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Colin de Verdière's meromorphic continuation of Eisenstein series

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We elaborate the brief note [Colin de Verdière 1981] on meromorphic continuation of Eisenstein series, and related harmonic analysis of automorphic forms. See also [Colin de Verdière 1982,83].

The context of [Colin de Verdière 1981] is not elementary: it uses technical aspects of [Friedrichs 1934,35]'s canonical *self-adjoint extensions* of symmetric unbounded operators on Hilbert spaces, and uses Sobolev spaces and Schwartz' distributions. The *compactness* of the inclusion map of Friedrichs-Sobolev spaces of automorphic forms with constant terms vanishing above $y = a$, into $L^2(\Gamma\backslash\mathfrak{H})$, proves the *compactness* of the resolvent of the Friedrichs self-adjoint extension $\tilde{\Delta}_a$ of the *restriction* of the invariant Laplacian to that subspace, giving its *meromorphy*. Eisenstein series differ from Eisenstein-series-like functions in the domain of $\tilde{\Delta}_a$ by elementary functions, giving the meromorphic continuation of the Eisenstein series.

A noteworthy preliminary result, reminiscent of [Avakumović 1956],[Roelcke 1956], [Selberg 1956], immediately extends Eisenstein series E_s to $\mathrm{Re}(s) > \frac{1}{2}$. Analytic continuation of the zeta function $\zeta(s)$ to $\mathrm{Re}(s) > 0$ is a corollary, the simplest example of [Langlands 1967/76] and [Langlands 1971] arguments about meromorphic continuation of automorphic L-functions.

The compactness of the imbedding of Friedrichs' L^2 Sobolev-like spaces of automorphic forms into L^2 also proves that the space of L^2 cuspforms decomposes discretely with respect to the invariant Laplacian, although this is not a trivial corollary, for reasons we explain.

The precise import of the compactness argument is widely misunderstood. Often, the description of the compactness argument (with corollaries about discrete decomposition of cuspforms) does not distinguish these (correct) arguments from similar (incorrect) arguments purportedly proving that truncated Eisenstein are eigenfunctions for the Laplacian. Yet, Colin-de-Verdière's argument *does* discretely decompose spaces containing truncated Eisenstein series, by self-adjoint extensions $\tilde{\Delta}_a$ of *restrictions* of the Laplacian Δ to subspaces. These operators $\tilde{\Delta}_a$ are not differential operators, as becomes clear below. in [Colin de Verdière 1982,83] these and other variants are usefully called *pseudo-Laplacians*.

As will be clarified later: for fixed cut-off height $y = a$, the pseudo-Laplacian constructed as the self-adjoint Friedrichs' extension $\tilde{\Delta}_a$ of the restriction of Δ, *does* have compact resolvent on the subspace $L^2(\Gamma\backslash\mathfrak{H})_a$ of $L^2(\Gamma\backslash\mathfrak{H})$ consisting of automorphic forms with constant term vanishing above $y = a$. Thus, $\tilde{\Delta}_a$ has a basis of eigenvectors. In particular, the orthogonal complement to cuspforms in $L^2(\Gamma\backslash\mathfrak{H})_a$ has an orthogonal basis of $\tilde{\Delta}_a$-eigenvectors, consisting of truncated Eisenstein $\wedge^a E_s$ whose constant term vanishes on $y = a$. There is no paradox, because $\tilde{\Delta}_a$ is designed to ignore order-zero distributions supported on the line $y = a$. Computed distributionally, $(\Delta - s(s-1)) \wedge^a E_s$ is a distribution supported on (images of) $y = a$. When the constant term does *not* vanish on that line, the resulting distribution is of order one, and $\wedge^a E_s$ is not in the domain of $\tilde{\Delta}_a$. When the constant term vanishes, the resulting distribution is of order zero, and the truncation $\wedge^a E_s$ is in the domain of $\tilde{\Delta}_a$, since the zero-order distribution is ignored.

The simplest example $\Gamma\backslash\mathfrak{H}$ with $\Gamma = SL_2(\mathbb{Z})$ and $\mathfrak{H} = SL_2(\mathbb{R})/SO(2)$ illustrates the mechanism.

1. *Harmonic analysis on* $\mathfrak{H}$

[1.1] Invariant Laplacian

The usual $SL_2(\mathbb{R})$-invariant Laplacian on the upper half-plane $\mathfrak{H} \approx G/K$ is

$$\Delta \;=\; y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

Parametrize Δ-eigenvalues as usual by

$$\lambda \;=\; \lambda_s \;=\; s(s-1)$$

Let

$$N \;=\; \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;:\; x \in \mathbb{R}\} \qquad\qquad A^+ \;=\; \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;:\; t > 0\}$$

[1.2] Density of automorphic test functions

Integration by parts on $C_c^\infty(\Gamma\backslash G)$ shows that Δ is a *symmetric* (unbounded) operator on $L^2(\Gamma\backslash\mathfrak{H})$. To show that it is *densely defined*, show that $C_c^\infty(\Gamma\backslash\mathfrak{H})$, defined to be right K-invariant functions in $C_c^\infty(\Gamma\backslash G)$, is dense in $L^2(\Gamma\backslash\mathfrak{H})$, as follows.

Fix $0 < 1 \le b < b' < \infty$, and take a smooth cut-off function $0 \le \tau \le 1$ on $(0,\infty)$ with

$$\tau(y) \;=\; \begin{cases} 1 & (\text{for } b' \le y) \\[2ex] 0 & (\text{for } 0 \le y \le b) \end{cases}$$

For $t > 0$, define a smooth cut-off by

$$\varphi_t(y) \;=\; \tau(y/t) \qquad\qquad (\text{for } t > 0)$$

Let $\Phi_t(z) = \varphi_t(\mathrm{Im}\,(z))$. With Γ_∞ the upper-triangular elements of $\Gamma = SL_2(\mathbb{Z})$, the corresponding pseudo-Eisenstein series is

$$\Psi_t(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Phi_t(\mathrm{Im}\,(\gamma\cdot z))$$

We claim that $(1-\Psi_t)\cdot f \to f$ in $L^2(\Gamma\backslash\mathfrak{H})$ as $t \to +\infty$, for all $f \in L^2(\Gamma\backslash\mathfrak{H})$. Indeed,

$$\int_{\Gamma\backslash\mathfrak{H}} \Big|(1-\Psi_t)f - f\Big|^2 \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Big|\Psi_t\cdot f\Big|^2 \;-\; \int_{\Gamma_\infty\backslash\mathfrak{H}} |\Phi_t\cdot f|^2 \;\le\; \int_{\Gamma_\infty\backslash\{y\ge t\}} |f|^2 \;\longrightarrow\; 0$$

because the tails of the integral of $|f|^2$ go to 0, by convergence of the integral of the L^2 norm of f.

[1.3] Friedrichs extension of Δ on $C_c^\infty(\Gamma\backslash\mathfrak{H})$

Precise discussion of an unbounded operator and its resolvent require a specified *domain*. Take[1] $C_c^\infty(\Gamma\backslash\mathfrak{H})$ as the domain of Δ.

[1] Generally, taking a domain to be *test functions* requires some sort of generalized vanishing on the boundary in the self-adjoint extension, if there is a boundary. In boundary-less situations such as $\Gamma\backslash\mathfrak{H}$, this is often appropriate. For the operators Δ_a later, the interaction with boundary properties is visible. For example, see [Grubb 2009], for extensive examples and a modern discussion of boundary conditions versus extensions of operators.

Let $\tilde{\Delta}$ be the *Friedrichs extension* of Δ to a *self-adjoint* (unbounded) operator on $L^2(\Gamma\backslash\mathfrak{H})$. The Friedrichs construction shows that the domain of $\tilde{\Delta}$ is *contained in* a Sobolev-like space:

$$\text{domain }\tilde{\Delta} \;\subset\; \text{Sob}(+1) \;=\; \Big(\text{completion of } C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ under } \;\; \langle v,w\rangle_{\text{Fr}} = \langle v,w\rangle + \langle -\Delta v, w\rangle\Big)$$

The domain of $\tilde{\Delta}$ *contains*[2] the smaller Sobolev space

$$\text{Sob}(+2) \;=\; \Big(\text{completion of } C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ under } \;\; \langle v,w\rangle_{\text{Sob}(+2)} = \langle v,w\rangle + \langle \Delta v, \Delta w\rangle\Big)$$

[1.3.1] Remark: The Sobolev spaces above are *defined* as completions of test functions, and there is no immediate need to make comparisons to other characterizations.

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2. *Meromorphic continuation up to the critical line*

The quotient $\Gamma\backslash\mathfrak{H}$ is the union of a *compact* part, whose (conceivably complicated) geometry does not matter, and a geometrically trivial *non-compact* part:

$$\Gamma\backslash\mathfrak{H} \;=\; X_{\text{cpt}} \cup X_\infty \qquad\qquad (\text{ compact } X_{\text{cpt}}, \text{ cusp neighborhood } X_\infty)$$

where

$$X_\infty \;=\; \text{image of } \{x+iy : y \ge y_o\} \;=\; \Gamma_\infty \;\; \{x+iy : y \ge y_o\} \;\approx\; \text{circle} \times \text{ray}$$

Define a smooth cut-off function τ as usual: fix $b < b'$ large enough so that the image of $\{z \in \mathfrak{H} : y > b\}$ in the quotient is in X_∞, let

$$\tau(y) \;=\; \begin{cases} 1 & (\text{for } y > b') \\ \\ 0 & (\text{for } y < b) \end{cases}$$

Form a pseudo-Eisenstein series h_s by automorphizing the smoothly cut-off function $\tau(\text{Im}\,(z)) \cdot y^s$:

$$h_s(z) \;=\; \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \tau(\text{Im}\,(\gamma z)) \cdot \text{Im}\,(\gamma z)^s$$

Since τ is supported on $y \ge b$ for large b, for any $z \in \mathfrak{H}$ there is at most one non-vanishing summand in the expression for h_s, and convergence is not an issue. Thus, the pseudo-Eisenstein series h_s is *entire* as a function-valued function of s. Let

$$\tilde{E}_s \;=\; h_s - (\tilde{\Delta} - \lambda)^{-1}\,(\Delta - \lambda)\,h_s \qquad\qquad (\text{where } \lambda = s(s-1))$$

[2.0.1] Remark: From Friedrichs, the resolvent $(\tilde{\Delta} - \lambda)^{-1}$ *exists* as a bounded operator for $s \in \mathbb{C}$ for λ_s not a non-positive real number, because of the non-positive-ness of Δ. Further, for λ_s not a non-positive real, this resolvent is a *holomorphic* operator-valued function. Thus, $\tilde{E}_s$ is holomorphic for $\text{Re}\,(s) > \frac{1}{2}$ and $\text{Im}\,(s) \ne 0$.

[2] In fact, any *self-adjoint* extension T of Δ will have domain containing Sob(+2), with T defined there by extending by continuity in the Sob(+2) topology. This is seen as follows. For $L^2(\Gamma\backslash\mathfrak{H})$-Cauchy v_i in the domain of T, if $\lim T v_i$ *exists* in the topology of $L^2(\Gamma\backslash\mathfrak{H})$, then $v_i \oplus T v_i$ is Cauchy in $L^2(\Gamma\backslash\mathfrak{H}) \oplus L^2(\Gamma\backslash\mathfrak{H})$. Graphs of self-adjoint operators, whether unbounded or bounded, are *closed*. Thus, but only because we *assumed* the limit *exists*, $\lim T v_i = T(\lim v_i)$. This argument does *not* touch upon $L^2(\Gamma\backslash\mathfrak{H})$-*continuity* of T, but, rather, proves that T is continuous in the Sob(+2) topology.

[2.0.2] Remark: The smooth function $(\Delta - \lambda)h_s$ is supported on the image of $b \le y \le b'$ in $\Gamma\backslash\mathfrak{H}$, which is *compact*. Thus, it is in $L^2(\Gamma\backslash\mathfrak{H})$. It might seem $\tilde{E}_s$ *vanishes*, if it is forgotten that the indicated resolvent maps to the domain of $\tilde{\Delta}$ *inside* $L^2(\Gamma\backslash\mathfrak{H})$, and that h_s is not in $L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(s) > \frac{1}{2}$. Indeed, since h_s is not in $L^2(\Gamma\backslash\mathfrak{H})$ and $(\tilde{\Delta} - \lambda)^{-1}(\Delta - \lambda)h_s$ is in $L^2(\Gamma\backslash\mathfrak{H})$, the difference cannot vanish.

[2.0.3] Theorem: With $\lambda = s(s-1)$ not non-positive real, $u = \tilde{E}_s - h_s$ is the unique element of the domain of $\tilde{\Delta}$ such that

$$(\tilde{\Delta} - \lambda)\, u \;=\; -(\Delta - \lambda)h_s$$

Thus, $\tilde{E}_s$ is the usual Eisenstein series E_s for $\mathrm{Re}(s) > 1$, and gives an analytic continuation of E_s to $\mathrm{Re}(s) > \frac{1}{2}$ with $s \notin (\frac{1}{2}, 1]$.

Proof: *Uniqueness* follows from Friedrichs' construction and construction of resolvents, because $\tilde{\Delta} - \lambda$ is a *bijection* of its domain to $L^2(\Gamma\backslash\mathfrak{H})$.

On the other hand, for $\mathrm{Re}(s) > \frac{1}{2}$ and $s \notin (0\frac{1}{2}, 1]$, $\tilde{E}_s - h_s$ is in $L^2(\Gamma\backslash\mathfrak{H})$, and is smooth, so is in the domain of $\tilde{\Delta}$. Abbreviate

$$H_s \;=\; (\Delta - \lambda)\, h_s$$

Then it is legitimate to compute

$$(\tilde{\Delta} - \lambda)(\tilde{E}_s - h_s) \;=\; (\tilde{\Delta} - \lambda)\Big((h_s - (\tilde{\Delta} - \lambda)^{-1}H_s) - h_s\Big) \;=\; (\tilde{\Delta} - \lambda)\Big(-(\tilde{\Delta} - \lambda)^{-1}H_s\Big) \;=\; -H_s$$

Thus, $\tilde{E}_s - hs$ is a solution. Certainly $E_s - h_s$ is a solution. ///

[2.0.4] Remark: Thus, the Eisenstein series E_s has an analytic continuation to $\mathrm{Re}(s) > \frac{1}{2}$ and $s \notin (\frac{1}{2}, 1]$ as an $h_s + L^2(\Gamma\backslash\mathfrak{H})$-valued function. Further, Friedrichs gives a bound for the L^2-norm of $E_s - h_s$ via an estimate on the operator norm of $(\tilde{\Delta} - \lambda)^{-1}$. The L^2-norm of $(\Delta - \lambda)h_s$ is not difficult to estimate, since its support is $b \le y \le b'$:

$$|(\Delta - \lambda)h_s|^2_{L^2} \;\le\; \int_0^1 \int_b^{b'} (|\Delta h_s| + |\lambda h_s|)^2\, \frac{dx\, dy}{y^2} \;\ll_{b,b'}\; |\lambda|^2$$

Since $\tilde{\Delta}$ is negative-definite, Friedrichs gives

$$\|(\tilde{\Delta} - \lambda)^{-1}\| \;\le\; \frac{1}{\mathrm{Im}(\lambda)} \;=\; \frac{1}{2(\sigma - \frac{1}{2})t} \qquad (\text{for } \sigma > \tfrac{1}{2},\ t \neq 0)$$

Thus,

$$|E_s - h_s|_{L^2} \;=\; \|(\tilde{\Delta} - \lambda)^{-1}\| \cdot |(\Delta - \lambda)h_s|_{L^2} \;\ll\; \frac{1}{(\sigma - \frac{1}{2})t} \cdot |s(s-1)|^{\frac{1}{2}}$$

[2.0.5] Remark: Granting that the Eisenstein series E_s has *constant term* $y^s + c_s y^{1-s}$, the analytic continuation of E_s to $\mathrm{Re}(s) > \frac{1}{2}$ analytically continues c_s to $\mathrm{Re}(s) > \frac{1}{2}$. Since $c_s = \xi(2s-1)/\xi(2s)$ with $\xi(s)$ the completed zeta-function

$$\xi(s) \;=\; \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$$

this yields the analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$, off the interval $[0, 1]$.

3. *Sobolev inequality/imbedding*

The self-adjoint extensions of differential operators typically have domains including not-necessarily-smooth functions, requiring a finer description of the spaces Sob(+1) occuring in Friedrichs' construction for the case of second-order operators.

In particular, as needed later, computations relevant to Sobolev-norm behavior of pseudo-Eisenstein series is clarified.

[3.1] Another description of $\mathrm{Sob}(+1)$

This description applies to general Γ, G, K.

Consider functions on $\Gamma\backslash\mathfrak{H} \approx \Gamma\backslash G/K$ as right K-invariant functions on $\Gamma\backslash G$. We use the G-invariant *trace pairing*[3]

$$\langle x, y\rangle \;=\; \mathrm{trace}\,(xy) \qquad\qquad (\text{with } x, y \in \mathfrak{g})$$

This pairing is *negative*-definite on the Lie algebra $\mathfrak{k}$ of K, and positive-definite on the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$. Thus, we can choose a negative-orthonormal basis $\{\theta_i\}$ of $\mathfrak{k}$, that is, with $\langle\theta_i, \theta_j\rangle = -\delta_{ij}$ with Kronecker delta. We can choose an orthonormal basis $\{x_j\}$ for $\mathfrak{p}$.

For any such choice, the Casimir element Ω in the universal enveloping algebra $U\mathfrak{g}$ is expressible as

$$\Omega \;=\; \sum_j x_j^2 \;-\; \sum_i \theta_i^2$$

The Lie algebra $\mathfrak{g}$ of G acts on the right on $\Gamma\backslash G$. The restriction of Ω to right K-invariant functions on G is the invariant Laplacian Δ on G/K, up to a constant. On test functions f on $\Gamma\backslash G$, integration by parts gives

$$\int_{\Gamma\backslash G} \Omega f \cdot \overline{f} \;=\; \sum_j \int_{\Gamma\backslash G} x_j^2 f \cdot \overline{f} \;-\; \sum_i \int_{\Gamma\backslash G} \theta_i^2 f \cdot \overline{f} \;=\; -\sum_j \int_{\Gamma\backslash G} x_j f \cdot x_j \overline{f} + \sum_i \int_{\Gamma\backslash G} \theta_i f \cdot \theta_i \overline{f}$$

For right K-invariant f, this computes

$$\int_{\Gamma\backslash G/K} -\Delta f \cdot \overline{f} \;=\; \sum_j \int_{\Gamma\backslash G} |x_j f|^2$$

Of course, typically the derivatives $x_j f$ are not right K-invariant, but this is harmless.

Thus, on one hand, a Sob(+1) norm $\langle,\rangle_1$ attached to Δ is expressible as

$$\langle f, f\rangle_1 \;=\; \int_{\Gamma\backslash G/K} (1-\Delta) f \cdot \overline{f} \;=\; \int_{\Gamma\backslash G/K} |f|^2 \;+\; \sum_j \int_{\Gamma\backslash G} |x_j f|^2$$

On the other hand, the computation shows that $L^2(\Gamma\backslash G)$ norms of first derivatives (given by $\mathfrak{g}$) of $f \in C_c^\infty(\Gamma\backslash G/K)$ are dominated by the Sob(+1) norm of f.

[3] For simple linear Lie algebras $\mathfrak{g}$, this $\mathbb{R}$-bilinear pairing is a multiple of the Killing form.

[3.2] Constant terms and local Sobolev spaces

Although $\Gamma_\infty\backslash N$ is compact, the constant term maps

$$f \longrightarrow \int_{\Gamma_\infty\backslash N} f(ng)\,dn$$

do *not* map $C_c^\infty(\Gamma\backslash G/K) \to C_c^\infty(N\backslash G/K)$. This prevents comparison of (global) Sobolev spaces. Nevertheless, *local* Sobolev spaces are readily compared: for compact $C \subset G$, let

$$\nu_C(f) \;=\; \int_C (1-\Omega)f\cdot\overline{f} \qquad\qquad (\text{for } f\in C^\infty(G/K))$$

Let

$$\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1) \;=\; \text{local } +1\text{-index Sobolev space on } N\backslash G/K$$

be the quasi-completion of $C^\infty(N\backslash G/K)$ with respect to the collection of these semi-norms. The constant-term map respects these semi-norms, since $\Gamma_\infty\backslash N$ is compact. Thus, we have a continuous map

$$c_P \;:\; \mathrm{Sob}(+1) \longrightarrow \mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1)$$

The dimension of $N\backslash G/K$ is much lower than that of $\Gamma\backslash G/K$. For $G = SL_2(\mathbb{R})$ or any real-rank 1 group, the dimension of $N\backslash G/K$ is 1. The (local) *Sobolev imbedding/inequality* shows that constant terms of $\mathrm{Sob}(+1)$ functions are continuous, since

$$\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1) \;\subset\; C^o(N\backslash G/K)$$

In fact, the local Sobolev theory shows that functions in $\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1)$ satisfy a non-trivial Lipschitz condition.

[3.3] Pseudo-Eisenstein series in $\mathrm{Sob}(+1)$

We need a simple sufficient condition for pseudo-Eisenstein series to be in $\mathrm{Sob}(+1)$. We revert to $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$, for simplicity.

With large $b > 0$, let $\varphi \in C^o[b,\infty)$ be *smooth*, except possibly at $y = a$ with fixed $a > b$, but continuous at $y = a$ and possessing left and right derivatives at $y = a$. We claim that the pseudo-Eisenstein series Ψ_φ is in $\mathrm{Sob}(+1)$ if

$$\int_0^\infty |\varphi|^2 + \Big|y\frac{\partial\varphi}{\partial y}\Big|^2\;\frac{dy}{y^2} \quad<\quad \infty$$

Proof: To discuss *right* derivatives, we must look at automorphic forms on the group G, rather than on the domain $\mathfrak{H}$. Let the Iwasawa decomposition of an element of G be $g = na(g)k$ with $n \in N$, $a(g) \in A^+$, and $k \in K$. Let

$$a_y \;=\; \begin{pmatrix}\sqrt{y} & 0\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}$$

and let $\Phi(g) = \varphi(y)$, where $a(g) = a_y$. The $\mathrm{Sob}(+1)$ hypothesis on φ implies that φ is *locally* in the $+1$ Sobolev space. Thus, locally, any first-derivative is in the 0^{th} Sobolev space, that is, locally L^2. This implies local integrability of φ and Φ.

The right action of $\alpha \in \mathfrak{g}$ on a smooth function f on G is

$$(\alpha f)(g) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(g\cdot e^{t\alpha})$$

The right action of $\mathfrak{g}$ commutes with the left action of G, so we can unwind:

$$\int_{\Gamma\backslash G} |\alpha\Psi_\varphi|^2 \;=\; \int_{\Gamma_\infty\backslash G} \alpha\Psi_\varphi\cdot\alpha\overline{\Phi} \;=\; \int_{\Gamma_\infty\backslash G}\;\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \alpha\Phi(\gamma g)\cdot\alpha\overline{\Phi}(g)\,dg$$

Since φ is supported on $y \geq b$, the same is true of $\alpha\varphi$, and by reduction theory $\alpha\Phi(\gamma g)\alpha\overline{\Phi}(a_g) \neq 0$ only for $\gamma \in \Gamma_\infty$. Thus,

$$\int_{\Gamma\backslash G} |\alpha\Psi_\varphi|^2 \;=\; \int_{\Gamma_\infty\backslash G} \alpha\Phi \cdot \alpha\overline{\Phi} \;=\; \int_{N\backslash G} |\alpha\Phi|^2$$

Let $(,)$ be the Killing form (or trace form) on $\mathfrak{g}$. It is negative-definite on the Lie algebra $\mathfrak{k}$ of K, and positive-definite on the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$. Modify $B(,)$ by reversing its sign on $\mathfrak{k}$, giving a positive-definite K-invariant form $B^+(,)$ on $\mathfrak{g}$, and corresponding K-invariant *length*.

Typically, the derivative αf of a right K-invariant function is no longer right K-invariant, but we still have

$$(\alpha f)(g\cdot k) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(gk\cdot e^{t\alpha}) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(g\cdot e^{t\cdot k\alpha k^{-1}}\cdot k)$$

$$=\; \frac{\partial}{\partial t}\Big|_{t=0} f(g\cdot e^{t\cdot k\alpha k^{-1}}) \qquad\qquad (\text{for } \alpha\in\mathfrak{g},\ k\in K,\ g\in G)$$

Let K have total measure 1. For $\alpha \in \mathfrak{g}$ with $B^+(\alpha,\alpha) \leq 1$, using an Iwasawa decomposition $G = NA^+K$, we have

$$\int_{N\backslash G} |\alpha\Phi|^2 \;\leq\; \int_0^\infty \int_K |\alpha\Phi(a_y k)|^2 \,\frac{dy}{y^2}\,dk \;\leq\; \int_0^\infty \sup_{\beta\in\mathfrak{g}:B^+(\beta,\beta)\leq 1} |\beta\Phi(a_y)|^2\,\frac{dy}{y^2}$$

Let

$$h \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \theta \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (\text{in } \mathfrak{g})$$

The elements h and $2X - \theta$ are in $\mathfrak{p}$, of length $\sqrt{2}$. The element θ is in $\mathfrak{k}$, of length $\sqrt{2}$. Any $\beta \in \mathfrak{g}$ is a linear combination,

$$\beta \;=\; ah + bX + c\theta \;=\; ah + \frac{b}{2}(2X-\theta) + (c+\frac{b}{2})\theta$$

Thus, for $B^+(\beta,\beta) \leq 1$, there is a uniform bound on the coefficients a, b, c. Thus, to uniformly bound $\beta\Phi$ it suffices to show $X\Phi(a_y) = 0$, $\theta\Phi(a_y) = 0$, and to bound $h\Phi(a_y)$.

Since Φ is right K-invariant, $\theta\Phi = 0$. Since Φ is left N-invariant,

$$X\Phi(a_y) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \Phi(a_y\, e^{tX}) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \Phi(e^{t\cdot yXy^{-1}}\, a_y) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \Phi(a_y) \;=\; 0$$

Finally,

$$h\Phi(a_y) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \Phi(a_y\, e^{th}) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \varphi(y\cdot e^{2t}) \;=\; 2y\frac{\partial\varphi}{\partial y}$$

Thus, in summary,

$$\int_{\Gamma\backslash G} |\alpha\Psi_\varphi|^2 \;\ll\; \int_0^\infty \Big| y\frac{\partial\varphi}{\partial y}\Big|^2\,\frac{dy}{y^2} \qquad\qquad (\text{uniform implied constant})$$

We should prove that Ψ_φ is a Sob(+1)-limit of elements of $C_c^\infty(\Gamma\backslash\mathfrak{H})$. In fact, as should be anticipated, it is a limit of elements Ψ_η with $\eta \in C_c^\infty(0,\infty)$. However, given the above comparison and prior development, the argument is straightforward. ///

4. *Eventually-vanishing constant terms*

Suitable restrictions Δ_a of Δ to subspaces of $L^2(\Gamma\backslash\mathfrak{H})$, where constant terms *vanishing* above a fixed height $y = a$, have Friedrichs extensions with *compact resolvents*.

[4.1] Constant terms vanishing for $y > a$

For $\varphi \in C_c^\infty(0,\infty)$, the corresponding pseudo-Eisenstein series is

$$\Psi_\varphi(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z)) \;\in\; C_c^\infty(\Gamma\backslash\mathfrak{H})$$

Fix $a > b'$. Denote the collection of all pseudo-Eisenstein series with test function φ supported on $[a,\infty)$ by

$$\Psi_{\geq a} \;=\; \{\Psi_\varphi : \varphi \text{ smooth on } (0,+\infty), \text{ compact support inside } [a,+\infty)\}$$

The collection of $L^2(\Gamma\backslash\mathfrak{H})$ functions with constant terms vanishing [4] in $y > a$ is best defined as

$$L^2(\Gamma\backslash\mathfrak{H})_a \;=\; \Psi_{\geq a}^\perp \;=\; \text{orthogonal complement to } \Psi_{\geq a} \text{ in } L^2(\Gamma\backslash\mathfrak{H})$$

Equivalently, since $\Psi_{\geq a} \subset C_c^\infty(\Gamma\backslash\mathfrak{H})$, we can also characterize $L^2(\Gamma\backslash\mathfrak{H})_a$ as the collection of *distributions* on $\Gamma\backslash\mathfrak{H}$ coming from elements of $L^2(\Gamma\backslash\mathfrak{H})$ and annihilating all pseudo-Eisenstein series in $\Psi_{\geq a}$.

[4.1.1] Proposition: Corresponding test functions are *dense* in $L^2(\Gamma\backslash\mathfrak{H})_a$, that is,

$$L^2(\Gamma\backslash\mathfrak{H})_a \;=\; L^2(\Gamma\backslash\mathfrak{H})\text{-closure of } \Big(L^2(\Gamma\backslash\mathfrak{H})_a \;\cap\; C_c^\infty(\Gamma\backslash\mathfrak{H})\Big)$$

Proof: As earlier, fix $0 < b < b' < \infty$, and take a smooth cut-off function $0 \leq \tau \leq 1$ on $(0,\infty)$ with

$$\tau(y) \;=\; \begin{cases} 1 & (\text{for } b' \leq y) \\ 0 & (\text{for } 0 \leq y \leq b) \end{cases}$$

let $\varphi_t(y) = \tau(y/t)$ and $\Phi_t(z) = \varphi_t(\mathrm{Im}(z))$. Form the corresponding pseudo-Eisenstein series

$$\Psi_t(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi_t(\mathrm{Im}(\gamma\cdot z))$$

We already proved that $(1-\Psi_t)\cdot f \to f$ in $L^2(\Gamma\backslash\mathfrak{H})$. We claim that, for $f \in L^2(\Gamma\backslash\mathfrak{H})$, the constant term of $(1-\Psi_t)\cdot f$ vanishes for $y \geq a$ for large t. Indeed, elementary reduction theory assures us that, for large t and $y \geq a$, $\Psi_t(\gamma\cdot z) \neq 0$ only for $\gamma \in \Gamma_\infty$. Then

[4] The *constant term* $c_P f$ of a function f on $\Gamma\backslash\mathfrak{H}$ is usually defined (somewhat imprecisely) by

$$c_P f(z) \;=\; \int_{N\cap\Gamma\backslash N} f(nz)\, dn \qquad\qquad (\text{with } N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix})$$

For fixed a, the usual characterization of $L^2(\Gamma\backslash\mathfrak{H})$ functions f with constant terms vanishing in $y \geq a$ would be that $c_P f(z) = 0$ for $y \geq a$. The intention is clear, but L^2 functions do not have pointwise values. The definition via pseudo-Eisenstein series avoids certain specious arguments.

$$c_P\big((1-\Psi_t)\cdot f\big)(iy) \;=\; \int_0^1 (1-\Psi_t) f(x+iy)\,dx$$

$$=\; (1-\varphi_t)(y)\int_0^1 f(x+iy)\,dx \;=\; (1-\varphi_t)(y)\cdot c_P f(iy)$$

Thus, for $y \ge a$ and large t, when $c_P f$ vanishes so does the constant term of $(1-\Psi_t)\cdot f$. Thus, test functions in $L^2(\Gamma\backslash\mathfrak{H})_a$ are dense in $L^2(\Gamma\backslash\mathfrak{H})_a$. ///

[4.2] The operators Δ_a, $\tilde{\Delta}_a$

Let

$$C_c^\infty(\Gamma\backslash\mathfrak{H})_a \;=\; L^2(\Gamma\backslash\mathfrak{H})_a \,\cap\, C_c^\infty(\Gamma\backslash\mathfrak{H})$$

Let Δ_a be the unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})_a$ defined by taking the operator Δ, but with domain $C_c^\infty(\Gamma\backslash\mathfrak{H})_a$. The density of test functions in $L^2(\Gamma\backslash\mathfrak{H})_a$ proves the symmetry of Δ_a, extending integration by parts on test functions. Let $\tilde{\Delta}_a$ be the Friedrichs extension of Δ_a to a self-adjoint unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})_a$. Let $\mathrm{Sob}(+1)_a$ be the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap L^2(\Gamma\backslash\mathfrak{H})_a$ with the $\mathrm{Sob}(+1)$-topology, and similarly for $\mathrm{Sob}(+2)_a$. By definition, the subspaces of test functions are dense in $\mathrm{Sob}(+1)_a$ and $\mathrm{Sob}(+2)_a$ with their finer topologies. Friedrichs' construction has the property

$$\mathrm{Sob}(+2)_a \;\subset\; \text{domain}\,\tilde{\Delta}_a \;\subset\; \mathrm{Sob}(+1)_a$$

[4.3] Distributional explication of $\tilde{\Delta}_a$

Let T_a be the order-zero distribution on $\Gamma\backslash\mathfrak{H}$ given by

$$T_a(f) \;=\; (c_P f)(a) \qquad\qquad (\text{for } f \in C_c^\infty(\Gamma\backslash\mathfrak{H})_a)$$

As observed earlier, the constant-term maps sends $\mathrm{Sob}(+1)$ to $\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1)$, and the latter is contained in continuous functions on $N\backslash G/K$, so T_a is a continuous functional on $\mathrm{Sob}(+1)$. Let $\mathscr{A}$ be the distributions on $(0,\infty)$ supported at $\{a\}$, and understand by $\mathscr{A}\circ c_P$ the composition of the constant-term map with distributions on $N\backslash G/K \approx (0,\infty)$ supported on $\{a\}$.

[4.3.1] Lemma: The domain in $L^2(\Gamma\backslash\mathfrak{H})_a$ of Friedrichs' extension $\tilde{\Delta}_a$ is

$$\text{domain}\,\tilde{\Delta}_a \;=\; \{f \in L^2(\Gamma\backslash\mathfrak{H})_a \;:\; \Delta f \in L^2(\Gamma\backslash\mathfrak{H})_a + \mathscr{A}\circ c_P\} \qquad (\text{distributional derivative } \Delta f)$$

The extension $\tilde{\Delta}_a$ is

$$\tilde{\Delta}_a f \;=\; g \qquad\qquad (\text{for } \Delta f \in g + \mathscr{A}\circ c_P \text{ with } g \in L^2(\Gamma\backslash\mathfrak{H})_a)$$

In fact, the same assertions hold with $\mathscr{A}\circ c_P$ replaced by $\mathbb{C}\cdot T_a$.

Proof: The proof consists of a review of Friedrichs' construction, computing the adjoint of a differential operator on test functions distributionally. Friedrichs characterizes the resolvent $(1-\tilde{\Delta}_a)^{-1}$ by requiring that it map to $\mathrm{Sob}(+1)_a$, and requiring

$$\langle (1-\tilde{\Delta}_a)^{-1} v, (1-\Delta) f\rangle \;=\; \langle v, f\rangle \qquad\qquad (\text{for } v \in L^2(\Gamma\backslash\mathfrak{H})_a, \text{ for } f \in C_c^\infty(\Gamma\backslash\mathfrak{H})_a)$$

The existence of $(1-\tilde{\Delta}_a)^{-1} v$ follows from Riesz-Fischer. Since f is a test function, we can compute distributionally:

$$\langle v, f\rangle \;=\; \langle (1-\tilde{\Delta}_a)^{-1} v, (1-\Delta) f\rangle \;=\; \langle (1-\Delta)(1-\tilde{\Delta}_a)^{-1} v, f\rangle$$

where the pairing is extended from $L^2(\Gamma\backslash\mathfrak{H})_a \times C_c^\infty(\Gamma\backslash\mathfrak{H})_a$ to

$$\big(\text{distributions on } \Gamma\backslash\mathfrak{H} \text{ vanishing on } \Psi_{\ge a}\big) \;\; \times \;\; C_c^\infty(\Gamma\backslash\mathfrak{H})_a$$

The distribution $u = (1-\Delta)(1-\tilde{\Delta}_a)^{-1}v$ is *not* completely determined by the conditions

$$\begin{cases} \langle u, f\rangle &=& \langle v, f\rangle & (\text{for all } f \in C_c^\infty(\Gamma\backslash\mathfrak{H})_a) \\ \\ \langle u, f\rangle &=& 0 & (\text{for all } f \in \Psi_{\ge a}) \end{cases}$$

Distributions $u-v$ annihilating $C_c^\infty(\Gamma\backslash\mathfrak{H})_a$ are necessarily supported on (the image of) the *tail* above $y = a$, namely, on

$$Y_\infty \;=\; \Gamma\backslash\big(\Gamma\cdot\{x+iy \in \mathfrak{H} : y \ge a\}\big)$$

Test functions and distributions on the tail Y_∞ can be decomposed into Fourier components, because $\Gamma_\infty\backslash N$ is *compact*. Thus, for a distribution $u-v$ to annihilate $C_c^\infty(\Gamma\backslash\mathfrak{H})_a$ requires not only that $u-v$ be supported on Y_∞, but, also, that all but the 0^{th} Fourier component of $u-v$ vanish. Thus, $u-v$ is equal to its 0^{th} Fourier component $c_P(u-v)$. Annihilation of $\Psi_{\ge a}$ implies that $u - v = c_P(u-v)$ is supported only on the boundary ∂Y_∞. Thus, the collection of possible distributions is contained in $\mathscr{A}\circ c_P$.

Conversely, if $w \in \mathrm{Sob}(+1)_a$ and $(1-\Delta)w - v = \eta\circ c_P$ with $\eta \in \mathscr{A}$, then $\langle(1-\Delta)w - v, f\rangle = 0$ for both $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})_a$ and $f \in \Psi_{\ge a}$, so $w = (\tilde{\Delta}_a - \lambda)^{-1}v$.

Identifying $N\backslash G/K \approx (0,+\infty)$ by taking the y-coordinate, the distributions $\mathscr{A}$ supported on the single point $\{a\}$ are finite linear combinations of Dirac delta (at a) and its derivatives. However, the specifics of the situation sharply limit the *order* of possible distributions, via *local* Sobolev theory, as follows. Application of the second-order differential operator $1-\Delta$ maps $\mathrm{Sob}(+1)_a$ to the local Sobolev space $\mathrm{Sob}^{\mathrm{loc}}_{\Gamma\backslash\mathfrak{H}}(-1)$ on $\Gamma\backslash\mathfrak{H}$. Application of the constant-term integral produces an element of the local Sobolev space $\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(-1)$, which we identify with $\mathrm{Sob}^{\mathrm{loc}}(-1)$ on $(0,+\infty)$. Standard Fourier series computations show that Dirac delta δ_a at $y = a$ is in $\mathrm{Sob}^{\mathrm{loc}}(-\frac{1}{2}-\varepsilon)$ for all $\varepsilon > 0$, but *not* in $\mathrm{Sob}^{\mathrm{loc}}(-\frac{1}{2})$. Thus, $\delta'_a \in \mathrm{Sob}^{\mathrm{loc}}(-\frac{3}{2}-\varepsilon)$ for all $\varepsilon > 0$, but $\delta'_a \notin \mathrm{Sob}^{\mathrm{loc}}(-\frac{3}{2})$, and so on. That is, only δ_a itself can arise in this fashion. Thus,

$$(1-\Delta)(1-\tilde{\Delta}_a)^{-1}v \;-\; v \;\;\in\;\; \mathbb{C}\cdot T_a \qquad\qquad (\text{for all } v \in L^2(\Gamma\backslash\mathfrak{H})_a)$$

as claimed. ///

[4.3.2] Remark: In particular, it is conceivable that $\tilde{\Delta}_a$ has eigenvectors whose distributional derivatives include multiples of the distribution T_a. Indeed, below we will discuss in some detail the fact that truncated Eisenstein series $\wedge^a E_s$ whose constant terms $y^s + c_s y^{1-s}$ vanish on the cut-off line $y = a$ are eigenfunctions for $\tilde{\Delta}_a$. Such truncated Eisenstein series are *not* eigenfunctions for Δ, nor for the Friedrichs self-adjoint extension $\tilde{\Delta}$ of Δ on $L^2(\Gamma\backslash\mathfrak{H})$.

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5. *Compactness of* $\mathrm{Sob}(+1)_a \to L^2(\Gamma\backslash\mathfrak{H})_a$

We claim that the inclusion $\mathrm{Sob}(+1)_a \to L^2(\Gamma\backslash\mathfrak{H})_a$, from $\mathrm{Sob}(+1)_a$ with its finer topology, is *compact*.

For proof, [Colin de Verdière 1981] cites [Lax-Phillips 1976] p. 206, to which we add some details. The *total boundedness* criterion for relative compactness requires that, given $\varepsilon > 0$, the image of the unit ball B in $\mathrm{Sob}(+1)_a$ in $L^2(\Gamma\backslash\mathfrak{H})_a$ can be covered by finitely-many balls of radius ϵ.

The idea is that the usual Rellich lemma reduces the issue to an estimate on the *tail*, which follows from the $\mathrm{Sob}(+1)_a$ condition.

The usual Rellich compactness lemma asserts the compactness of proper inclusions of Sobolev spaces on products of circles. Given $c \geq a$, cover the image Y_o of $\frac{\sqrt{3}}{2} \leq y \leq c+1$ in $\Gamma\backslash\mathfrak{H}$ by small coordinate patches U_i, and one large open U_∞ covering the image Y_∞ of $y \geq c$. Invoke compactness of Y_o to obtain a finite sub-cover of Y_o. Choose a smooth partition of unity $\{\varphi_i\}$ subordinate to the finite subcover along with U_∞, letting φ_∞ be a smooth function that is identically 1 for $y \geq c$. A function f in the Sobolev $+1$-space on Y_o is a finite sum of functions $\varphi_i \cdot f$. The latter can be viewed as having compact support on small opens in $\mathbb{R}^2$, thus identified with functions on products of circles, and lying in Sobolev $+1$-spaces there. Apply the Rellich compactness lemma to each of the finitely-many inclusion maps of Sobolev $+1$-spaces on product of circles. Thus, certainly, $\varphi_i \cdot B$ is totally bounded in $L^2(\Gamma\backslash\mathfrak{H})$.

Thus, to prove compactness of the global inclusion, it suffices to prove that, given $\varepsilon > 0$, the cut-off c can be made sufficiently large so that $\varphi_\infty \cdot B$ lies in a single ball of radius ε inside $L^2(\Gamma\backslash\mathfrak{H})$. That is, it suffices to show that

$$\lim_{c\to\infty} \int_{y>c} |f(z)|^2 \frac{dx\,dy}{y^2} \longrightarrow 0 \qquad \text{(uniformly for } |f|_{\mathrm{Fr}} \leq 1)$$

As a preliminary, we prove a reassuring, if unsurprising, lemma asserting that the Sob(+1)-norms of systematically specified families of smooth *tails* are dominated by the Sob(+1)-norms of the original functions.

Let ψ be a smooth real-valued function on $(0,+\infty)$ with

$$\begin{cases} \psi(y) = 0 & (\text{for } 0 < y \leq 1) \\ 0 \leq \psi(y) \leq 1 & (\text{for } 1 < y < 2) \\ 1 \leq \psi(y) & (\text{for } 1 \leq y) \end{cases}$$

[5.0.1] Claim: For fixed η, for $t \geq 1$, the smoothly cut-off tail $f^{[t]}(x+iy) = \psi\Big(\dfrac{y}{t}\Big) \cdot f(x+iy)$ has Sob(+1)-norm dominated by that of f itself:

$$|f^{[t]}|_{\mathrm{Sob}(+1)} \ll_\eta |f|_{\mathrm{Sob}(+1)} \qquad \text{(implied constant independent of } f \text{ and } t \geq 1)$$

Proof: Since $|a+bi|^2 = a^2+b^2$ and Δ has real coefficients, it suffices to treat real-valued f. Since $0 \leq \psi \leq 1$, certainly $|\psi f|_{L^2} \leq |f|_{L^2}$. For the other part of the Sob(+1)-norm,

$$\langle -\Delta f^{[t]}, f^{[t]}\rangle = -\int_{S^1}\int_{y\geq t} \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f^{[t]} \cdot f^{[t]}\, dx\,dy$$

$$= -\int_{S^1}\int_{y\geq t} \psi^2\Big(\frac{y}{t}\Big) f_{xx} f + \frac{1}{t^2}\psi''\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big) f^2 + \frac{2}{t}\psi'\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big) f_y f + \psi\Big(\frac{y}{t}\Big)^2 f_{yy} f \;\; dx\,dy$$

Some terms are easy to estimate: using the fact that ψ' and ψ'' are supported on $[1,2]$,

$$\int_{S^1}\int_{y\geq t} -\psi\Big(\frac{y}{t}\Big)^2 f_{xx} f + \Big|\frac{1}{t^2}\psi''\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big) f^2\Big| - \psi\Big(\frac{y}{t}\Big)^2 f_{yy} f \;\; dx\,dy \ll_\psi \int_{S^1}\int_{t\leq y\leq 2t} \frac{f^2}{t^2} - (f_{xx}f + f_{yy}f)\, dx\,dy$$

$$\leq \int_{S^1}\int_{t\leq y\leq 2t} \frac{(2t)^2 f^2}{t^2} - y^2\Big(f_{xx} + f_{yy}\Big) f \;\; \frac{dx\,dy}{y^2} \leq 4|f|^2_{L^2} - \int_{\Gamma\backslash\mathfrak{H}} \Delta f \cdot f\, \frac{dx\,dy}{y^2} \ll |f|^2_{\mathrm{Sob}(+1)}$$

with a uniform implied constant. The remaining term is usefully transformed by an integration by parts:

$$\int_{S^1}\int_{y\geq t} \frac{2}{t}\psi'\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big) f_y f \;\; dx\,dy = \int_{S^1}\int_{t\leq y\leq 2t} \frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big) \cdot \frac{\partial}{\partial y}(f^2) \;\; dx\,dy$$

$$= \int_{S^1}\int_{t\leq y\leq 2t} \frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\psi\Big(\frac{y}{t}\Big)\Big) \cdot f^2 \;\; dx\,dy$$

and then is dominated by

$$\int_{S^1}\int_{t\le y\le 2t}\Big|\frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\Big)\Big|\cdot f^2\;\;dx\,dy \;\le\; \int_{S^1}\int_{t\le y\le 2t}\Big|\frac{\partial}{\partial y}\Big(\frac{1}{t}\psi'\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)\Big)\Big|\cdot f^2\cdot(2t)^2\;\;\frac{dx\,dy}{y^2}$$

$$=\;4\int_{S^1}\int_{t\le y\le 2t}\Big|\psi''\Big(\frac{y}{t}\Big)\,\psi\Big(\frac{y}{t}\Big)+\psi'\Big(\frac{y}{t}\Big)^2\Big|\cdot f^2\;\;\frac{dx\,dy}{y^2}\;\ll_\psi\;|f|^2_{L^2}$$

with implied constant independent of f and $t\ge 1$. ///

Let the Fourier coefficients of f be $\widehat{f}(n)$. Take $c>a$ so that the 0^{th} Fourier coefficient $\widehat{f}(0)$ vanishes identically.

[5.0.2] Remark: To legitimize the following computation, recall that we proved above that $f\in\mathrm{Sob}(+1)$ has *square-integrable* first derivatives, where this differentiation is necessarily in an L^2 sense.

By Plancherel for the Fourier expansion in x, and then elementary inequalities: integrating over the part of Y_∞ above $y=c$, letting $\mathscr{F}$ be Fourier transform in x,

$$\int\int_{y>c}|f|^2\;\frac{dx\,dy}{y^2}\;\le\;\frac{1}{c^2}\int\int_{y>c}|f|^2\,dx\,dy\;=\;\frac{1}{c^2}\sum_{n\ne 0}\int_{y>c}|\widehat{f}(n)|^2\,dy$$

$$\le\;\frac{1}{c^2}\sum_{n\ne 0}(2\pi n)^2\int_{y>c}|\widehat{f}(n)|^2\,dy\;=\;\frac{1}{c^2}\sum_{n\ne 0}\int_{y>c}\Big|\mathscr{F}\frac{\partial f}{\partial x}(n)\Big|^2\,dy\;=\;\frac{1}{c^2}\int\int_{y>c}\Big|\frac{\partial f}{\partial x}\Big|^2\,dx\,dy$$

$$=\;\frac{1}{c^2}\int\int_{y>c}-\frac{\partial^2 f}{\partial x^2}\cdot\overline{f}(x)\,dx\,dy\;\le\;\frac{1}{c^2}\int\int_{y>c}-\frac{\partial^2 f}{\partial x^2}\cdot\overline{f}(x)-\frac{\partial^2 f}{\partial y^2}\cdot\overline{f}(x)\,dx\,dy$$

$$=\;\frac{1}{c^2}\int\int_{y>c}-\Delta f\cdot\overline{f}\;\frac{dx\,dy}{y^2}\;\le\;\frac{1}{c^2}\int\int_{\Gamma\backslash\mathfrak{H}}-\Delta f\cdot\overline{f}\;\frac{dx\,dy}{y^2}\;=\;\frac{1}{c^2}|f|^2_{\mathrm{Fr}}\;\le\;\frac{1}{c^2}$$

This uniform bound completes the proof that the image of the unit ball in $\mathrm{Sob}(+1)_a$ in $L^2(\Gamma\backslash\mathfrak{H})_a$ is *totally bounded*. Thus, the inclusion is a compact map. ///

[5.0.3] Corollary: For λ off a *discrete* set of points in $\mathbb{C}$, $\tilde{\Delta}_a$ has *compact resolvent* $(\tilde{\Delta}_a-\lambda)^{-1}$, and the parametrized family of compact operators

$$(\tilde{\Delta}_a-\lambda)^{-1}\;:\;L^2(\Gamma\backslash\mathfrak{H})_a\;\longrightarrow\;L^2(\Gamma\backslash\mathfrak{H})_a$$

is *meromorphic* in $\lambda\in\mathbb{C}$.

Proof: Friedrichs' construction shows that $(\tilde{\Delta}_a-\lambda)^{-1}:L^2(\Gamma\backslash\mathfrak{H})_a\to\mathrm{Sob}(+1)_a$ is continuous even with the stronger topology of $\mathrm{Sob}(+1)_a$. Thus, the composition

$$L^2(\Gamma\backslash\mathfrak{H})_a\longrightarrow\mathrm{Sob}(+1)_a\subset L^2(\Gamma\backslash\mathfrak{H})_a\qquad\text{by}\qquad f\longrightarrow(\tilde{\Delta}_a-\lambda)^{-1}f\longrightarrow(\tilde{\Delta}_a-\lambda)^{-1}f$$

is the composition of a continuous operator with a compact operator, so is compact. Thus,

$$(\tilde{\Delta}_a-\lambda)^{-1}\;:\;L^2(\Gamma\backslash\mathfrak{H})_a\longrightarrow L^2(\Gamma\backslash\mathfrak{H})_a\qquad\text{is a compact operator}$$

We claim that, for a (not necessarily bounded) normal operator T, if T^{-1} exists and is *compact*, then $(T-\lambda)^{-1}$ exists and is a compact operator for λ off a *discrete* set in $\mathbb{C}$, and is *meromorphic* in λ.[5] To

[5] This assertion and its proof are standard. For a similar version in a standard source, see [Kato 1966], p. 187 and preceding. The same compactness and meromorphy assertion plays a role in the (somewhat apocryphal) Selberg-Bernstein treatment of the meromorphic continuation of Eisenstein series.

prove the claim, first recall from the spectral theory of normal compact operators, the non-zero spectrum of compact T^{-1} is all *point spectrum*. We claim that the spectrum [6] of T and non-zero spectrum of T^{-1} are in the bijection $\lambda \leftrightarrow \lambda^{-1}$. From the algebraic identities

$$T^{-1} - \lambda^{-1} \;=\; T^{-1}(\lambda - T)\lambda^{-1} \qquad\qquad T - \lambda \;=\; T(\lambda^{-1} - T^{-1})\lambda$$

failure of either $T - \lambda$ or $T^{-1} - \lambda^{-1}$ to be *injective* forces the failure of the other, so the point spectra are identical. For (non-zero) λ^{-1} not an eigenvalue of *compact* T^{-1}, $T^{-1} - \lambda^{-1}$ is injective *and* has a continuous, everywhere-defined inverse. [7] For such λ, inverting the relation $T - \lambda = T(\lambda^{-1} - T^{-1})\lambda$ gives

$$(T - \lambda)^{-1} \;=\; \lambda^{-1}(\lambda^{-1} - T^{-1})^{-1}T^{-1}$$

from which $(T-\lambda)^{-1}$ is continuous and everywhere-defined. That is, λ is *not* in the spectrum of T. Finally, $\lambda = 0$ is not in the spectrum of T, because T^{-1} exists and is continuous. This establishes the bijection.

Thus, when T^{-1} is compact, the spectrum of T is *countable*, with no accumulation point in $\mathbb{C}$. Letting $R_\lambda = (T - \lambda)^{-1}$, the resolvent relation

$$R_\lambda \;=\; (R_\lambda - R_0) + R_0 \;=\; (\lambda - 0)R_\lambda R_0 + R_0 \;=\; (\lambda R_\lambda + 1)\circ R_0$$

expresses R_λ as the composition of a continuous operator with a compact operator, proving its compactness. ///

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6. *Discreteness of cuspforms*

We claim that the space $L^2_{\mathrm{cfm}}(\Gamma\backslash\mathfrak{H})$ has a Hilbert space basis of eigenfunctions for Δ.

The compactness of the inclusion $j_a : \mathrm{Sob}(+1)_a \to L^2(\Gamma\backslash\mathfrak{H})_a \subset L^2(\Gamma\backslash\mathfrak{H})$, proven above, is the bulk of the proof. Nevertheless, the argument should be made sufficiently clear to distinguish it from fallacious arguments that may seem to prove that truncated Eisenstein series decompose discretely in $L^2(\Gamma\backslash\mathfrak{H})$, or are eigenfunctions for Δ or its self-adjoint extension [8] $\tilde\Delta$.

[6.1] $(\tilde\Delta - \lambda)^{-1}$ does not stabilize $L^2(\Gamma\backslash\mathfrak{H})_a$

Friedrichs' construction shows that $(\tilde\Delta - \lambda)^{-1} : L^2(\Gamma\backslash\mathfrak{H}) \to \mathrm{Sob}(+1)$ is continuous even with the stronger topology of $\mathrm{Sob}(+1)$. Thus, on a subspace $H \subset L^2(\Gamma\backslash\mathfrak{H})$ mapped by $(\tilde\Delta - \lambda)^{-1}$ to

$$\mathrm{Sob}(+1)_a \;=\; \mathrm{Sob}(+1) \,\cap\, L^2(\Gamma\backslash\mathfrak{H})_a$$

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[6] Recall that for a possibly-unbounded operator T with *dense domain* D_T, the *point spectrum* or *discrete spectrum* (set of eigenvalues) consists of $\lambda \in \mathbb{C}$ such that $T - \lambda$ fails to be *injective*. The *continuous* spectrum consists of λ with $T - \lambda$ is *injective* and *dense, non-closed image*, but *bounded* inverse $(T-\lambda)^{-1}$ on $(T-\lambda)D_T$. The *residual spectrum* consists of λ with $T - \lambda$ injective, but $(T-\lambda)D_T$ not dense. The definition of *continuous spectrum* simplifies for *closed* T: we claim that for $(T-\lambda)^{-1}$ densely defined and continuous, the image $(T-\lambda)D_T$ is the whole space, so $(T-\lambda)^{-1}$ is *everywhere defined*. Indeed, the continuity gives a constant C such that $|x| \le C\cdot|(T-\lambda)x|$ for all $x \in D_T$. Then $(T-\lambda)x_i$ Cauchy implies x_i Cauchy, and closedness of the graph of T implies that $T(\lim x_i) = \lim Tx_i$. Since $(T-\lambda)D_T$ is dense, it is the whole space.

[7] That $S - \lambda$ is *surjective* for compact normal S and $\lambda \neq 0$ not an eigenvalue is an easy part of *Fredholm theory*.

[8] It suffices to argue in terms of the Friedrichs extension $\tilde\Delta$ of Δ. In fact, various (non-trivial) arguments prove that the (graph) *closure* of Δ is its unique self-adjoint extension. This has the extra interest of distinguishing $\tilde\Delta$ from the operators $\tilde\Delta_a$.

the composite is compact:

$$j_a \circ (\tilde{\Delta} - \lambda)^{-1} \;:\; H \longrightarrow \text{Sob}(+1)_a \longrightarrow L^2(\Gamma\backslash\mathfrak{H}) \qquad = \text{ compact}$$

However, as observed below, no individual $L^2(\Gamma\backslash\mathfrak{H})_a$ is stable under $(\tilde{\Delta} - \lambda)^{-1}$. Only the *intersection*

$$\bigcap_{a>0} L^2(\Gamma\backslash\mathfrak{H})_a \;=\; L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$$

is $(\tilde{\Delta} - \lambda)^{-1}$-stable. Indeed, for $f \in L^2(\Gamma\backslash\mathfrak{H})_a$ and $\Psi_\varphi \in \Psi_{\geq a}$, noting that Ψ_φ is smooth,

$$\langle (\tilde{\Delta} - \lambda)^{-1} f, \Psi_\varphi \rangle \;=\; \langle f, (\tilde{\Delta} - \overline{\lambda})^{-1} \Psi_\varphi \rangle \;=\; \langle f, (\Delta - \overline{\lambda})^{-1} \Psi_\varphi \rangle \;=\; \langle f, \Psi_{(\Delta - \overline{\lambda})^{-1}\varphi} \rangle$$

by the G-invariance of Δ. As we show, $(\Delta - \overline{\lambda})^{-1}\varphi$ rarely has compact support, but will still have sufficient decay to form corresponding pseudo-Eisenstein series. Pseudo-Eisenstein series formed with a broader class of data φ enter the proof (just below) that $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is $(\tilde{\Delta} - \lambda)^{-1}$-stable.

Thus, for $L^2(\Gamma\backslash\mathfrak{H})_a$ to be $(\tilde{\Delta} - \lambda)^{-1}$-stable would require that the support of $(\Delta - \overline{\lambda})^{-\ell}\varphi$ be inside $[a, +\infty)$ for all $0 < \ell \in \mathbb{Z}$. This fails for essentially all test functions φ, verified as follows.

The question is of the support of a solution F to differential equations of the form

$$\Big(y^2 \frac{d^2}{dy^2} - \lambda\Big)^\ell F \;=\; \varphi \qquad\qquad (\text{with } 0 < \ell \in \mathbb{Z})$$

or

$$\Big(\Big(y\frac{d}{dy}\Big)^2 - y\frac{d}{dy} - \lambda\Big)^\ell F \;=\; \varphi$$

with $\varphi \in C_c^\infty[a, +\infty)$. In coordinates $y = e^x$, with $u(x) = F(e^x)$ and $v(x) = \varphi(e^x)$, the equation is

$$\Big(\frac{d^2}{dx^2} - \frac{d}{dx} - \lambda\Big)^\ell u \;=\; v$$

Taking Fourier transforms, this is

$$(-x^2 + ix - \lambda)^\ell\, \widehat{u} \;=\; \widehat{v}$$

and

$$\widehat{u} \;=\; \frac{\widehat{v}}{(-x^2 + ix - \lambda)^\ell}$$

The compact support of v implies that $\widehat{v}$ extends to $\mathbb{C}$ and is *entire*, with explicable growth.

For u to be supported on a half-line $[-A, +\infty)$ with $A > 0$, $\widehat{u}$ must extend to the lower half-plane in $\mathbb{C}$, with growth constraint

$$\widehat{u}(\xi + i\eta) \;\ll_\xi\; e^{A|\eta|} \qquad\qquad (\text{for } \eta < 0)$$

Letting $\lambda = s(s-1)$, the zeros of $x^2 - ix + \lambda = 0$ are at $x = is, i(1-s)$. Given the relation to $\widehat{v}$, for $\widehat{u}$ to be supported on $[-A, \infty)$, $\widehat{v}$ must vanish to order at least ℓ at whichever of $is, i(1-s)$ is in the lower half-plane. However, for given v, this must be true for *all* ℓ, which is impossible.

That is, there is no non-trivial space of $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ consisting of functions supported on $[-A, +\infty)$ and stable under solution of $u'' - u' - \lambda u = v$. Thus, under $(\tilde{\Delta} - \lambda)^{-1}$ the space $\Psi_{\geq a}$ of pseudo-Eisenstein series is mapped to a space of automorphic forms with supports (in a fundamental domain) *not* confined to $y \geq a$. That is, no individual space $L^2(\Gamma\backslash\mathfrak{H})_a$ is $(\tilde{\Delta} - \lambda)^{-1}$-stable.

[6.2] $(\tilde{\Delta}-\lambda)^{-1}$ stabilizes $L^2_{\rm cfm}(\Gamma\backslash\mathfrak{H})$

This stability property implies that $(\tilde{\Delta}-\lambda)^{-1}$ restricted to $L^2_{\rm cfm}(\Gamma\backslash\mathfrak{H})$ is a compact operator.

Proof: The space of L^2 cuspforms can be characterized as the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to the space of pseudo-Eisenstein series Ψ_φ with arbitrary data $\varphi\in C^\infty_c(0,+\infty)$. However, the relation

$$\langle(\tilde{\Delta}-\lambda)^{-1}f,\Psi_\varphi\rangle \;=\; \langle f,(\tilde{\Delta}-\overline{\lambda})^{-1}\Psi_\varphi\rangle \;=\; \langle f,(\Delta-\overline{\lambda})^{-1}\Psi_\varphi\rangle \;=\; \langle f,\Psi_{(\Delta-\overline{\lambda})^{-1}\varphi}\rangle$$

suggests considering a class of data φ closed under solution of the corresponding differential equation. Letting $y=e^x$ and $\varphi(e^x)=v(x)$, as above, the differential equation is

$$u''-u'-\lambda u \;=\; v$$

Taking Fourier transform,

$$\widehat{u} \;=\; \frac{-\widehat{v}}{x^2-ix+\lambda}$$

With $\lambda=s$, the zeros of the denominator are at is and $i(1-s)$. Taking s large positive real moves these poles as far away from the real line as desired. Thus, from Paley-Wiener-type considerations, if $\widehat{v}$ were holomorphic on the strip $|{\rm Im}\,(\xi)|\le N$, and integrable and square-integrable on horizontal lines inside that strip, certainly the same will be true of $\widehat{u}$. The inverse Fourier transform will have a bound $e^{-N|x|}$.

The corresponding function u on $(0,+\infty)$ will be bounded by y^N as $y\to 0^+$, and by y^{-N} as $y\to+\infty$. A soft argument (for example, via *gauges*) proves good convergence of the associated pseudo-Eisenstein series.

Thus, we can redescribe the space of cuspforms to make visible the stability under $(\tilde{\Delta}-\lambda)^{-1}$. This completes the proof that $L^2_{\rm cfm}(\Gamma\backslash\mathfrak{H})$ decomposes discretely, that is, has an orthonormal Hilbert space basis of $\tilde{\Delta}$-eigenvectors. ///

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7. *Meromorphic continuation beyond the critical line*

[7.1] Unique characterization

Similar to the description of E_s as $\tilde{E}_s$ above, but with $\tilde{\Delta}_a$ in place of $\tilde{\Delta}$, with the pseudo-Eisenstein series h_s as earlier, put

$$\tilde{E}_{a,s} \;=\; h_s \;-\; (\tilde{\Delta}_a-\lambda)^{-1}\,(\Delta-\lambda)\,h_s \qquad\qquad (\text{with } \lambda=s(s-1))$$

Indeed, $(\Delta-\lambda)h_s$ is in $L^2(\Gamma\backslash\mathfrak{H})_a$. For $\lambda=s(s-1)$ not a non-positive real, $(\tilde{\Delta}_a-\lambda)^{-1}$ is a bijection of $L^2(\Gamma\backslash\mathfrak{H})_a$ to the domain of $\tilde{\Delta}_a$, so $u=\tilde{E}_{s,a}-h_s$ is the *unique* element of the domain of $\tilde{\Delta}_a$ satisfying

$$(\tilde{\Delta}_a-\lambda)\,u \;\;=\;\; -(\Delta-\lambda)\,h_s$$

[7.2] Meromorphy

Since the pseudo-Eisenstein series h_s is entire, the meromorphy of the resolvent $(\tilde{\Delta}_a-\lambda)^{-1}$ yields the meromorphy of $\tilde{E}_{a,s}$.

[7.3] Constant term of $\tilde{E}_{a,s}$

By Friedrichs' construction, the domain of $\tilde{\Delta}_a$ is inside $\mathrm{Sob}(+1)_a$. The constant-term projection [9] maps to the local Sobolev space $\mathrm{Sob}^{\mathrm{loc}}_{N\backslash G/K}(+1)$. On the one-dimensional $N\backslash G/K$, this Sobolev space is inside *continuous* functions, by Sobolev imbedding.

Since $(\tilde{\Delta}_a - \lambda)^{-1}$ maps $(\Delta - \lambda)h_s$ to a function with constant term vanishing above $y = a$, above $y = a$ the constant term of $\tilde{E}_{a,s}$ is that of h_s, namely, y^s. More generally, evaluate $\tilde{\Delta}_a - \lambda$ distributionally by application of $\Delta - \lambda$: for some constant C_s,

$$-(\Delta - \lambda)h_s \;=\; (\tilde{\Delta}_a - \lambda)(\tilde{E}_{a,s} - h_s) \;=\; (\Delta - \lambda)(\tilde{E}_{a,s} - h_s) + C_s \cdot T_a \qquad \text{(as distributions)}$$

Thus,

$$(\Delta - \lambda)\tilde{E}_{a,s} \;=\; -C_s \cdot T_a \qquad \text{(as distributions)}$$

Since Δ is invariant, it commutes with the constant-term map, and the distribution $(\Delta - \lambda)c_P\tilde{E}_{a,s}$ is 0 away from $y = a$. The distributional differential equation

$$\Big(y^2 \frac{\partial^2}{\partial y^2} - s(s-1)\Big)\, u \;=\; 0 \qquad \text{(on } 0 < y < a\text{)}$$

has solutions $A_s y^s + B_s y^{1-s}$ for some A_s, B_s. Since $\tilde{E}_{a,s}$ is meromorphic, so are A_s, B_s. In summary,

$$c_P\tilde{E}_{a,s}(z) \;=\; \begin{cases} y^s & \text{(for } y > a\text{)} \\ A_s y^s + B_s y^{1-s} & \text{(for } 0 < y < a\text{)} \end{cases}$$

The *continuity* of the constant term explains what happens at $y = a$:

$$A_s \cdot a^s + B_s \cdot a^{1-s} \;=\; a^s$$

The latter relation shows that neither A_s nor B_s is identically 0.

[7.4] Meromorphic continuation of E_s

Let $\mathrm{ch}_{[a,\infty)}$ be the characteristic function of $[a, \infty)$, let

$$\varphi_s(y) \;=\; \mathrm{ch}_{[a,\infty)}(y) \cdot \big(A_s y^s + B_s y^{1-s} - y^s\big)$$

and form a pseudo-Eisenstein series

$$\Phi_s(z) \;=\; \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_s(\mathrm{Im}(\gamma z))$$

The support of φ_s is inside $[a, \infty)$, so for each $z \in \mathfrak{H}$ the series has at most one non-zero summand, so converges for all $s \in \mathbb{C}$.

[7.4.1] Theorem:

$$A_s \cdot E_s \;=\; \tilde{E}_{a,s} + \Phi_s$$

This gives the meromorphic continuation of E_s.

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[9] We do not lose anything in the Sobolev index when *projecting* to the constant term, in contrast to general *trace theorems*, in which one loses an index of $m/2$ by restricting by codimension m.

Proof: We have shown that $u = E_s - h_s$ is the unique solution in Sob(+1) to

$$(\tilde{\Delta} - \lambda)\, u \;=\; -(\Delta - \lambda)\, h_s$$

Thus, multiplying through by A_s, it suffices prove that $\tilde{E}_{a,s} + \Phi_s - A_s \cdot h_s$ is in Sob(+1) and satisfies

$$(\tilde{\Delta} - \lambda)\,(\tilde{E}_{a,s} + \Phi_s - A_s \cdot h_s) \;=\; -(\Delta - \lambda)\,(A_s \cdot h_s)$$

The fact that $\tilde{E}_{a,s} - h_s$ is in $\mathrm{Sob}(+1)_a$ motivates the rearrangement

$$\tilde{E}_{a,s} + \Phi_s - A_s \cdot h_s \;=\; (\tilde{E}_{a,s} - h_s) + (\Phi_s - A_s\, h_s + h_s)$$

Thus, we must show that the pseudo-Eisenstein series $F = \Phi_s - A_s h_s + h_s$ is in Sob(+1).

Regarding integrability, by reduction theory, Φ_s is just φ_s on $y > a$, so

$$F \;=\; \Phi_s - A_s h_s + h_s \;=\; (A_s y^s + B_s y^{1-s} - y^s) - A_s y^s + y^s \;=\; B_s y^{1-s} \qquad \text{(for } y > a\text{)}$$

For $\mathrm{Re}(s) > 1$, y^{1-s} is square-integrable on $y > a$, so F is in $L^2(\Gamma\backslash\mathfrak{H})$.

To demonstrate the additional smoothness required for F to be in Sob(+1), from the discussion of Sobolev semi-norms, it suffices to show that the right-derivatives αF are in $L^2(\Gamma\backslash G)$ for $\alpha \in \mathfrak{g}$. By the left invariance of the right action of $\mathfrak{g}$, it suffices to prove square-integrability, on standard Siegel sets, of the derivatives of the data $\varphi_s - (A_s - 1)\tau y^s$ used to form the pseudo-Eisenstein series. This data is smooth everywhere but at $y = a$, where it is *continuous*, since $A_s a^s + B_s a^s - a^s = 0$. Further, it possesses continuous left and right derivatives at $y = a$, so is locally in a +1-index Sobolev space at $y = a$. The data is left N-invariant and right K-invariant, and A^+ normalizes N, so we need only consider the differential operator $y\frac{\partial}{\partial y}$ coming from the Lie algebra of A^+: the derivative is discontinuous at $y = a$, and as a distribution it is

$$y\frac{\partial}{\partial y}\Big(\varphi_s + (A_s - 1)\tau y^s\Big) \;=\; \begin{cases} B_s(1-s)y^{1-s} & \text{(for } y > a\text{)} \\ -s(A_s - 1)\tau y^s & \text{(for } b' \le y < a\text{)} \\ (A_s - 1)(\tau y^s + y\tau' y^s) & \text{(for } b \le y \le b'\text{)} \\ 0 & \text{(for } y \le b\text{)} \end{cases}$$

For $\mathrm{Re}(s) > 1$ this derivative is square-integrable on standard Siegel sets. Thus, $\Phi_s - A_s h_s + h_s$ is in Sob(+1), proving that $\tilde{E}_{a,s} + \Phi_s - A_s h_s$ is in Sob(+1).

To show that $\tilde{E}_{a,s} + \Phi_s - A_s h_s$ satisfies the expected equation, we justify computing the effect of differential operators on $\tilde{E}_{a,s} + \Phi_s - A_s h_s$ distributionally, as follows. For $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, using complex-bilinear pairings,

$$\langle(\tilde{\Delta} - \lambda)(\tilde{E}_{a,s} + \Phi_s - A_s h_s),\, f\rangle \;=\; \langle \tilde{E}_{a,s} + \Phi_s - A_s h_s,\, (\Delta - \lambda)f\rangle \;=\; \langle(\Delta - \lambda)(\tilde{E}_{a,s} + \Phi_s - A_s h_s),\, f\rangle$$

By design, using the invariance of Δ,

$$(\Delta - \lambda)(\tilde{E}_{a,s} + \Phi_s) \;=\; (\Delta - \lambda)\tilde{E}_{a,s} + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Delta - \lambda)\varphi_s\circ\gamma \;=\; -C_s\cdot T_a + C_s\cdot T_a \;=\; 0 \qquad \text{(as distributions)}$$

Thus,

$$(\tilde{\Delta} - \lambda)(\tilde{E}_{a,s} + \Phi_s - A_s h_s) \;=\; (\Delta - \lambda)(\tilde{E}_{a,s} + \Phi_s - A_s h_s) \;=\; 0 - A_s(\Delta - \lambda)h_s$$

as desired, proving $\tilde{E}_{a,s} + \Phi_s = E_s$ for $\mathrm{Re}(s) > 1$. The meromorphic continuation of $\tilde{E}_{a,s} + \Phi_s$ then gives that of E_s. ///

[7.5] Functional equation

For $\mathrm{Re}(1-s) > 1$, by the same argument, $\tilde{E}_{a,s} + \Phi_s = B_s E_{1-s}$. Thus, with $a(s) = B_s/A_s$,

$$\begin{cases} E_{1-s} &= a(s)\cdot E_s \\ \varphi_s \cdot a(1-s) &= 1 \end{cases}$$

Since $c_P E_s = y^s + c_s y^{1-s}$, apparently $c_s = a(s) = B_s/A_s$. We have

$$a(\overline{s}) \;=\; \overline{a(s)}$$

Thus, on $\mathrm{Re}(s) = \frac{1}{2}$, where $\overline{s} = 1-s$,

$$|a(s)| \;=\; 1 \qquad\qquad (\text{on } \mathrm{Re}(s) = \tfrac{1}{2})$$

In particular, $c_s a(s)$ has no pole on $\mathrm{Re}(s) = \frac{1}{2}$. Since $a(s)$ has no poles on $\mathrm{Re}(s) = \frac{1}{2}$, via *Maass-Selberg relations*, computing the L^2 norm of the *truncated* Eisenstein series, E_s itself has no poles on $\mathrm{Re}(s) = \frac{1}{2}$.

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8. *Discrete decomposition of truncated Eisenstein series*

The space $L^2(\Gamma\backslash\mathfrak{H})_a$ of L^2 automorphic forms with constant terms varnishing about $y = a$ is much larger than the discrete spectrum (cuspforms and constants) of Δ or $\tilde{\Delta}$ on $L^2(\Gamma\backslash\mathfrak{H})$. Yet $\tilde{\Delta}_a$ decomposes the whole space $L^2(\Gamma\backslash\mathfrak{H})_a$ *discretely*. How can this be?

[8.1] Certain truncated Eisenstein series are eigenfunctions

Truncated Eisenstein series $\wedge^a E_s$ are not eigenfunctions for the differential operator Δ. See [Garrett 2009] for a discussion.

Nevertheless, for fixed a, for s such that $a^s + c_s a^{1-s} = 0$, the truncation $\wedge^a E_s$ is in $\mathrm{Sob}(+1)_a$, and *is* an eigenfunction for $\tilde{\Delta}_a$, since with $a^s + c_s a^{1-s} = 0$ the distributional derivative $(\Delta - \lambda_s)\wedge^a E_s$ is indeed a constant multiple of the distribution T_a considered above. Thus, from that discussion, $(\tilde{\Delta}_a - \lambda_s)\wedge^a E_s = 0$.

Note that $\tilde{\Delta}_a$ is not quite a differential operator.

Still, $\tilde{\Delta}_a$ is non-negative and self-adjoint, so all eigenvalues are non-positive real. Thus, the only truncated Eisenstein series $\wedge^a E_s$ which are eigenfunctions for $\tilde{\Delta}_a$ for *some* a must have $s \in \frac{1}{2} + i\mathbb{R}$ or $s \in [0,1]$.

On the other hand, given an Eisenstein series E_s with $\mathrm{Re}(s) = \frac{1}{2}$ and $s \neq \frac{1}{2}$, there are infinitely-many cut-off values $a > 1$ for which $\wedge^a E_s$ is an eigenfunction for $\tilde{\Delta}_a$. This amounts to solving for $a > 1$ in $a^{2s-1} = -c_s$, that is,

$$\log a \;=\; \frac{\log(-c_s)}{2s-1}$$

Conveniently, by various arguments, (including Colin de Verdière's, as below), $|c_s| = 1$ on $\mathrm{Re}(s) = \frac{1}{2}$. The ambiguity of $\log(-c_s)$ by $2\pi i\mathbb{Z}$ gives infinitely-many values $a > 1$ satisfying the condition, with logarithms differing by integer multiples of $\pi/\mathrm{Im}(s)$.

Real s in $(\frac{1}{2}, 1]$ behaves differently. For these simplest Eisenstein series, by various means one can show that c_s has a simple pole at $s = 1$, with positive residue. Thus, as $s \to 1^-$ the function c_s is real-valued and goes to $-\infty$, so

$$\log a \;=\; \frac{-c_s}{2s-1} \;=\; \frac{c_s}{1-2s} \;\longrightarrow\; +\infty \qquad\qquad (\text{as } s \to 1^-)$$

In the case at hand, c_s has no poles in $(\frac{1}{2}, 1)$. Thus, every large-enough $a > 1$ has exactly one $\frac{1}{2} < s < 1$ with $\wedge^a E_s$ an eigenfunction for $\tilde{\Delta}_a$.

[8.2] Orthogonality and inner products

For $s \neq z, 1-z$, the eigenvalues $\lambda_s = s(s-1)$ and $\lambda_z = z(z-1)$ are distinct, so when both $a^s + c_s a^{1-s} = 0$ and $a^z + c_s a^{1-z} = 0$, by properties of self-adjoint operators,

$$\langle \wedge^a E_s, \wedge^a E_z \rangle \;=\; 0 \qquad\qquad (\text{for } a^s + c_s a^{1-s} = 0 \text{ and } a^z + c_s a^{1-z} = 0 \text{ and } s \neq z, 1-z)$$

For general s, z with $s \neq z, 1-z$, the Maaß-Selberg relation (see appendix) with $\mathbb{C}$-bilinear pairing is

$$\langle \wedge^a E_s, \wedge^a E_z \rangle \;=\; \frac{a^{s+z-1}}{s+z-1} + c_s \frac{a^{(1-s)+z-1}}{(1-s)+z-1} + c_z \frac{a^{s+(1-z)-1}}{s+(1-z)-1} + c_s c_z \frac{a^{(1-s)+(1-z)-1}}{(1-s)+(1-z)-1}$$

For $s \neq z, 1-z$, the denominators do not vanish. With the conditions $a^s + c_s a^{1-s} = 0$ and $a^z + c_s a^{1-z} = 0$, the right-hand side of the Maaß-Selberg relation is easily seen to vanish, giving another proof of the orthogonality of $\wedge^a E_s$ and $\wedge^a E_s$.

The L^2 norm of truncated Eisenstein series $\wedge^a E_s$ with $a^s + c_s a^{1-s} = 0$ and $\mathrm{Re}(s) = \frac{1}{2}$ is readily computed by a limiting process in the Maaß-Selberg relation (see appendix):

$$\| \wedge^a E_{\frac{1}{2}+it} \|^2 \;=\; 2\log a + c_{\frac{1}{2}+it} \frac{a^{-2it}}{-2it} + c_{\frac{1}{2}-it} \frac{a^{2it}}{2it} \;-\; \tfrac{1}{2}\Big(c'_{\frac{1}{2}+it} c_{\frac{1}{2}-it} + c_{\frac{1}{2}+it} c'_{\frac{1}{2}-it} \Big)$$

For $a^s + c_s a^{1-s} = 0$ but general z, there is only partial collapse of the Maaß-Selberg relation, to

$$\begin{aligned} \langle \wedge^a E_s, \wedge^a E_z \rangle \quad &= \quad a^{s+z-1} \Big(\frac{1}{s+z-1} - \frac{1}{(1-s)+z-1} \Big) \\ + \; c_z a^{s+(1-z)-1} & \Big(\frac{1}{s+(1-z)-1} - \frac{1}{(1-s)+(1-z)-1} \Big) \end{aligned}$$

[8.3] Discrete decomposition versus continuous decomposition

The orthogonal complement to cuspforms in $L^2(\Gamma\backslash\mathfrak{H})_a$ is large, and it decomposes *discretely* for $\tilde{\Delta}_a$, that is, is spanned by genuine eigenfunctions for $\tilde{\Delta}_a$. Our prior discussion shows that the truncated Eisenstein series lying in $\mathrm{Sob}(+1)_a$, that is, with $a^s + c_s a^{1-s}$, are eigenvectors for $\tilde{\Delta}_a$.

In fact, conversely, at least assuming $\lambda_z < -1/4$, any $\tilde{\Delta}_a$ λ_z-eigenfunction f is such a truncated Eisenstein series.

A complete proof of this requires an extension of the usual automorphic Plancherel theorem to a global automorphic L^2 Sobolev theory, so we merely sketch the argument here.

The distribution T_a is compactly supported on $\Gamma\backslash\mathfrak{H}$, and is in $\mathrm{Sob}(-\frac{1}{2} - \varepsilon)$ for all $\varepsilon > 0$, so is in a *global* automorphic Sobolev space $\mathrm{Sob}^{\mathrm{afc}}(-\frac{1}{2} - \varepsilon)$. That is,

$$\frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \int_0^\infty \frac{|T_a(E_{1-s})|^2}{|\lambda_s|^{2(\frac{1}{2}+\varepsilon)}} \, ds \;<\; \infty \qquad\qquad (\text{for all } \varepsilon > 0)$$

and in the corresponding topology

$$T_a \;=\; \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} T_a(E_{1-s}) \, E_s \, ds \;=\;=\; \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (a^{1-s} + c_{1-s} a^s) \, E_s \, ds$$

For a $\tilde{\Delta}_a$ λ_z-eigenfunction f, solving the equation $(\Delta - \lambda_z) f = T_a$ gives

$$f \;=\; \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \frac{a^{1-s} + c_{1-s} a^s}{\lambda_s - \lambda_z} \, E_s \, ds$$

Since f is in $L^2(\Gamma\backslash\mathfrak{H})$, Plancherel gives

$$\|f\|^2 \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \Big|\frac{a^{1-s}+c_{1-s}a^s}{\lambda_s-\lambda_z}\Big|^2\,ds$$

Since the denominator vanishes at $s = 1-z$, it must be that the numerator vanishes there also. Thus, $a^z + c_z a^{1-z} = 0$, and $\wedge^a E_z$ is an eigenfunction.

Away from (images of) $y = a$ the function f is locally a genuine eigenfunction for Δ. Further, $c_P f = 0$ above $y = a$, and $(y^2\frac{\partial^2}{\partial y^2} - \lambda_z)c_P f = 0$ below $y = a$. The solutions to the latter equation are of the form $Ay^z + By^{1-z}$. Since eigenfunctions are in $\mathrm{Sob}(+1)_a$, their constant terms are *continuous*, so $Aa^z + Ba^{1-z} = 0$. Since $a^z + c_z a^{1-z} = 0$, necessarily $f - A\cdot\wedge^a E_z$ has vanishing constant term. Since this is orthogonal to cuspforms, it is 0. This completes the sketch of the argument.

[8.3.1] Remark: This discrete decomposition does not contradict the standard *continuous* decomposition of this orthogonal complement as integrals of E_s on the critical line.

[8.3.2] Remark: Just to be clear: even truncated Eisenstein series $\wedge^a E_z$ with $\lambda_z \notin \mathbb{R}$ decompose discretely in $L^2(\Gamma\backslash\mathfrak{H})_a$ for $\tilde{\Delta}_a$, as linear combinations of those $\wedge^a E_z$ with $a^s + c_s a^{1-s} = 0$.

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9. *Appendix: Friedrichs extensions*

We recall Friedrichs' construction of *self-adjoint extensions* of symmetric, half-bounded, densely-defined (unbounded) operators on Hilbert spaces.

[9.1] Symmetric operators and adjoints

Write $A \subset B$ for not-everywhere-defined operators on a Hilbert space when the domain of A is a subset of the domain of B and A, B agree on the domain of A. An operator T is *symmetric* when $T \subset T^*$, and *self-adjoint* when $T = T^*$. These comparisons refer to the *domains* of these not-everywhere-defined operators. In the following claim and its proof, the domain of a map S on V is incorporated in a reference to its *graph*

$$\text{graph } S \;=\; \{v \oplus Sv \;:\; v \in \text{domain } S\} \;\subset\; V \oplus V$$

The direct sum $V \oplus V$ is a Hilbert space, with natural inner product

$$\langle v \oplus v', w \oplus w'\rangle \;=\; \langle v, v'\rangle + \langle w, w'\rangle$$

Define an isometry U of $V \oplus V$ by

$$U \;:\; V \oplus V \longrightarrow V \oplus V \qquad \text{by} \qquad v \oplus w \longrightarrow -w \oplus v$$

The adjoint T^* is characterized by its graph, which is the orthogonal complement in $V \oplus V$ to an image of the graph of T, namely,

$$\text{graph } T^* \;=\; \text{orthogonal complement of } U(\text{graph } T)$$

[9.1.1] Corollary: For $T_1 \subset T_2$ with dense domains, $T_2^* \subset T_1^*$, and $T_1 \subset T_1^{**}$. ///

[9.1.2] Corollary: A self-adjoint operator has a closed graph. ///

[9.1.3] Remark: The closed-ness of the graph of a self-adjoint operator is essential in proving existence of *resolvents*, below.

[9.1.4] Proposition: Eigenvalues for symmetric operators T, D are *real.*

Proof: Suppose $0 \neq v \in D$ and $Tv = \lambda v$. Then

$$\lambda\langle v, v\rangle \;=\; \langle \lambda v, v\rangle \;=\; \langle Tv, v\rangle \;=\; \langle v, T^* v\rangle \qquad \text{(because } v \in D \subset D^*\text{)}$$

Further, because T^* agrees with T on D,

$$\langle v, T^* v\rangle \;=\; \langle v, \lambda v\rangle \;=\; \overline{\lambda}\bar{v}, v\rangle$$

Thus, λ is real. ///

A densely-defined symmetric operator T, D is *positive* (or *non-negative*) when

$$\langle Tv, v\rangle \;\geq\; 0 \qquad \text{(for all } v \in D\text{)}$$

Certainly all the eigenvalues of a positive operator are non-negative real.

[9.2] Friedrichs' extension

[9.2.1] Theorem: (Friedrichs) A *positive*, densely-defined, symmetric operator T, D has a positive *self-adjoint* extension.

Proof: [10] Define a new hermitian form $\langle,\rangle_1$ and corresponding norm $\|\cdot\|_1$ by

$$\langle v, w\rangle_1 \;=\; \langle v, w\rangle + \langle Tv, w\rangle \qquad \text{(for } v, w \in D\text{)}$$

The symmetry and non-negativity of T make this positive-definite hermitian on D. Note that $\langle v, w\rangle_1$ makes sense when at least one of v, w is in D.

Let D_1 be the closure in V of D with respect to the metric d induced by $\|\cdot\|$. We claim that D_1 is also the d-completion of D. Indeed, for v_i a d-Cauchy sequence in D, v_i is Cauchy in V in the original topology, since

$$|v_i - v_j| \;\leq\; |v_i - v_j|_1$$

For two sequences v_i, w_j with the same d-limit v, the d-limit of $v_i - w_i$ is 0. Thus,

$$|v_i - w_i| \;\leq\; |v_i - w_i|_1 \;\longrightarrow\; 0$$

For $h \in V$ and $v \in D_1$, the functional $\lambda_h : v \to \langle v, h\rangle$ has a bound

$$|\lambda_h v| \;\leq\; |v| \cdot |h| \;\leq\; |v|_1 \cdot |h|$$

Thus, the norm of the functional λ_h on D_1 is at most $|h|$. By Riesz-Fischer, there is unique Bh in the Hilbert space D_1 with $|Bh|_1 \leq |h|$, such that

$$\lambda_h v \;=\; \langle Bh, v\rangle_1 \qquad \text{(for } v \in D_1\text{)}$$

Thus,

$$|Bh| \;\leq\; |Bh|_1 \;\leq\; |h|$$

[10] We essentially follow [Riesz-Nagy 1955], pages 329-334.

The map $B : V \to D_1$ is verifiably linear. There is an obvious *symmetry* of B:

$$\langle Bv, w\rangle \;=\; \lambda_w Bv \;=\; \langle Bv, Bw\rangle_1 \;=\; \overline{\langle Bw, Bv\rangle_1} \;=\; \overline{\lambda_v Bw} \;=\; \overline{\langle Bw, v\rangle} \;=\; \langle v, Bw\rangle \qquad \text{(for } v, w \in V\text{)}$$

Positivity of B is similar:

$$\langle Bv, v\rangle \;=\; \lambda_v Bv \;=\; \langle Bv, Bv\rangle_1 \;\geq\; \langle Bv, Bv\rangle \geq 0$$

Finally B is *injective*: if $Bw = 0$, then for all $v \in D_1$

$$0 \;=\; \langle v, 0\rangle_1 \;=\; \langle v, Bw\rangle_1 \;=\; \lambda_w v \;=\; \langle v, w\rangle$$

Since D_1 is dense in V, $w = 0$. Similarly, if $w \in D_1$ is such that $\lambda_v w = 0$ for all $v \in V$, then $0 = \lambda_w w = \langle w, w\rangle$ gives $w = 0$. Thus, $B : V \to D_1$ is bounded, symmetric, positive, injective, with dense image. In particular, B is self-adjoint.

Thus, B has a possibly *unbounded* positive, symmetric inverse A. Since B injects V to a dense subset D_1, necessarily A *surjects* from its domain (inside D_1) to V. We claim that A is *self-adjoint*. Let $S : V \oplus V \to V \oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A \;=\; S(\text{graph } B)$$

Also, in computing orthogonal complements $X^\perp$, clearly

$$(S\, X)^\perp \;=\; S\big(X^\perp\big)$$

From the obvious $U \circ S = -S \circ U$, compute

$$\text{graph } A^* \;=\; (U \text{ graph } A)^\perp \;=\; (U \circ S \text{ graph } B)^\perp \;=\; (-S \circ U \text{ graph } B)^\perp$$

$$=\; -S\big((U \text{ graph } B)^\perp\big) \;=\; -\,\text{graph} A \;=\; \text{graph } A$$

since the domain of B^* is the domain of B. Thus, A is self-adjoint.

We claim that for v in the domain of A, $\langle Av, v\rangle \geq \langle v, v\rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av\rangle \;=\; \langle Bw, w\rangle \;=\; \lambda_w Bw \;=\; \langle Bw, Bw\rangle_1 \;\geq\; \langle Bw, Bw\rangle \;=\; \langle v, v\rangle$$

Similarly, with $v' = Bw'$, and $v \in D_1$,

$$\langle v, Av'\rangle \;=\; \langle v, w'\rangle \;=\; \lambda_{w'} v \;=\; \langle v, Bw'\rangle_1 \;=\; \langle v, v'\rangle_1 \qquad (v \in D_1,\ v' \text{ in the domain of } A)$$

Since B maps V to D_1, the domain of A is contained in D_1. We claim that the domain of A is dense in D_1 in the d-topology, not merely in the subspace topology from V. Indeed, for $v \in D_1$ $\langle,\rangle_1$-orthogonal to the domain of A, for v' in the domain of A, using the previous identity,

$$0 \;=\; \langle v, v'\rangle_1 \;=\; \langle v, Av'\rangle$$

Since B injects V to D_1, A surjects from its domain to V. Thus, $v = 0$.

Last, prove that A is an extension of $S = 1_V + T$. On one hand, as above,

$$\langle v, Sw\rangle \;=\; \lambda_{Sw} v \;=\; \langle v, BSw\rangle_1 \qquad \text{(for } v, w \in D\text{)}$$

On the other hand, by definition of $\langle,\rangle_1$,

$$\langle v, Sw\rangle \;=\; \langle v, w\rangle_1 \qquad \text{(for } v, w \in D\text{)}$$

Thus,

$$\langle v, w - BSw \rangle_1 \;=\; 0 \qquad\qquad (\text{for all } v, w \in D)$$

Since D is d-dense in D_1, $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of B, so is in the domain of A, and

$$Aw \;=\; A(BSw) \;=\; Sw$$

Thus, the domain of A contains that of S and extends S. ///

[9.3] The resolvent $R_\lambda = (T - \lambda)^{-1}$

Let $R_\lambda = (T - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$ when this inverse exists as a linear operator defined at least on a dense subset of V.

[9.3.1] Theorem: Let T be self-adjoint and densely defined. For $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}$, the operator R_λ is everywhere defined on V, and the operator norm admits an estimate

$$\|R_\lambda\| \;\le\; \frac{1}{|\mathrm{Im}\,\lambda|}$$

For T *positive*, for $\lambda \notin [0, +\infty)$, R_λ is everywhere defined on V, and the operator norm is estimated by

$$\|R_\lambda\| \;\le\; \begin{cases} \dfrac{1}{|\mathrm{Im}\,\lambda|} & (\text{for } \mathrm{Re}\,(\lambda) \le 0) \\[2mm] \dfrac{1}{|\lambda|} & (\text{for } \mathrm{Re}\,(\lambda) \ge 0) \end{cases}$$

Proof: For $\lambda = x + iy$ off the real line and v in the domain of T,

$$|(T-\lambda)v|^2 \;=\; |(T+x)v|^2 + \langle (T-x)v, iyv \rangle + \langle iyv, (T-x)v \rangle + y^2|v|^2$$

$$=\; |(T+x)v|^2 - iy\langle (T-x)v, v \rangle + iy\langle v, (T-x)v \rangle + y^2|v|^2$$

The symmetry of T, and the fact that the domain of T^* contains that of T, implies that

$$\langle v, Tv \rangle \;=\; \langle T^* v, v \rangle \;=\; \langle Tv, v \rangle$$

Thus,

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2 \;\ge\; y^2|v|^2$$

Thus, for $y \neq 0$, $(T - \lambda)v \neq 0$. Let D be the domain of T. On $(T - \lambda)D$ there is an inverse R_λ of $T - \lambda$, and for $w = (T - \lambda)v$ with $v \in D$,

$$|w| \;=\; |(T-\lambda)v| \;\ge\; |y| \cdot |v| \;=\; |y| \cdot |R_\lambda (T-\lambda)v| \;=\; |y| \cdot |R_\lambda w|$$

which gives

$$|R_\lambda w| \;\le\; \frac{1}{|\mathrm{Im}\,\lambda|} \cdot |w| \qquad\qquad (\text{for } w = (T-\lambda)v,\ v \in D)$$

Thus, the operator norm on $(T - \lambda)D$ satisfies $\|R_\lambda\| \le 1/|\mathrm{Im}\,\lambda|$ as claimed.

We must show that $(T - \lambda)D$ is the whole Hilbert space V. If

$$0 \;=\; \langle (T-\lambda)v, w \rangle \qquad\qquad (\text{for all } v \in D)$$

then the adjoint of $T - \lambda$ can be defined on w simply as $(T - \lambda)^* w = 0$, since

$$\langle Tv, w \rangle \;=\; 0 \;=\; \langle v, 0 \rangle \qquad\qquad (\text{for all } v \in D)$$

Thus, $T^* = T$ is defined on w, and $Tw = \overline{\lambda}w$. For λ not real, this implies $w = 0$. Thus, $(T - \lambda)D$ is *dense* in V.

Since T is self-adjoint, it is *closed*, so $T - \lambda$ is closed. The equality

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2$$

gives

$$|(T-\lambda)v|^2 \;\ll_y\; |v|^2$$

Thus, for fixed $y \neq 0$, the map

$$v \oplus (T-\lambda)v \;\longrightarrow\; (T-\lambda)v$$

respects the metrics, in the sense that

$$|(T-\lambda)v|^2 \;\le\; |(T-\lambda)v|^2 + |v|^2 \;\ll_y\; |(T-\lambda)v|^2 \qquad \text{(for fixed } y \neq 0)$$

The graph of $T - \lambda$ is *closed*, so is a *complete* metric subspace of $V \oplus V$. Since F respects the metrics, it preserves completeness. Thus, the metric space $(T-\lambda)D$ is *complete*, so is a closed subspace of V. Since the closed subspace $(T - \lambda)D$ is dense, it is V. Thus, for $\lambda \notin \mathbb{R}$, R_λ is everywhere-defined. Its norm is bounded by $1/|\mathrm{Im}\,\lambda|$, so it is a continuous linear operator on V.

Similarly, for T *positive*, for $\mathrm{Re}\,(\lambda) \le 0$,

$$|(T-\lambda)v|^2 \;=\; |Tv|^2 - \lambda\langle Tv, v\rangle - \overline{\lambda}\langle v, Tv\rangle + |\lambda|^2\cdot|v|^2 \;=\; |Tv|^2 + 2|\mathrm{Re}\,\lambda|\langle Tv, v\rangle + |\lambda|^2\cdot|v|^2 \;\ge\; |\lambda|^2\cdot|v|^2$$

Then the same argument proves the existence of an everywhere-defined inverse $R_\lambda = (T - \lambda)^{-1}$, with $\|R_\lambda\| \le 1/|\lambda|$ for $\mathrm{Re}\,\lambda \le 0$. ///

[9.4] Holomorphy of the resolvent

[9.4.1] Theorem: (Hilbert) For points λ, μ off the real line, or, for T *positive*, for λ, μ off $[0, +\infty)$,

$$R_\lambda - R_\mu \;=\; (\lambda - \mu)R_\lambda R_\mu$$

For the operator-norm topology, $\lambda \to R_\lambda$ is *holomorphic* at such points.

Proof: Applying R_λ to

$$1_V - (T-\lambda)R_\mu \;=\; \big((T-\mu) - (T-\lambda)\big)R_\mu \;=\; (\lambda-\mu)R_\mu$$

gives

$$R_\lambda(1_V - (T-\lambda)R_\mu) \;=\; R_\lambda\big((T-\mu) - (T-\lambda)\big)R_\mu \;=\; R_\lambda(\lambda-\mu)R_\mu$$

Then

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} \;=\; R_\lambda R_\mu$$

For holomorphy, with $\lambda \to \mu$,

$$\frac{R_\lambda - R_\mu}{\lambda-\mu} - R_\mu^2 \;=\; R_\lambda R_\mu - R_\mu^2 \;=\; (R_\lambda - R_\mu)R_\mu \;=\; (\lambda-\mu)R_\lambda R_\mu R_\mu$$

Taking operator norm, using $\|R_\lambda\| \le 1/|\mathrm{Im}\,\lambda|$,

$$\Big\|\frac{R_\lambda - R_\mu}{\lambda-\mu} - R_\mu^2\Big\| \;\le\; \frac{|\lambda-\mu|}{|\mathrm{Im}\,\lambda|\cdot|\mathrm{Im}\,\mu|^2}$$

Thus, for $\mu \notin \mathbb{R}$, as $\lambda \to \mu$, this operator norm goes to 0, demonstrating the holomorphy. For *positive* T, the estimate $\|R_\lambda\| \le 1/|\lambda|$ for $\mathrm{Re}\,\lambda \le 0$ yields holomorphy on the negative real axis. ///

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10. *Appendix: simplest Maaß-Selberg relation*

According to Borel, Harish-Chandra gave the name *Maaß-Selberg relation* to the formula for the inner product of truncated Eisenstein series, though the systematic computation is due to Langlands. A crucial technical issue is a precise notion of *truncation* of Eisenstein series. We recall the simplest possible example.

[10.1] Truncation

The *constant term* $c_P f$ of f on $\Gamma\backslash\mathfrak{H}$ along the upper-triangular subgroup P is its 0^{th} Fourier coefficient

$$c_P f(g) \;=\; \int_0^1 f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\, dx$$

The *truncation operators* $\wedge^T$ for large positive real T act on an automorphic form f by killing off f's constant term for large $y > T$. The naive definition

$$(\text{naive } T\text{-truncation of } f)(x+iy) \;=\; \begin{cases} f(x+iy) & (\text{for } y \le T) \\ \\ f(x+iy) - c_P f(y) & (\text{for } y > T) \end{cases}$$

fails to describe the truncated function as a Γ-invariant function on $\mathfrak{H}$. More carefully, first define the *tail* $c_P^T f$ of the constant term $c_P f$ of f by

$$c_P^T f(y) \;=\; \begin{cases} 0 & (\text{for } y < T) \\ \\ c_P f(y) & (\text{for } y \ge T) \end{cases}$$

Make a pseudo-Eisenstein series

$$\Psi(c_P^T f) \;=\; \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (c_P^T f) \circ \gamma$$

and define the *truncation operator* $\wedge^T$ by

$$\wedge^T f \;=\; f - \Psi(c_P^T f)$$

By reduction theory, for large-enough T,

$$\Psi(c_P^T f) \;=\; c_P^T f \qquad\qquad (\text{for } y > T)$$

so for $y > T$ we do obtain the desired annihilation of the constant term:

$$c_P(\wedge^T f) \;=\; c_P^T f - c_P^T \Psi(c_P^T f) \;=\; c_P^T f - c_P^T f \;=\; 0 \qquad\qquad (\text{for } y > T)$$

[10.2] Maaß-Selberg relation off the critical line

Let $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ be the zeta function with its gamma factor attached. It is standard and elementary that

$$c_P E_s \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}$$

Abbreviate $c_s = \xi(2s-1)/\xi(2s)$. Let $\langle , \rangle$ be the complex-bilinear pairing

$$\langle f, g\rangle \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(x+iy)\cdot g(x+iy)\,\frac{dx\,dy}{y^2}$$

[10.2.1] Theorem: *(Maaß-Selberg relation)* For $s, z \in \mathbb{C}$, so that no denominators in the following vanish,

$$\langle \wedge^T E_s, \wedge^T E_z\rangle \;=\; \frac{T^{s+z-1}}{s+z-1} + c_s\frac{T^{(1-s)+z-1}}{(1-s)+z-1} + c_z\frac{T^{s+(1-z)-1}}{s+(1-z)-1} + c_s c_z\frac{T^{(1-s)+(1-z)-1}}{(1-s)+(1-z)-1}$$

Proof: The proof is a direct computation. First,

$$\langle \wedge^T E_s, \wedge^T E_z\rangle \;=\; \langle \wedge^T E_s, E_z\rangle$$

because the tail of the constant term of E_z is orthogonal to the truncated version $\wedge^T E_s$ of E_s. Then

$$\langle \wedge^T E_s, \wedge^T E_z\rangle \;=\; \langle \wedge^T E_s, E_z\rangle \;=\; \langle E_s - \Psi((y^s + c_s y^{1-s})^T), E_z\rangle \;=\; \langle \Psi(\begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases}), E_z\rangle$$

The usual unwinding trick applied to the awkward pseudo-Eisenstein series in the first argument of $\langle , \rangle$ transforms the last expression into

$$\int_{\Gamma_\infty\backslash\mathfrak{H}} \begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases} \cdot c_P(E_z)\,\frac{dx\,dy}{y^2} \;=\; \int_0^\infty \begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases} \cdot (y^z + c_z y^{1-z})\cdot \frac{dy}{y^2}$$

$$= \int_0^T y^s\cdot(y^z + c_z y^{1-z})\,\frac{dy}{y^2} \;-\; \int_T^\infty c_s y^{1-s}\,(y^z + c_z y^{1-z})\,\frac{dy}{y^2}$$

Take $\mathrm{Re}(z)$ is bounded above and below, so $\mathrm{Re}(1-z)$ is also bounded, and take $\mathrm{Re}(s)$ sufficiently large so that all the integrals converge. The above becomes

$$\int_0^T y^{s+z-1}\,\frac{dy}{y} + c_z\int_0^T y^{s+(1-z)-1}\,\frac{dy}{y} - c_s\int_T^\infty y^{(1-s)+z-1}\,\frac{dy}{y} - c_s c_z\int_T^\infty y^{(1-s)+(1-z)-1}\,\frac{dy}{y}$$

which gives the theorem. By analytic continuation, it is valid everywhere it makes sense. ///

[10.3] On the critical line

For $\overline{s} \neq s, 1-s$, the Maaß-Selberg relation computes the length of $\wedge^T E_s$ by taking $z = \overline{s}$: putting $s = \sigma + it$,

$$\| \wedge^T E_s\|^2_{L^2(\Gamma\backslash\mathfrak{H})} \;=\; \frac{T^{2\sigma-1}}{2\sigma-1} + c_{\sigma+it}\frac{T^{-2it}}{-2it} + c_{\sigma-it}\frac{T^{2it}}{2it} + c_{\sigma+it}c_{\sigma-it}\frac{T^{1-2\sigma}}{1-2\sigma}$$

As $\sigma \to \frac{1}{2}$, the blow-up in the first and last terms must cancel. Observe:

$$\frac{T^{2\sigma-1}}{2\sigma-1} \;=\; \frac{1}{2\sigma-1} + \log T + O(\sigma - \tfrac{1}{2})$$

and, since $c_{\frac{1}{2}+it}c_{\frac{1}{2}-it} = 1$ for t real,

$$c_{\sigma+it}c_{\sigma-it}\frac{T^{1-2\sigma}}{1-2\sigma} \;=\; \frac{-1}{2\sigma-1} \;-\; \tfrac{1}{2}\frac{\partial}{\partial\sigma}\Big|_{\sigma=\frac{1}{2}}\Big(c_{\sigma+it}c_{\sigma-it}T^{1-2\sigma}\Big) \;+\; O(\sigma - \tfrac{1}{2})$$

$$= \frac{-1}{2\sigma-1} \;-\; \tfrac{1}{2}\Big(c'_{\frac{1}{2}+it}c_{\frac{1}{2}-it} + c_{\frac{1}{2}+it}c'_{\frac{1}{2}-it} - 2\log T\Big) \;+\; O(\sigma - \tfrac{1}{2})$$

Taking the limit,

$$\| \wedge^T E_{\frac{1}{2}+it} \|^2 \;=\; 2\log T + c_{\frac{1}{2}+it} \frac{T^{-2it}}{-2it} + c_{\frac{1}{2}-it} \frac{T^{2it}}{2it} \;-\; \tfrac{1}{2}\Big(c'_{\frac{1}{2}+it} c_{\frac{1}{2}-it} + c_{\frac{1}{2}+it} c'_{\frac{1}{2}-it} \Big)$$

Bibliography

[Avakumović 1956] V. G. Avakumović, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. **65** (1956), 327-344.

[Colin de Verdière 1981] Y. Colin de Verdière, *Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 7, 361-363.

[Colin de Verdière 1982] Y. Colin de Verdière, *Pseudo-laplaciens, I*, Ann. Inst. Fourier (Grenoble) **32** (1982) no. 3, xiii, 275-286.

[Colin de Verdière 1983] Y. Colin de Verdière, *Pseudo-laplaciens, II*, Ann. Inst. Fourier (Grenoble) **32** (1983) no. 2, 87-113.

[Faddeev 1967] L. Faddeev, *Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobacevskii plane*, AMS Transl. Trudy (1967), 357-386.

[Friedrichs 1934] K.O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren*, Math. Ann. **109** (1934), 465-487, 685-713,

[Friedrichs 1935] K.O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren*, Math. Ann. **110** (1935), 777-779.

[Garrett 2009] P. Garrett, *An iconic error*, http://www.math.umn.edu/∼garrett/m/v/iconic_error.pdf (retrieved 03 June 2011).

[Godement 1966a] R. Godement, *Decomposition of $L^2(\Gamma\backslash G)$ for $\Gamma = SL(2,\mathbb{Z})$*, in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Grubb 2009] G. Grubb, *Distributions and operators*, Springer-Verlag, 2009.

[Hejhal 1976,83], D. Hejhal, *The Selberg trace formula for $PSL_2(\mathbb{R})$, I,II*, SLN **548, 1001**, Springer-Verlag, 1976, 1983.

[Hejhal 198], D. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series*, in *Recent Progress in Analytic Number Theory*, ed. H. Halberstam and C. Hooley, vol. 2, Academic Press, NY, 1981, 95-110.

[Kato 1966] T. Kato, *Perturbation theory for linear operators*, Springer, 1966, second edition, 1976, reprinted 1995.

[Langlands 1967/1976] R.P.Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, Berlin and New York, 1976.

[Langlands 1971] R. Langlands, *Euler Products*, Yale Univ. Press, New Haven, 1971.

[Lax-Phillips 1976] P. Lax, R. Phillips, *Scattering theory for automorphic functions*, Annals of Math. Studies, Princeton, 1976.

[Moeglin-Waldspurger 1989] C. Moeglin, J.-L. Waldspurger, *Le spectre résiduel de GL_n*, with appendix *Poles des fonctions L de pairs pour GL_n*, Ann. Sci. École Norm. Sup. **22** (1989), 605-674.

[Moeglin-Waldspurger 1995] C. Moeglin, J.-L. Waldspurger, *Spectral Decompositions and Eisenstein series*, Cambridge Univ. Press, Cambridge, 1995.

[Roelcke 1956] W. Roelcke, *Analytische Fortsetzung der Eisensteinreihen zu den parabolischen Spitzen von Grenzkreisgruppen erster Art*, Math. Ann. **132** (1956), 121-129.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87

[Venkov 1979] A. Venkov, *Spectral theory of automorphic functions, the Selberg zeta-function, and some problems of analytic number theory and math. physics*, Russian Math. Surveys **34** (1979) no. 3, 79-153.