:\; \gamma\mathfrak{S}_{t_1} \cap \mathfrak{S}_{t_1} \neq \phi\}$$

By its definition η is invariant under Γ_∞, which in this example is simply translation by $\mathbb{Z}$. Thus, η is well-defined on $\Gamma\backslash(\Gamma \cdot \mathfrak{S}_{t_1})$.

Then η lifts to $\Gamma\backslash G$.

The derivatives by $U\mathfrak{g}$ of η on the right are easy to estimate: from the definition, in the usual Iwasawa coordinates NMK, $\eta(nmk) = \eta(m)$. The relevant derivative is from the Lie algebra of M, namely, $y\frac{\partial}{\partial y}$ in Iwasawa coordinates on G/K. By design, this derivative has a bound independent of R, because

$$y\frac{\partial}{\partial y}\,\eta(\log y - R) \;=\; \eta'(\log y - R)$$

[1.6] $H^\ell_o = H^\ell$ The idea is to prove that test functions are dense in $H^\ell(\Gamma\backslash G/K)$, by showing that *smooth cut-offs* approach a given smooth function.

Let $\eta = \eta_R$ be as above. We claim that the *smooth cut-offs* (point-wise products, not mollifications) ηf approach f in H^ℓ. For fixed $g = nmk$, from Leibniz' rule, the derivative of ηf by an element of $\alpha \in U\mathfrak{g}^{\le \ell}$ is a finite sum of derivatives $\alpha\eta \cdot \beta f$ where $\alpha \in U\mathfrak{g}^{\le \ell_1}$ and $\beta \in U\mathfrak{g}^{\le \ell_2}$ with $\ell_1 + \ell_2 = \ell$.

Further, we only need consider differential operators from a *compact* subset of $U\mathfrak{g}^{\le \ell}$. The Adjoint action by the compact K produces another compact subset of $U\mathfrak{g}^{\le \ell}$, for fixed k acting on the right on $\eta(nm)$. The differential operators coming from the Lie algebras of K and N act trivially on the right on $\eta(nm)$. The only non-trivial differential operators are from the Lie algebra of M, namely, iterates of $y\frac{\partial}{\partial y}$, and

$$\big(y\frac{\partial}{\partial y}\big)^\ell \eta_R(nm) \;=\; \eta^{(\ell)}(\log y - R)$$

By design, this is bounded, independent of R.

Any extreme term where η is *not* differentiated appears only as $\eta \cdot \beta f - \beta f = (\eta - 1)\beta f$.

Thus, all terms vanish on $\{t \le \log y \le R\}$, and the L^2 norms of the rest are estimated uniformly in R by the tails of the integrals of (norms-squared of) derivatives of f, which go to 0 by square-integrability . Thus, $H^{2\ell}_o = H^{2\ell}$ for $2\ell \ge 0$.

[1.7] Reduction to Laplacian The topologies on Sobolev spaces $H^\ell(X)$ can be given more simply by the seminorms from a smaller class of differential operators, namely, the polynomials in Δ or Casimir Ω. This is fortunate, because the preferred functions in spectral expansions are eigenfunctions for Δ and/or Ω.

Let V be a Hilbert-space representation of G. Suppose that the subspace V^K of K-fixed (that is, *spherical* vectors is not trivial. Let $U\mathfrak{g}^{\le n}$ be the usual filtration on the universal enveloping algebra. We claim that, given n, there is a constant C such for $\alpha \in U\mathfrak{g}^{\le n}$ and $f \in V^K$

$$\langle \alpha f, \alpha f\rangle \;\le\; \langle (-\Omega + C)^n\, f, f\rangle$$

Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition with $\mathfrak{k}$ the Lie algebra of K. By Poincaré-Birkhoff-Witt, $U\mathfrak{g}$ is spanned by monomials of the form $x_1 \ldots x_m y_1 \ldots y_n$ with $x_i \in \mathfrak{p}$ and $y_i \in \mathfrak{k}$. Since $\mathfrak{k}$ annihilates spherical vectors f, the effect of $U\mathfrak{g}$ on spherical vectors f is that of linear combinations of monomials from $\mathfrak{p}$.

Let $\Omega_\mathfrak{p}$ and $\Omega_\mathfrak{k}$ be the corresponding components of Casimir Ω. The Killing form is positive-definite on $\mathfrak{p}$, and negative definite on $\mathfrak{k}$, so $\Omega_\mathfrak{p}$ is a non-positive symmetric operator in any Hilbert-space representation V of the group G, and $\Omega_\mathfrak{k}$ is a non-negative symmetric operator. An element $x \in \mathfrak{p}$ of length 1 can be extended to a self-dual basis $\{x_i\}$ for $\mathfrak{p}$, and

$$\|xf\|^2 \;=\; \int xf \cdot x\overline{f} \;\le\; \sum_i \int x_i f \cdot x_i \overline{f} \;=\; \int \sum_i -x_i^2 f \cdot \overline{f} \;=\; \int -\Omega_\mathfrak{p} f \cdot \overline{f}$$

For fixed n, let C be a large positive constant, strictly larger than the largest eigenvalue of $\pm\Omega_\mathfrak{k}$ on $\mathcal{U}\mathfrak{g}^{\le n}$. Then $-\Omega_\mathfrak{k} + C$ is a non-negative, symmetric operator, and

$$T \;=\; -\Omega_\mathfrak{p} - \Omega_\mathfrak{k} + C \;=\; -\Omega + C$$

is a *strictly* positive symmetric (unbounded) operator on the Hilbert-space closure of $U\mathfrak{g}^{\le n} \cdot V^K$.

By Friedrichs, T has an (everywhere-defined) *inverse* R, a positive, symmetric *bounded* operator. By spectral theory for bounded, symmetric operators, there is a positive, symmetric $\sqrt{R}$ in the closure of the polynomial algebra $\mathbb{C}[R]$. Thus, $1\sqrt{R}$ is a symmetric positive $\sqrt{T}$ commuting with all operators commuting with T. Thus,

$$\begin{aligned}\langle -\Omega_\mathfrak{p}\alpha f, \alpha f\rangle \;&\le\; \langle(-\Omega_\mathfrak{p} - \Omega_\mathfrak{k} + C)\alpha f, \alpha f\rangle \;=\; \langle(-\Omega + C)\alpha f, \alpha f\rangle \;=\; \langle\sqrt{T}\,\sqrt{T}\,\alpha f, \alpha f\rangle \\ &=\; \langle\sqrt{T}\,\alpha f, \sqrt{T}\alpha f\rangle \;=\; \langle\alpha\sqrt{T} f, \alpha\sqrt{T}\rangle\end{aligned}$$

By induction on the degree of α,

$$\langle\alpha\sqrt{T}f, \alpha\sqrt{T}f\rangle \;\le\; \langle T^{\deg\alpha}\sqrt{T}f, \sqrt{T}f\rangle \;=\; \langle T^{1+\deg\alpha} f, f\rangle$$

That is,

$$\langle x\alpha f, x\alpha f\rangle \;\le\; \langle(-\Omega + C)^{1+\deg\alpha} f, f\rangle$$

Since we want a comparison of *topologies*, not *metrics*, the salient feature is that, for $\alpha \in U\mathfrak{g}^n$,

$$\langle\alpha f, \alpha f\rangle \;\le\; \langle(-\Omega + C)^{\deg\alpha} f, f\rangle \;\ll\; \langle(-\Omega + 1)^{\deg\alpha} f, f\rangle$$

Since test functions are dense in $H^\ell(X)$, the same relation holds for all vectors in $H^\ell(X)$.

In particular, this gives $H^\ell(X)$ a Hilbert-space structure, with respect to the norm

$$\|f\|_\ell^2 \;=\; \langle(-\Omega + 1)^\ell f,\, f\rangle \qquad\qquad (\text{for } f \in C_c^\infty(X))$$

and then completing.

[1.7.1] Remark: The integration-by-parts argument does *not* immediately apply to merely *smooth* functions, rather than *test* functions. Thus, we do *not* immediately obtain the dominance of all the Sobolev seminorms by the special sub-family involving only the Laplacian for *smooth* functions.

[1.7.2] Remark: In the Euclidean case, invocation of standard facts about Schwartz spaces and tempered distributions resolves the discrepancy mentioned in the previous remark. However, the analogous considerations for automorphic forms are less standard, less easily applicable, and less well understood.

[1.8] Large Sobolev space For $\ell \ge 0$ the *large* Sobolev space $W^{2,\ell} = W^{2,\ell}(X)$ is the collection of $u \in L^2(X)$ such that u and its distributional derivatives αu with $\alpha \in U\mathfrak{g}^{\le\ell}$ are in $L^2(X)$. Give $W^{2,\ell}$ the topology from the Sobolev norm $\|\cdot\|_\ell$. We claim that

$$W^{2,\ell}(X) \;=\; H^\ell(X) \qquad\qquad (\text{for } \ell \ge 0)$$

In fact, proving $W^{2,\ell}(\Gamma\backslash G) = H^\ell(\Gamma\backslash G)$ is convenient, because differentiations on the right by $U\mathfrak{g}$ disrupt the right K-invariance. Taking right K-fixed elements at the end suffices.

Urysohn's lemma and the definition of integral show that $C^0_o(\Gamma\backslash G)$ is dense in $L^2(\Gamma\backslash G)$. From this, the group G acts continuously on $L^2(\Gamma\backslash G)$ by translation

$$(g \cdot f)(x) \;=\; f(x\,g) \qquad\qquad (\text{for } f \in L^2(\Gamma\backslash G),\, g \in G)$$

The right action of G does *not* commute with the right action of $U\mathfrak{g}$, but G *does* map bounded subsets of $U\mathfrak{g}^{\le\ell}$ to bounded subsets, so the action of G *does* preserve the topology given by seminorms ν_B for B ranging over bounded subsets of $U\mathfrak{g}^{\le\ell}$. Then the continuity of G on L^2 gives the continuity of G on $W^{2,\ell}$.

From the continuity of the group action, a typical application of Gelfand-Pettis integrals proves density of smooth functions in $W^{2,\ell}$, as follows. Let $\{\eta\}$ be a smooth *approximate identity.* For $u \in W^{2,\ell}$, the basic estimate on Gelfand-Pettis integrals [1] shows that the averaged translations

$$\eta \cdot u \;=\; \int_G \eta(g)\, g\cdot u\, dg$$

often called *packets*, approach f in the topology on the representation space $W^{2,\ell}$.

At the same time, taking a genuine measurable function u as a representative for an L^2 function,

$$(\eta\cdot u)(x) \;=\; \int_G \eta(g)\, u(xg)\, dg \;=\; \int_G \eta(x^{-1}g)\, u(g)\, dg$$

The literal definition of derivative can be applied: for a fixed vector $v \in G$ and t real, the differentiability

$$|\eta(g\, \exp(tv)) - \eta(g) - t\,\nabla\eta(g)\cdot v| \;=\; o(t)$$

gives

$$\Big|\int_G \eta(g\,\exp(tv))\, u(g)\, dg - \int_G \eta(g)\, u(g)\, dg - \int_G t\,\nabla\eta(g)\cdot v\; u(g)\, dg\Big| \;=\; o(t)$$

Thus,

$$\frac{\partial}{\partial t}\Big|_{t=0} \;=\; \frac{1}{t}\int_G \eta(g\,\exp(tv))\, u(g)\, dg - \int_G \eta(g)\, u(g)\, dg \;=\; \int_G \nabla\eta(g)\cdot v\; u(g)\, dg$$

Since $\nabla\eta\cdot v$ is again a test function, we see that $\eta\cdot u$ is *smooth.*

Thus, $H^\ell_o = H^\ell = W^{2,\ell}$ for $\ell \ge 0$.

[1.9] Differentiation and spectral expansion

Since $\mathscr{F} : L^2(X) \to L^2(\Xi)$ is an isometric isomorphism obtained by extension by continuity from $\mathscr{F}$ on $C^\infty_c(X)$, defining Δ by extension by continuity, the computation for test functions

$$\mathscr{F}\Delta f \;=\; \int_X \Delta f\,\Phi_\xi \;=\; \int_X f\,\Delta\overline{\Phi}_\xi \;=\; \lambda_\xi\int_X f\,\overline{\Phi}_\xi \;=\; \lambda_\xi\mathscr{F} f \qquad\qquad (\text{for } f\in C^\infty_c(X))$$

gives the analogous assertion for f in $H^\ell_o = H^\ell = W^{2,\ell}$ with $\ell \ge 2$. This is **L^2-differentiation** for non-negative index Sobolev spaces. *Term-wise differentiation* in the L^2 sense is valid:

$$\Delta f \;=\; \mathscr{F}^{-1}\mathscr{F}\Delta f \;=\; \mathscr{F}^{-1}\lambda_\xi\mathscr{F} f \qquad\qquad (\text{for } f\in H^\ell = W^{2,\ell} \text{ with } \ell\ge 2)$$

[1] The basic estimate on Gelfand-Pettis integrals is that an integral $\int_X f$ over a probability space X of a continuous V-valued function f lies in the closure of the convex hull of the image $f(X)$. This is valid for V quasi-complete and locally convex.

This differentiation is in a non-classical sense, and the maps $\mathscr{F}$ and $\mathscr{F}^{-1}$ are *not* literal integrals.

[1.10] Characterization of Sobolev spaces by spectral transforms

By expressing Sobolev norm and differentiation via spectral transforms $\mathscr{F}$, certainly $\mathscr{F}W^{2,\ell}$ is *contained in*

$$V^\ell \;=\; \{v \;:\; (1-\lambda_\xi)^{\ell/2}\, v \in L^2(\Xi)\} \qquad\qquad (\text{for } \ell \ge 0)$$

Give V^ℓ the Hilbert-space structure from the expected norm

$$\|v\|^2_{V^\ell} \;=\; \int_\Xi (1-\lambda_\xi)^\ell\, |v|^2$$

inherited from the Sobolev space. We claim that Fourier transform gives a Hilbert space *isomorphism*

$$\mathscr{F} \;:\; W^{2,\ell} \;\longrightarrow\; V^\ell \qquad\qquad (\text{an } \textit{isomorphism})$$

The key point is to show that, for $(1-\lambda_\xi)v \in L^2(\Xi)$,

$$(1-\Delta)\mathscr{F}^{-1}v \;=\; \mathscr{F}^{-1}\big((1-\lambda_\xi)v\big) \qquad\qquad (\text{distributional derivative})$$

To verify this equality, evaluate on test functions φ:

$$\Big((1-\Delta)\mathscr{F}^{-1}v\Big)(\varphi) \;=\; \mathscr{F}^{-1}v\big((1-\Delta)\varphi\big) \;=\; v\Big(\mathscr{F}(1-\Delta)\varphi\Big)$$

by Plancherel for $L^2(X)$ and $L^2(\Xi)$. Using the identities for test functions, this is

$$v\Big((1-\lambda_\xi)\mathscr{F}\varphi\Big) \;=\; (1-\lambda_\xi)v\big(\mathscr{F}\varphi\big) \;=\; \Big(\mathscr{F}^{-1}\big((1-\lambda_\xi)v\big)\Big)(\varphi)$$

again by L^2 Plancherel. This proves the above equality, namely, that the distribution $(1-\Delta)\mathscr{F}^{-1}v$ is given by integration against the L^2 function $\mathscr{F}^{-1}\big((1-\lambda_\xi)v\big)$.

This proves the necessary intertwining of multiplication and differentiation by inverse Fourier transform on V^ℓ.

Since the seminorms $\|(1-\Delta)^\ell f\|$ give the topology on $H^{2\ell}_o = H^{2\ell} = W^{2,2\ell}$, the above intertwining yields the desired equality $\mathscr{F}W^{2,2\ell} = V^{2\ell}$, as follows. The case $2\ell = 0$ is Plancherel. For arbitrary $2\ell \ge 0$, for $v \in V^{2\ell}$, for $k \le \ell$,

$$(1-\Delta)^k\mathscr{F}^{-1}v \;=\; \mathscr{F}^{-1}(1-\lambda_\xi)^k v \;\in\; F^{-1}V^{2\ell-2k} \;\subset\; F^{-1}L^2 \;=\; L^2$$

That is, all these distributional derivatives $(1-\Delta)^k\mathscr{F}^{-1}v$ are in L^2, so $\mathscr{F}^{-1}v$ is in $W^{2,2\ell}$.

Thus, we have identified the precise image of the Sobolev space under the spectral transform $\mathscr{F}$.

[1.11] Negative-index Sobolev spaces

Obviously we cannot easily describe negative-index Sobolev spaces in terms of negative-order differential operators. Instead, characterize negative-index Sobolev spaces as Hilbert-space duals

$$H^{-\ell} \;=\; \text{Hilbert-space dual to } H^\ell \qquad\qquad (\text{for } \ell \ge 0)$$

The continuous L^2-differentiation $\Delta : H^\ell \to H^{\ell-2}$ for $\ell \ge 2$ on positive-index Sobolev spaces gives an adjoint, still denoted Δ,

$$\Delta \;:\; H^{-(\ell-2)} \;\longrightarrow\; H^{-\ell} \qquad\qquad (\text{for } \ell \ge 2)$$

For $\ell \ge 0$ the spectral transform $\mathscr{F}$ is an isomorphism $\mathscr{F} : H^\ell \to V^\ell$, where, as above,

$$V^\ell \;=\; \{\text{measurable } v \;:\; (1-\lambda_\xi)^{\ell/2}\cdot v \in L^2(\Xi)\} \qquad\qquad (\text{for } \textit{any } \ell \in \mathbb{Z})$$

with corresponding Hilbert space structure.

The space $V^{-\ell}$ is naturally the Hilbert space dual $V^{\ell *}$ to V^ℓ, by integration:

$$\langle v, w \rangle \;=\; \int_X v(\xi)\, w(\xi)\, d\xi \qquad\qquad \text{(complex-bilinear, for } v \in V^\ell \text{ and } w \in V^{-\ell}\text{)}$$

The adjoint $\mathscr{F}^*$ of the *restriction* of $\mathscr{F}$ to H^ℓ gives an isomorphism

$$\mathscr{F}^* \;:\; V^{-\ell} \;\longrightarrow\; H^{-\ell} \qquad\qquad (\text{for } \ell \geq 0)$$

The density of $V^{\ell+1}$ in V^ℓ and the auto-duality of $V^0 = L^2$ implies that the chain of *inclusions*

$$\ldots \subset V^2 \subset V^1 \subset V^0 \approx V^{0*} \subset V^{-1} \subset V^{-2} \subset \ldots$$

really does give the maps $V^{-\ell} \to V^{-\ell-1}$ adjoint to the inclusions $V^{\ell+1} \to V^\ell$ for $\ell \geq 0$. That is, these adjoint maps can be identified with the literal inclusions of spaces of functions. Visibly, $V^{-\ell}$ is dense in $V^{-\ell-1}$.

Further, these inclusions and their adjoints are compatible with the multiplication operator(s) $\mu : V^\ell \to V^{\ell-2}$ multiplying by $1 - \lambda_\xi$. Indeed, each $\mu : V^\ell \to V^{\ell-2}$ is obviously an isomorphism compatible with the inclusions.

Plancherel asserts that the complex-linear adjoint of $\mathscr{F} : L^2(X) \longrightarrow L^2(\Xi)$ is simply $\mathscr{F}^{-1}$. This allows gluing together the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 \\
& \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx \\
\cdots \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0
\end{array}$$

and the adjoint diagram

$$\begin{array}{ccccccc}
H^0 & \xrightarrow{\text{inc}^*} & H^{-1} & \xrightarrow{\text{inc}^*} & H^{-2} & \longrightarrow \cdots \\
\mathscr{F}^*\big\uparrow\approx & & \mathscr{F}^*\big\uparrow\approx & & \mathscr{F}^*\big\uparrow\approx & \\
V^0 & \xrightarrow{\text{inc}^*} & V^{-1} & \xrightarrow{\text{inc}^*} & V^{-2} & \longrightarrow \cdots
\end{array}$$

Using the observation that the adjoints of inclusions of the V^ℓ are again inclusions, we have a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 & \xrightarrow{\text{inc}^*} & H^{-1} & \xrightarrow{\text{inc}^*} & H^{-2} & \xrightarrow{\text{inc}^*} \cdots \\
& \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx & & \mathscr{F}^*\big\uparrow\approx & & \mathscr{F}^*\big\uparrow\approx & & \mathscr{F}^*\big\uparrow\approx & \\
\cdots \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0 & \xrightarrow{\text{inc}} & V^{-1} & \xrightarrow{\text{inc}} & V^{-2} & \xrightarrow{\text{inc}} \cdots
\end{array}$$

The density of V^ℓ in $V^{\ell-1}$ for every $\ell \in \mathbb{Z}$ is converted by Fourier transform to density of H^ℓ in $H^{\ell-1}$ for every ℓ. Thus, the adjoint maps inc^* on the top row are genuine inclusions. On the negative-index spaces, *define* spectral inversions $\mathscr{F}^{-1}$ as the inverses of the isomorphism $\mathscr{F}^*$. Thus, we have a diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 & \xrightarrow{\text{inc}} & H^{-1} & \xrightarrow{\text{inc}} & H^{-2} & \longrightarrow \cdots \\
& \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx & & \mathscr{F}\big\downarrow\approx & & & \\
\cdots \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0 & \xrightarrow{\text{inc}} & V^{-1} & \xrightarrow{\text{inc}} & V^{-2} & \longrightarrow \cdots
\end{array}$$

The adjoint $\Delta^* : H^{-\ell+2} \longrightarrow H^{-\ell}$ of the L^2 differentiation $\Delta : H^\ell \longrightarrow H^{\ell-2}$ is *still* converted by Fourier transforms to multiplication by λ_ξ, and the adjoint of this multiplication map is multiplication by the same scalar. Thus, **define** the L^2 **differentiation** on negative-index spaces to be these adjoints. This gives a commutative diagram

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^4 & \xrightarrow{\Delta} & H^2 & \xrightarrow{\Delta} & H^0 & \xrightarrow{\Delta} & H^{-2} & \xrightarrow{\Delta} & H^{-4} & \longrightarrow \cdots \\ & \mathscr{F}\downarrow\approx & & \mathscr{F}\downarrow\approx & & \mathscr{F}\downarrow\approx & & \mathscr{F}\downarrow\approx & & & \\ \cdots \longrightarrow & V^4 & \xrightarrow{\times\lambda_\xi} & V^2 & \xrightarrow{\times\lambda_\xi} & V^0 & \xrightarrow{\times\lambda_\xi} & V^{-2} & \xrightarrow{\times\lambda_\xi} & V^{-4} & \longrightarrow \cdots \end{array}$$

and a similar diagram for odd-index spaces.

Since Fourier transform converts $1-\Delta$ to multiplication by $1-\lambda_\xi$, and this multiplication gives isomorphisms $V^\ell \to V^{\ell-2}$ for all $\ell \in \mathbb{Z}$, $1-\Delta$ is an isomorphism $H^\ell \to H^{\ell-2}$ for all $\ell \in \mathbb{Z}$.

The density of test functions in negative-index Sobolev spaces follows from the density of V^0 in every $V^{-\ell}$, and the density of test functions in H^0.

This density implies the extension by continuity to *all* Sobolev spaces of the integration by parts adjoint relation

$$\langle \Delta f, v\rangle_{H^{\ell-2}\times H^{-\ell+2}} \;=\; \langle f, \Delta v\rangle_{H^\ell \times H^{-\ell}} \qquad \text{(for } f \in H^\ell \text{ and } v \in H^{-\ell})$$

[Avakumović 1956] V. G. Avakumović, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. **65** (1956), 327-344.

[Bargmann 1947] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. Math. **48** (1947), 568-640.

[Berezin 1957] F.A. Berezin, *Laplace operator on semisimple Lie groups*, Trudy Moscow Mat. Obsc. **6** (1857) 371-463. English translation in Amer. Math. Soc. Transl. **21** (1962), 239-339.

[Berezin 1956a] F.A. Berezin, *Laplace operators on semisimple Lie groups*, Dokl. Akad. Nauk SSSR **107** (1956), 9-12.

[Berezin 1956b] F.A. Berezin, *Representations of complex semisimple Lie groups in Banach spaces*, Dokl. Akad. Nauk SSSR **110** (1956), 897-900.

[Berezin 1957] F.A. Berezin, *Laplace operator on semisimple Lie groups*, Trudy Moscow Mat. Obsc. **6** (1857) 371-463. English translation in Amer. Math. Soc. Transl. **21** (1962), 239-339.

[Cogdell-PS 1990] J. Cogdell, I. Piatetski-Shapiro, *The arithmetic and spectral analysis of Poincaré series*, Perspectives in Mathematics, Academic Press, 1990.

[DKV 1979] J. Duistermaat, J. Kolk, V. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Inv. Math. **52** (1979), 27-93.

[DKV 1983] J. Duistermaat, J. Kolk, V. Varadarajan, *Functions, flows, and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*, Comp. Math. **49** (1983), 309-398.

[Gangolli-Varadarajan 1988] R. Gangolli, V.S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*, Springer-Verlag, 1988.

[Gelfand-Naimark 1950] I.M. Gelfand, M.A. Naimark, *Unitary representations of the classical groups*, Trudy Mat.Inst. Steklova **36** (1950), 1-288.

[Gelfand-Naimark 1952] I.M. Gelfand, M.A. Naimark, *Unitary representations of the unimodular group containing the identity representation of the unitary subgroup*, Trudy Moscow Mat. Obsc. **1** (1952), 423-475.

[Harish-Chandra 1954] Harish-Chandra, *The Plancherel formula for complex semisimple Lie groups*, Trans. AMS **76** (1954), 485-528.

[HarishChandra 1958] Harish-Chandra, *Spherical functions on semisimple Lie groups I*, Amer. J. Math. **79** (1958), 241-310.

[Helgason 1984] S. Helgason, *Groups and geometric analysis*, Academic Press, 1984.

[Iwaniec 2002] H. Iwaniec, *Spectral methods of automorphic forms*, Graduate Studies in Mathematis **53**, AMS, 2002.

[Jorgenson-Lang 2001] S. Lang, J. Jorgenson, *Spherical inversion for* $SL_n(\mathbb{R})$, Springer, 2001.

[Jorgenson-Lang 2008] S. Lang, J. Jorgenson, *The heat kernel and theta inversion on* $SL_2(\mathbb{C})$, Springer, 2008.

[Jorgenson-Lang 2009] S. Lang, J. Jorgenson, *Heat Eisenstein series on* $SL_n(\mathbb{C})$, Mem. AMS **201**, no. 946, 2009.

[Knapp 1986] A. Knapp, *Representation theory of semi-simple real Lie groups: an overview based on examples*, Princeton University Press, 1986.

[Langlands 1976] R. Langlands, *On the functional equations satisfied by Eisenstein series*, SLN 544, 1975.

[Meyers-Serrin 1964] N. Meyers, J. Serrin, $H = W$, Proc. Nat. Acad. Sci. **51** (1964), 1055-1056.

[Moeglin-Waldspurger 1989] C. Moeglin, J.-L. Waldspurger, *Le spectre résiduel de* GL_n, with appendix *Poles des fonctions L de pairs pour* GL_n, Ann. Sci. École Norm. Sup. **22** (1989), 605-674.

[Moeglin-Waldspurger 1995] C. Moeglin, J.-L. Waldspurger, *Spectral Decompositions and Eisenstein series*, Cambridge Univ. Press, Cambridge, 1995.

[Roelcke 1956] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, S.-B. Heidelberger Akad. Wiss. Math.Nat.Kl. 1953/1955 (1956), 159-267.

[Schwartz 1950] L. Schwartz, *Théorie des Distributions*, I, Hermann, Paris, 1950.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces*, J. Indian Math. Soc. **20** (1956).

[Selberg 1991] A. Selberg, with a foreword by K. Chandrasekharan, Collected papers, Vol. II (Springer, Berlin, 1991).

[Sobolev 1937] S.L. Sobolev, *On a boundary value problem for polyharmonic equations (Russian)*, Mat. Sb. **2** (44) (1937), 465-499.

[Sobolev 1938] S.L. Sobolev, *On a theorem of functional analysis (Russian)*, Mat. Sb. N.S. **4** (1938), 471-497.

[Varadarajan 1989] V. S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, Cambridge University Press, 1989.