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The Constant Term

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- Basic estimates
- The hierarchy of constant terms

Let G be a reductive real Lie group, for example $G = GL(n, \mathbb{R})$. Let A be a maximal $\mathbb{R}$-split torus, in the case of $GL(n, \mathbb{R})$ the diagonal matrices, with connected component of the identity A^+. Let K be a maximal compact subgroup of G, for $GL(n, \mathbb{R})$ the standard orthogonal group $O(n)$. Let N be the unipotent radical of a minimal parabolic containing A, in the case of $GL(n, \mathbb{R})$ upper-triangular unipotent matrices. The Iwasawa decomposition of G is with respect to this data is

$$G = N \cdot A^+ \cdot K$$

Thus, the function $g \longrightarrow a_g$ defined by expressing $g = na_gk$ with $n \in N$, $a \in A^+$, $k \in K$ is well-defined.

Let $\log : A^+ \longrightarrow \mathfrak{a}$ be the inverse of the Lie exponential map from the Lie algebra $\mathfrak{a}$ of A^+ to A^+ itself. For λ in the complexification $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ of the group of characters of $\mathfrak{a}$, keeping in mind that $a_g \in A^+$, write

$$a_g^\lambda = e^{\lambda(\log a_g)}$$

For brevity, we may abbreviate the function $g \longrightarrow a_g^\lambda$ simply as a^λ.

[0.1] **Lemma:** Let C be a compact set in G, $x \in G$. Then there is a compact subset C_A of A^+ such that $y \in xC$ implies $a_y \in a_x \cdot C_A$.

Proof: Let $G = N \cdot A^+ \cdot K$ be an Iwasawa decomposition as above. Given a compact subset C of G, $C \cdot K$ is still compact and contains C, and is right K-stable. For a right K-stable compact subset C

$$C \subset (NA^+ \cap C) \cdot K$$

since in Iwasawa coordinates $pk \in C$ with $p \in NA^+$ and $k \in K$ implies by right K-stability that $p = (pk) \cdot k^{-1}$ is also in C. There are compact subsets $C_N \subset N$, $C_A \subset A^+$ so that

$$K \cdot C \subset C_N \cdot C_A \cdot K$$

Then

$$xC \subset Na_xK \cdot C \subset Na_x \cdot C_NC_AK \subset N \cdot (a_xNa_x^{-1}) \cdot (a_xC_A) \cdot K \subset N \cdot (a_xC_A) \cdot K$$

which shows that for $y \in xC$ the element a_y is in a_xC_A. ///

A left $N \cap \Gamma$-invariant function $\mathbb{C}$-valued f on G is said to be *of moderate growth of exponent* λ on a fixed Siegel set

$$S_t = \{x \in G : a_x^\alpha \geq t \text{ for all positive simple roots } \alpha\}$$

if

$$f(g) = O(a_g^\lambda) \quad (\text{ for } g \in S_t\ )$$

[0.2] **Corollary:** Fix an exponent λ. For any $\varphi \in C_c^\infty(G)$ there is a constant c and constant $0 < \mu$ so that, for any f of moderate growth of exponent λ on a Siegel set S_t, $\varphi \cdot f$ is of moderate growth of exponent λ on the Siegel set $S_{\mu t}$.

Proof: Let C be a compact set containing the support of φ. Then

$$\varphi f(x) = \int_G f(xg)\,\varphi(g)\,dg = \int_G f(g)\,\varphi(x^{-1}g)\,dg = \int_{xC} f(g)\,\varphi(x^{-1}g)\,dg$$

By the previous lemma, there is a compact subset C_A of A^+ such that for $y \in xC$ we have $a_y \in a_x C_A$. Thus, there is a constant c such that, in absolute value,

$$|\varphi f(x)| \le \sup|\varphi| \int_{xC} |f(g)|\,dg \le \sup|\varphi| \cdot c \cdot \int_{xC} a_x^\lambda\,dg \le \sup|\varphi| \cdot c \cdot \mathrm{meas}\,(C) \cdot a_x^\lambda$$

by invoking the previous lemma. ///

[0.3] **Corollary:** If f is smooth and of moderate growth with exponent λ on Siegel sets, and if $\varphi \cdot f = f$ for some $\varphi \in C_c^\infty(G)$, then f is of **uniform moderate growth** of exponent λ, in the sense that for any differential operator L in the universal enveloping algebra of the Lie algebra of G, Lf is of moderate growth with exponent λ on Siegel sets.

Proof: The point is that the left-G-invariant differential operators X 'on the right' attached to the right regular representation of G, arising from X in the Lie algebra of G by

$$Xf(x) = \frac{\partial}{\partial s}\Big|_{s=0} f(x \cdot e^{sX})$$

interact nicely with the action of $\varphi \in C_c^\infty(G)$ on f, as follows.

$$X(\varphi \cdot f)(x) = \frac{\partial}{\partial s}\Big|_{s=0} \int_G f(x \cdot e^{sX} g)\,\varphi(g)\,dg = \frac{\partial}{\partial s}\Big|_{s=0} \int_G f(xg)\,\varphi(e^{-sX}g)\,dg$$

by replacing g by $e^{-sX}g$. Then this is

$$\int_G f(xg)\,\frac{\partial}{\partial s}\Big|_{s=0} \varphi(e^{-sX}g)\,dg = \int_G f(xg)\,X^{\mathrm{left}}\varphi(g)\,dg$$

where X^{left} is the (*right*-G-invariant) differential operator 'on the left' naturally attached to X via the *left* regular representation. [1] Thus, since $\varphi \cdot f = f$,

$$Xf(x) = X(\varphi \cdot f)(x) = ((X^{\mathrm{left}}\varphi) \cdot f)(x)$$

which is of moderate growth of exponent λ, by the previous corollary. Thus, by induction on the degree of the differential operator L, Lf is of moderate growth of exponent λ. ///

[0.4] **Proposition:** Let P be an arbitrary *maximal* (proper) parabolic (containing the maximal split torus A) with Levi component M and unipotent radical N. Let f be smooth and left $(N \cap \Gamma)$-invariant. Let

$$\alpha : a_x \longrightarrow a_x^\alpha$$

be the simple positive root in (the Lie algebra of) N so that every root β in N satisfies $\beta \ge \alpha$. Suppose that for any Y in the Lie algebra of G the (right) Lie derivative Yf is of moderate growth of exponent λ in Siegel sets. Then

$$(f - f_P)(x) = O(a_x^{\lambda - \alpha})$$

[1] The interchange of differentiation and integration is justified by observing that the integral is compactly supported, continuous, and takes values in a quasi-complete locally convex topological vector space on which differentiation is a continuous linear map.

[0.5] **Remark:** For $G = GL(n)$, the standard simple positive roots are

$$\alpha_i \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix} = m_i/m_{i+1}$$

for $1 \le i \le n-1$.

[0.6] **Remark:** For non-maximal parabolics there is *not* the same sort of clear decrease of the exponent of growth. Instead, a somewhat more complicated estimate holds.

Proof: First, we give a proof for $G = GL(2)$. Normalizing the measure of $(\Gamma \cap N)\backslash N$ to be 1,

$$(f_P - f)(x) = \int_{(\Gamma\cap N)\backslash N} f(nx) - f(x)\, dn = \int_{0 \le t \le 1} f(e^{tX} \cdot x) - f(x)\, dt$$

where X is the element

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in the Lie algebra of N. By the fundamental theorem of calculus

$$f(e^{tX} \cdot x) - f(x) = \int_0^t \frac{\partial}{\partial r}\Big|_{r=0} f(e^{(r+s)X} \cdot x)\, ds = \int_0^t -X^{\text{left}} f(e^{sX} \cdot x)\, ds$$

where X^{left} is the natural right-G-invariant operator attached to X via the *left* regular representation. The main mechanism of this proof resides in the conversion of this operator to a *left-G*-invariant operator attached to the *right* regular representation, as follows.

$$X^{\text{left}} f(e^{sX} \cdot x) = (\frac{\partial}{\partial r}\Big|_{r=0} f)(e^{rX+sX} \cdot x) = (\frac{\partial}{\partial r}\Big|_{r=0} f)(e^{sX} \cdot x \cdot e^{r \cdot \operatorname{Ad} x^{-1}(X)}) = \operatorname{Ad} x^{-1}(X)\, f(e^{sX} \cdot x)$$

where $\operatorname{Ad} x^{-1}(X)$ is the *left-G*-invariant operator attached to X via the *right* regular representation. Let

$$x = n_x\, a_x\, \theta_x$$

with $n_x \in N$, $a_x \in M$, $\theta_x \in K$. Then

$$\operatorname{Ad} x^{-1}(X) = \operatorname{Ad}(\theta_x^{-1}\, a_x^{-1}\, n_x^{-1})(X) = \operatorname{Ad}(\theta_x^{-1}\, a_x^{-1})(X)$$

Further,

$$\operatorname{Ad} a_x^{-1}(X) = (a_x)^{-2} \cdot X$$

since X is in the $a_x \longrightarrow a_x^2$ rootspace. Then

$$\operatorname{Ad}(\theta_x^{-1}\, a_x^{-1})(X) = a_x^{-2} \cdot \operatorname{Ad} \theta_x^{-1}(X) = a_x^{-2} \cdot \sum_i c_i(\theta_x) Y_i$$

where the c_i are continuous functions (depending upon X) on K and $\{Y_i\}$ is a basis for the Lie algebra of G. Since the c_i are continuous on the compact set K, they have a uniform bound c. Altogether,

$$(f_P - f)(x) = \int_{0 \le t \le 1} \int_{0 \le s \le t} a_x^{-2} \cdot \left(-\sum_i c_i(\theta_x) Y_i \right) f(e^{sX} \cdot x)\, ds\, dt$$

$$= a_x^{-2} \cdot \sum_i c_i(\theta_x) \int_{0 \le t \le 1} \int_{0 \le s \le t} (-X_i f)(e^{sX} \cdot x)\, ds\, dt = a_x^{-2} \cdot \sum_i c_i(\theta_x) \int_{0 \le t \le 1} (-Y_i f)(e^{tX} \cdot x)\, dt$$

$$= a_x^{-2} \cdot \sum_i c_i(\theta_x) (-Y_i f)_P(x)$$

In this case the only root in N is $a_x \longrightarrow a_x^2$, so the assertion of the proposition holds in this case (where $G = GL(2)$).

Next, we redo the proof to work at least for maximal proper parabolics P having *abelian* unipotent radicals N. (The general case is complicated only in aspects somewhat irrelevant to the main point.) Normalizing the measure of $(\Gamma \cap N)\backslash N$ to be 1, we can write

$$(f - f_P)(x) = \int_{(\Gamma \cap N)\backslash N} f(nx) - f(x)\, dn = \int_{[0,1]^k} f(e^{t_1 X_1 + \cdots + t_k X_k} \cdot x) - f(x)\, dt_1 \ldots dt_k$$

where $X_1, \ldots, X_k$ is a basis for the Lie algebra of N so that

$$\{t_1 X_1 + \cdots + t_k X_k \,:\, 0 \le t_i \le 1,\ 1 \le i \le k\}$$

maps bijectively to $(\Gamma \cap N)\backslash N$, using the abelian-ness to know that this is possible.

By the fundamental theorem of calculus, for X in the Lie algebra,

$$f(e^{tX} \cdot x) - f(x) = \int_0^t \frac{\partial}{\partial r}\Big|_{r=0} f(e^{(r+s)X} \cdot x)\, ds = \int_0^t -X^{\rm left} f(e^{sX} \cdot x)\, ds$$

where $X^{\rm left}$ is the natural *right-G*-invariant operator attached to X. (The main mechanism of this proof resides in the conversion of such operators to *left-G*-invariant operators.) Rewrite this integral (by untelescoping) as a sum of k integrals of the form

$$\int_{[0,1]^k} f(e^{t_1 X_1 + \cdots + t_i X_i} \cdot x) - f(e^{t_1 X_1 + \cdots + t_{i-1} X_{i-1}} \cdot x)\, dt_1 \ldots dt_k$$

Fix the index i, and abbreviate

$$Y = t_1 X_1 + \cdots + t_{i-1} X_{i-1}$$

and let $t = t_i$, $X = X_i$. Then, by the fundamental theorem of calculus, the previous integrand integrated just in $t = t_i$ is

$$\int_{0 \le t \le 1} f(e^{Y+tX} \cdot x) - f(e^Y \cdot x)\, dt = \int_{0 \le t \le 1} \int_{0 \le s \le t} \frac{\partial}{\partial s}\Big|_{s=0} f(e^{Y+sX+tX} \cdot x)\, ds\, dt$$

$$= \int_{0 \le t \le 1} \int_{0 \le s \le t} (-X^{\rm left} f)(e^{Y+sX} \cdot x)\, ds\, dt$$

We convert the operator $X^{\rm left}$ to an operator on the right (and, thus, *left* G-invariant), as follows.

$$(X^{\rm left} f)(e^{Y+sX} \cdot x) = (\frac{\partial}{\partial r}\Big|_{r=0} f)(e^{Y+rX+sX} \cdot x)$$

$$= (\frac{\partial}{\partial r}\Big|_{r=0} f)(e^{Y+sX} \cdot x \cdot e^{r \cdot {\rm Ad}\, x^{-1}(X)}) = {\rm Ad}\, x^{-1}(Y)\, f(e^{Y+sX} \cdot x)$$

where ${\rm Ad}\, x^{-1}(X)$ is the *left-G*-invariant operator attached to Y via the *right* regular representation. Let

$$x = n_x\, a_x\, \theta_x$$

with $n_x \in N$, $a_x \in M$, $\theta_x \in K$. Then

$$\mathrm{Ad}\, x^{-1}(X) = \mathrm{Ad}\,(\theta_x^{-1}\, a_x^{-1}\, n_x^{-1})(X) = \mathrm{Ad}\,(\theta_x^{-1}\, a_x^{-1})(X)$$

using again the assumed abelian-ness of the Lie algebra of N. Now suppose further that X lies in the β rootspace in the Lie algebra of N. Then

$$\mathrm{Ad}\, a_x^{-1}(X) = \beta(a_x)^{-1} \cdot X$$

and

$$\mathrm{Ad}\,(\theta_x^{-1}\, a_x^{-1})(X) = \beta(a_x)^{-1} \cdot \mathrm{Ad}\,\theta_x^{-1}(X) = \beta(a_x)^{-1} \cdot \sum_{1 \le i \le k} c_i(\theta_x) Y_i$$

where the c_i are continuous functions (depending upon X) on K and $\{Y_i\}$ is a basis for the Lie algebra of G. Since the c_i are continuous on the compact set K, they have a uniform bound c (depending on X). Then altogether

$$\int_{0 \le t \le 1} f(e^{Y+tX} \cdot x) - f(e^{Y} \cdot x)\, dt = \beta(a_x)^{-1} \cdot \sum_{1 \le i \le k} c_i(\theta_x) \int_{0 \le t \le 1} \int_{0 \le s \le t} (-Y_i f)(e^{Y+sX} \cdot x)\, ds\, dt$$

On Siegel sets, for all such β,

$$\beta(a_x)^{-1} = O(a_x^{-\alpha})$$

Thus, using the exponent λ moderate growth of each of the functions $Y_i f$, we have found

$$\int_{0 \le t \le 1} f(e^{Y+tX} \cdot x) - f(e^{Y} \cdot x)\, dt = O(a_x^{\lambda - \alpha})$$

or, in the original notation,

$$\int_{0 \le t \le 1} f(e^{t_1 X_1 + \cdots + t_i X_i} \cdot x) - f(e^{t_1 X_1 + \cdots + t_{i-1} X_{i-1}} \cdot x)\, dt_i = O(a_x^{\lambda - \alpha})$$

Then, integrating in $dt_1, \ldots, dt_{i-1}$ and in $dt_{i+1}, \ldots, dt_k$ over copies of $[0,1]$ gives the same estimate for the k-fold integral:

$$\int_{[0,1]^k} f(e^{t_1 X_1 + \cdots + t_i X_i} \cdot x) - f(e^{t_1 X_1 + \cdots + t_{i-1} X_{i-1}} \cdot x)\, dt_1 \ldots dt_k = O(a_x^{\lambda - \alpha})$$

This is the assertion. ///

[0.7] **Corollary:** Let P be a maximal proper parabolic, with α the unique simple positive root in N. For f smooth of moderate growth of exponent λ in Siegel sets, and for $\varphi \cdot f = f$ for some $\varphi \in C_c^\infty(G)$, $f - f_P$ is of exponent $\lambda - \ell\alpha$ for all positive integers ℓ.

Proof: If $\varphi f = f$ then the previous corollary on *uniform* moderate growth asserts that Lf is of moderate growth exponent λ for every L in the universal enveloping algebra. On the other hand, the previous proposition shows that since every Xf is of exponent λ, $f - f_P$ is of exponent $\lambda - \alpha$. But then the uniform moderate growth assures that every $X(f - f_P)$ is of exponent $\lambda - \alpha$, as well. Applying the last proposition again, we find that

$$(Xf - Xf_P) - (Xf - Xf_P)_P = Xf - Xf_P = X(f - f_P)$$

is of exponent $\lambda - 2 \cdot \alpha$. This begins an induction which proves the corollary. ///

1. *The hierarchy of constant terms*

Let Δ denote the collection of simple (positive) roots. For each $\alpha \in \Delta$, there is a maximal proper parabolic P_α whose unipotent radical N^P has Lie algebra $\mathfrak{n}$ containing the α^{th} root space $\mathfrak{g}_\alpha$ in the Lie algebra $\mathfrak{g}$ of G. In particular, the Lie algebra $\mathfrak{n}$ is exactly the sum of all the rootspaces $\mathfrak{g}_\beta$ with $\beta \geq \alpha$.

Let c_α be the mapping which computes the P_α constant term

$$c_\alpha f(g) = \int_{\Gamma_{N_\alpha}\backslash N_\alpha} f(ng)\, dn$$

for locally integrable f left-invariant under a co-compact subgroup Γ_{N_α} of N_α. The group N_α has Haar measure normalized so that

$$\mathrm{meas}\,(\Gamma_{N_\alpha}\backslash N_\alpha) = 1$$

In particular, for simplicity we assume a consistency relation among these co-compact subgroups Γ_{N_α} by letting $\Gamma_{N_{\min}}$ be a cocompact subgroup of the unipotent radical of a minimal parabolic

$$P_{\min} = \cap_{\alpha\in\Delta} P_\alpha$$

and take

$$\Gamma_{N_\alpha} = N_\alpha \cap \Gamma_{N_{\min}}$$

A simple example is to take $G = GL(n, \mathbb{R})$ and

$$\Gamma_{N_{\min}} = \text{ upper-triangular unipotent matrices with integer entries}$$

[1.1] **Lemma:** For simple roots α, β,

$$c_\alpha \circ c_\beta = c_\beta \circ c_\alpha$$

Proof: A direct computation, changing variables in the integrals definitions of these operators, using the unimodularity of the groups, etc. ///

[1.2] **Proposition:** Let P_S be the parabolic whose unipotent radical contains exactly the simple roots S. Let c_P be the constant term operator for P. Then

$$1 - c_P = \prod_{\alpha\in S} (1 - c_\alpha)$$

[1.3] **Corollary:** Let f be left $\Gamma_{N_{\min}}$-invariant and Z-finite and K-finite. Then

$$\left(\prod_{\alpha\in\Delta} (1 - c_\alpha)\right) f$$

is of rapid decay in any Siegel set aligned with the implied family of parabolic subgroups.