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Bernstein's analytic continuation of complex powers

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Let f be a polynomial in $x_1, \ldots, x_n$ with real coefficients. For complex s, let f_+^s be the function defined by

$$f_+^s(x) \;=\; \begin{cases} f(x)^s & (\text{for } f(x) \geq 0) \\ 0 & (\text{for } f(x) \leq 0) \end{cases}$$

Certainly for $\mathrm{Re}(s) \geq 0$ the function f_+^s is *locally integrable*. For s in this range, define a *distribution*, also denoted f_+^s, by

$$f_+^s(\phi) \;=\; \int_{\mathbb{R}^n} f_+^s(x)\,\phi(x)\,dx \qquad\qquad (\text{for } \phi \in C_c^\infty(\mathbb{R}^n))$$

The object is to analytically continue the distribution f_+^s, as a meromorphic (distribution-valued) function of s. One should also ask about analytic continuation as a *tempered* distribution. Several provocative examples appear in [Gelfand-Shilov 1964]. In a lecture at the 1963 Amsterdam Congress, I.M. Gelfand refined this question to require further that one show that the poles lie in a finite number of arithmetic progressions.

[Bernstein 1968] gave a *first* proof, under a regularity hypothesis on the zero-set of the polynomial f. We essentially reproduce that argument here, with supporting material from complex function theory and from the theory of distributions.

[Bernstein-Gelfand 1969] proved the meromorphic continuation without any hypotheses, but assuming desingularization [Hironaka 1966]. [Atiyah 1970] gave a similar argument. Although this resolved the original question, invocation of desingularization was unsatisfactory.

[Bernstein 1971] created the theory of D-modules, and [Bernstein 1972] used it to prove the existence of the *Bernstein polynomial* $F_f(s)$ for any f, and so on. Invocation of desingularization was avoided, and much more was done, besides.

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1. *Analytic continuation of distributions*

We recall the nature of the topologies on test functions and on distributions. Let $C_c^\infty(U)$ be the collection of compactly-supported smooth functions with support inside a set $U \subset \mathbb{R}^n$. As usual, for U *compact*, we have a countable family of seminorms on $C_c^\infty(U)$:

$$\mu_\nu(f) \;=\; \sup_x |D^\nu f|$$

where for $\nu = (\nu_1, \ldots, \nu_n)$, as usual,

$$D^\nu \;=\; \left(\frac{\partial}{\partial x_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\nu_n}$$

It is elementary that (for U compact) $C_c^\infty(U)$ is a Fréchet space. To treat U not necessarily compact (e.g., $\mathbb{R}^n$ itself) let

$$U_1 \subset U_2 \subset \ldots$$

be compact subsets of U whose union is U. Then $C_c^\infty(U)$ is the union of the spaces $C_c^\infty(U_i)$, with the locally convex *colimit (direct limit) topology.* There is a *countable* cofinal sub-colimit, and each Fréchet space is closed in the next, so the colimit is called an *LF-space* (strict-(co-)limit-of-Fréchet). LF-spaces are not complete metrizable, but are *quasi-complete*, meaning that *bounded Cauchy nets converge.*

The spaces $\mathcal{D}^*(U)$ and $\mathcal{D}^*(\mathbb{R}^n)$ of *distributions* on U and on $\mathbb{R}^n$, respectively, are the continuous duals of $\mathcal{D}(U) = C_c^\infty(U)$ and $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$. The topology on the continuous dual space V^* of a topological vectorspace V is the *weak-$*$ topology:* a sub-basis near 0 in V^* consists of sets

$$U_{v,\epsilon} \;=\; \{\lambda \in V^* : |\lambda(v)| < \epsilon\}$$

A V^*-valued function f on an open subset Ω of $\mathbb{C}$ is *holomorphic* on Ω when, for every $v \in V$, the $\mathbb{C}$-valued function

$$z \to f(z)(v)$$

on Ω is holomorphic in the usual sense. More precisely, this is *weak-$*$ holomorphy*, referring to the topology.

Let $z_o \in \Omega$ for an open subset Ω of $\mathbb{C}$ and f a holomorphic V^*-valued function on $\Omega - z_o$. Say that f is *weakly meromorphic* at z_o when, for every $v \in V$, the $\mathbb{C}$-valued function $z \to f(z)(v)$ has a *pole* (as opposed to essential singularity) at z_o. Say that f is *strongly meromorphic* at z_o if the orders of these poles are *bounded independently of* v. That is, f is strongly meromorphic at z_o if there is an integer n and an open set Ω containing z_o so that, for all $v \in V$ the $\mathbb{C}$-valued function

$$z \to (z - z_o)^n f(z)(v)$$

is holomorphic on Ω. If n is the least integer f so that $(z - z_o)^n f$ is holomorphic at z_o, then f *is of order* $-n$ *at* z_o, etc.

To say that f is *strongly meromorphic* on an open set Ω is to require that there be a set S of points of Ω *with no accumulation point in* Ω so that f is holomorphic on $\Omega - S$, and so that f is strongly meromorphic at each point of S.

For brevity, but risking some confusion, we will often say *meromorphic* rather than *strongly meromorphic.*

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2. *Statements of theorems on analytic continuation*

Let $\mathcal{O}$ be the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$. For $z \in \mathbb{R}^n$, let $\mathcal{O}_z$ be the *local ring* at z, i.e., the ring of ratios P/Q of polynomials where the denominator does not vanish at z. Let $\mathfrak{m}_z$ be the maximal ideal of $\mathcal{O}_z$ consisting of elements of $\mathcal{O}_z$ whose numerator vanishes at z. Let I_z (depending upon f) be the ideal in $\mathcal{O}_z$ generated by

$$\frac{\partial f}{\partial x_1}, \;\ldots,\; \frac{\partial f}{\partial x_n}$$

A point $z \in \mathbb{R}^n$ is said to be *simple* with respect to the polynomial f when
- $f(z) \;=\; 0$
- $I_z \supset \mathfrak{m}_z^N$ for some N
- There are $\alpha_i \in \mathfrak{m}_z$ so that $f = \sum_i \alpha_i \frac{\partial f}{\partial x_i}$

[2.1.1] Remark: The second condition is equivalent to the $\mathcal{O}_z/I_z$ being finite-dimensional. The simplest situation in which the second condition holds is when $I_z = \mathcal{O}_z$, i.e., some partial derivative of f is non-zero at z. The third condition does not follow from the first two: Bernstein notes that the first two conditions hold but the third does not for

$$f(x,y) \;=\; x^5 + y^5 + x^2 y^2$$

[2.1.2] Theorem: (local version) For z a simple point with respect to f, there is a neighborhood U of x so that the distribution

$$f^s_{+,U}(\phi) \;=\; \int f^s_+(x)\,\phi(x)\,dx$$

on test functions $\phi \in C^\infty_c(U)$ on U has an analytic continuation to a meromorphic $C^\infty_c(U)^*$-valued function.

[2.1.3] Theorem: (global version) If *all* real zeros of $f(x)$ are *simple* (with respect to f), then f^s_+ is a meromorphic (distribution-valued) function of $s \in \mathbb{C}$.

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3. *Bernstein's proof*

Let R_z be the ring of linear differential operators with coefficients in $\mathcal{O}_z$. Note that R_z is both a left and a right $\mathcal{O}$-module: for $D \in R_z$, for $f, g \in \mathcal{O}$ and ϕ a smooth function near z, the definition is

$$(fDg)(\phi) \;=\; f\,D(g\phi)$$

[3.1.1] Lemma: There is a differential operator $D \in R_z$ and a non-zero *Bernstein polynomial* H in a single variable so that

$$D(f^{n+1}) \;=\; H(n) f^n$$

for any natural number n. (*Proof below.*)

Proof of Local Theorem from Lemma: Let U be a small-enough neighborhood of z so that on it all coefficients of D are holomorphic on U. For sufficiently large n the function f^{n+1}_+ is continuously differentiable, so

$$Df^{n+1}_+ \;=\; H(n) f^n_+$$

For each fixed $\phi \in C^\infty_c(U)$ consider the function

$$g(s) \;=\; (Df^{s+1}_+ - H(s) f^s_+)(\phi)$$

The hypotheses of the proposition below are satisfied, so the equality for all large-enough natural numbers implies equality everywhere:

$$Df^{s+1}_+ \;=\; H(s) f^s_+$$

This gives

$$f^s_+ \;=\; \frac{Df^{s+1}_+}{H(s)}$$

Now claim that for any $0 \le n \in \mathbb{Z}$ the distribution f^s_+ on $C^\infty_c(U)$ is meromorphic for $\mathrm{Re}(s) > -n$. For $n = 0$ this is clear. The formula just derived gives the induction step. Further, this argument shows that the poles of f^s_+ *restricted to $C^\infty_c(U)$* are concentrated on the finite collection of arithmetic progressions

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \ldots$$

where the λ_i are the roots of $H(s)$. In particular, the *order* of the pole of f^s_+ at a point s_o is equal to the number of roots λ_i so that s_o lies among

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \ldots$$

In particular, the distribution f_+^s really is (strongly) meromorphic. This proves the Theorem, granting the Lemma and granting the Proposition. ///

[3.1.2] Proposition: (attributed by Bernstein to Carlson) For g analytic for $\mathrm{Re}\, s > 0$ and $|g(s)| < be^{c\mathrm{Re}\, s}$ and $g(n) = 0$ for all sufficiently large natural numbers n, we have $g \equiv 0$.

Proof: (of Global Theorem) Invoking the Local Theorem *and its proof above*, for each $z \in \mathbb{R}^n$ choose a neighborhood U_z of z in which f_+^s is meromorphic, so that U_z is *Zariski-open*, i.e., is the complement of a finite union of zero sets of polynomials. Indeed, writing

$$f(x) \;=\; \sum \alpha_i \, \frac{\partial f}{\partial x_i}$$

with $\alpha_i \in \mathcal{O}_z$, as in the proof of the Local Theorem, let $\alpha_i = g_i/h_i$ with polynomials g_i and h_i, and take U_z to be the complement of the union of the zero-sets of the denominators h_i.

By Hilbert's Basis Theorem, $\mathbb{R}^n$ is covered by finitely-many $U_{z_1}, \ldots, U_{z_n}$. Make a partition of unity subordinate to this finite cover, i.e., take $\psi_1, \ldots, \psi_n$ so that $\psi_i \geq 0$, $\sum \psi_i \equiv 1$, and $\mathrm{spt}\psi_i \subset U_{z_i}$. Then

$$f_+^s \;=\; \sum_i \psi_i f_+^s$$

By choice of the U_{z_i}, the right-hand side is a finite sum of meromorphic (distribution-valued) functions.

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4. *Proof of the Lemma: the Bernstein polynomial*

Now we prove existence of the differential operator D and the *Bernstein polynomial* H. This is the most serious part of the argument. (The complex function theory proposition is not trivial, but is standard).

Proof: (of Lemma) Let

$$P \;=\; \sum \alpha_i \, \frac{\partial}{\partial x_i} \in R_z$$

where the $\alpha_i \in \mathfrak{m}_z$ are so that

$$f \;=\; \sum \alpha_i \, \frac{\partial f}{\partial x_i} \in R_z$$

Put

$$S_i \;=\; \frac{\partial f}{\partial x_i} P - f \frac{\partial}{\partial x_i} \;=\; \frac{\partial f}{\partial x_i}(P+1) - \frac{\partial}{\partial x_i} Q$$

Then

$$P(f) \;=\; \sum_i \alpha_i \, \frac{\partial f}{\partial x_i} \;=\; f \qquad\qquad S_i f \;=\; \frac{\partial f}{\partial x_i} f - f \frac{\partial f}{\partial x_i} \;=\; 0$$

By Leibniz' formula,

$$P(f^n) \;=\; n f^n \qquad\qquad S_i(f^n) \;=\; 0$$

[4.1.1] Sublemma: There is a non-zero polynomial M in one variable so that

$$M(P) \;=\; \sum_i J_i \frac{\partial f}{\partial x_i} \qquad\qquad \text{(for some } J_i \in R_z\text{)}$$

Proof: (of Sublemma) As usual, write $|\nu| = \nu_1 + \ldots + \nu_n$. For a natural number m, move all the coefficients to the right of the differential operators, that is, write

$$P^m \;=\; \sum_{|\nu| \leq m} D^\nu \, \gamma_{m,\nu} \qquad\qquad \text{(where } \gamma_{m,\nu} \in \mathcal{O}_z\text{)}$$

That this is possible is easy to see:

$$x_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} x_i = \begin{cases} 0 & (\text{for } i \neq j) \\ -1 & (\text{for } i = j) \end{cases}$$

The coefficients $\gamma_{m,\nu}$ are *polynomials* in the α_i, so $\gamma_{m,\nu} \in \mathfrak{m}_z^{|\nu|}$. Taking $M(P)$ of the form

$$M(P) = \sum_{m \leq q} b_m P^m = \sum_{m,\nu} D^\nu b_m \gamma_{m,\nu} \qquad (\text{with } b_m \in \mathbb{R})$$

the condition of the sublemma will be met if

$$\sum_m b_m \gamma_{m,\nu} \in I_z \qquad (\text{for all indices } \nu)$$

If $|\nu| \geq N$, where $I \supset \mathfrak{m}^N$, then this condition is automatically fulfilled. Thus, there are finitely-many conditions

$$\sum_m b_m \bar{\gamma}_{m,\nu} = 0$$

where $\bar{\gamma}_{m,\nu}$ is the image of $\gamma_{m,\nu}$ in $\mathcal{O}_z/\mathfrak{m}_z^N$. Since the latter quotient is, by hypothesis, a finite-dimensional vector space, the collection of such conditions gives a finite collection of homogeneous equations in the coefficients b_m. More specifically, the number of such conditions is

$$\dim \mathcal{O}_z/\mathfrak{m}_z^N \times \operatorname{card}\{\nu : |\nu| < N\}$$

Taking q large enough assures the existence of a non-trivial solution $\{b_m\}$, proving the sublemma. ///

Returning to the proof of the lemma: as an equation in R_z

$$M(P)(P+1) = \sum J_i \frac{\partial}{\partial x_i}(P+1) = \sum J_i S_i + \sum J_i \frac{\partial}{\partial x_i} f$$

Put

$$D = \sum J_i \frac{\partial}{\partial x_i} \qquad H(P) = M(P)(P+1)$$

Then, as desired,

$$D(f^{n+1}) = (\sum J_i \frac{\partial}{\partial x_i} f)(f^n) = H(P)(f^n) = H(n) f^n$$

This proves the Lemma, constructing the differential operator D. ///

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5. *Proof of the Proposition: estimates on zeros*

The necessary result is standard, although not so elementary as to be an immediate corollary of Cauchy's Theorem:

[5.1.1] Proposition: For g an analytic function for $\operatorname{Re} s > 0$ and $|g(s)| < b e^{c \operatorname{Re} s}$, if $g(n) = 0$ for all sufficiently large natural numbers n, then $g \equiv 0$.

Proof: Consider

$$G(z) = e^{-c}\, g(\frac{z+1}{z-1})$$

Then g is turned into a bounded function G on the disc, with zeros at points $(n-1)/(n+1)$ for sufficiently large natural numbers n.

We claim that, for a bounded function G on the unit disc with zeros ρ_i, either $G \equiv 0$ or

$$\sum_i (1 - |\rho_i|) < +\infty$$

If we prove this, then in the situation at hand the natural numbers are mapped to

$$\rho_n \;=\; (n-1)/(n+1) \;=\; 1 - \frac{1}{n+1}$$

so here

$$\sum_n (1 - |\rho_n|) \;=\; \sum_n \frac{1}{n+1} \;=\; +\infty$$

Thus, we would conclude $G \equiv 0$ as desired.

Recall *Jensen's formula:* for any holomorphic function G on the unit disc with $G(0) \neq 0$ and with zeros $\rho_1, \ldots$, for $0 < r < 1$ we have

$$|G(0)| \prod_{|\rho_i| \le r} \frac{r}{|\rho_i|} \;=\; \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |G(re^{i\theta})| \, d\theta \right\}$$

Granting this, the assumed boundedness of G on the disc gives an absolute constant C so that for all N

$$|G(0)| \prod_{|\rho_i| \le r} \frac{r}{|\rho_i|} \;\le\; C$$

(We can harmlessly divide by a suitable power of z to guarantee that $G(0) \neq 0$.) Then, letting $r \to 1$,

$$\prod |\rho_i| \;\le\; |G(0)|^{-1} C^{-1}$$

For an infinite product of positive real numbers $|\rho_i|$ less than 1 to have a value > 0, it is elementary that we must have

$$\sum_i (1 - |\rho_i|) < +\infty$$

as claimed. This proves the proposition. ///

While we're here, let's recall the proof of Jensen's formula (e.g., as in [Rudin 1966], for example. Fix $0 < r < 1$ and let

$$H(z) \;=\; G(z) \prod \frac{r^2 - \bar{\rho} z}{r(\rho - z)} \prod \frac{\rho}{\rho - z}$$

where the first product is over roots ρ with $|\rho| < r$ and the second is over roots with $|\rho| = r$. Then H is holomorphic and non-zero in an open disk of radius $r + \epsilon$ for some $\epsilon > 0$. Thus, $\log |H|$ is *harmonic* in this disk, and has the mean value property

$$\log |H(0)| \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |H(re^{i\theta})| \, d\theta$$

On one hand,

$$|H(0)| \;=\; |G(0)| \prod \frac{r}{|\rho|}$$

On the other hand, if $|z| = r$ the factors

$$\frac{r^2 - \bar{\rho} z}{r(\rho - z)}$$

have absolute value 1. Thus,

$$\log|H(re^{i\theta})| \;=\; \log|G(re^{i\theta})| - \sum_{|\rho|=r} \log|1 - e^{i(\theta - \arg\rho)}| \qquad\qquad (\text{where } e^{i\arg\rho} \;=\; \rho)$$

As noted in Rudin (see below),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|1 - e^{i\theta}|\, d\theta \;=\; 0$$

Therefore, the integral in the assertion of the mean value property is unchanged upon replacing H by G. Putting this all together gives Jensen's formula. ///

We execute the integral computation, following Rudin. Since the disk is simply-connected, there is a function $\lambda(z)$ on the open unit disc so that

$$\exp(\lambda(z)) \;=\; 1 - z$$

We completely specify this λ by requiring $\lambda(0) = 0$. We have $\mathrm{Re}\,\lambda(z) = \log|1 - z|$ and $|\mathrm{Im}\,\lambda(z)| < \frac{\pi}{2}$. Let $\delta > 0$ be small. Let $\Gamma = \Gamma_\delta$ be the (counterclockwise) path around the unit circle from $e^{i\delta}$ to $e^{(2\pi-\delta)i}$ and let $\gamma = \gamma_\delta$ be the (clockwise) path around a small circle centered at 1 from $e^{(2\pi-\delta)i}$ to $e^{i\delta}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta \;=\; \lim_{\delta\to 0} \frac{1}{2\pi}\int_\delta^{2\pi-\delta} \log|1 - e^{i\theta}|\, d\theta$$

$$=\; \mathrm{Re}\left[\frac{1}{2\pi i}\int_\Gamma \lambda(z)\,\frac{dz}{z}\right] \;=\; \mathrm{Re}\left[\frac{1}{2\pi i}\int_\gamma \lambda(z)\,\frac{dz}{z}\right]$$

by Cauchy's theorem. Elementary estimates show that the latter integral has a bound of the form $C\delta\log(1/\delta)$, which goes to 0 as $\delta \to 0$.

[Atiyah 1970] M.F. Atiyah *Resolution of singularities and division of distributions*, Comm. Pure. Appl. Math. **23** (1970), 145-150.

[Bernstein 1968] I.N. Bernstein, *The possibility of analytic continuation of f_+^λ for certain polynomials f* (Russian), Fun. An. and Applications **2** (1968) no. 1, 92-93.

[Bernstein-Gelfand 1969] I.N. Bernstein and S.I. Gelfand, *Meromorphy of the function P^λ* (Russian), Fun. An. and Applications **3** (1969), no. 1, 84-85.

[Bernstein 1971] I.N. Bernstein, *Modules over rings of differential operators; an investigation of the fundamental solutions of equations with constant coefficients* (Russian), Fun. An. and Applications **5** (1971) no. 2, 1-16.

[Bernstein 1972] I.N. Bernstein, *Analytic continuation of generalized functions with respect to a parameter* (Russian), Fun. An. and Applications **6** (1972) no. 4, 26-40.

[Gelfand-Shilov 1964] I.M. Gelfand, G.E. Shilov, *Generalized functions, vol. I: properties and operations*, Academic Press, 1964 (Russian original, Fizmatgiz, 1958).

[Hironaka 1964] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II*, Ann. of Math. **79** (1964), 109-203, 205-326.

[Rudin 1966] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.
