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Compactness of anisotropic arithmetic quotients

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These notes give the beginning of a treatment of reduction theory for classical groups following Tamagawa-Mostow and Godement's Bourbaki article. For the moment, the non-compact case is neglected.

- Affine heights
- Minkowski reduction
- Imbeddings of arithmetic quotients
- Mahler's criterion for compactness
- Compactness of anisotropic quotients of orthogonal groups

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1. *Affine heights*

Let K_v be the standard compact subgroup of $GL(n, \mathbb{Q}_v)$: namely, for $\mathbb{Q}_v \approx \mathbb{R}$ the usual orthogonal group $O(n)$, and for $\mathbb{Q}_v$ non-archimedean it is $GL(n, \mathbb{Z}_v)$. (The fact that theses subgroups are *maximal* compact will not be needed.) Let $V = \mathbb{Q}^n$, and $V_\mathbb{A} = V \otimes \mathbb{A}$. Let $GL(n, \mathbb{A})$ act on the right on $\mathbb{A}^n$ by matrix multiplication.

For the *real* prime v of $\mathbb{Q}$ define the **local height** function η_v on $x = (x_1, \ldots, x_n) \in V_{\mathbb{Q}_v} = \mathbb{Q}_v^n$ by

$$\eta_v(x) \;=\; \sqrt{x_1^2 + \ldots + x_n^2}$$

For a *non-archimedean* prime v of $\mathbb{Q}$ define the **local height** function η_v on $x = (x_1, \ldots, x_n) \in V_{\mathbb{Q}_v} = \mathbb{Q}_v^n$ by

$$\eta_v(x) \;=\; \sup_i |x_i|_v$$

A vector $x \in V_\mathbb{A}$ is **primitive** if it is of the form $x_o g$ where $g \in GL(n, \mathbb{A})$ and $x_o \in V_\mathbb{Q}$. That is, it is an image of a *rational* point of the vectorspace by an element of the *adele* group. For $x = (x_1, \ldots, x_n) \in V_\mathbb{Q}$, at almost all non-archimedean primes v the x_i's are in $\mathbb{Z}_v$ and have greatest common divisor 1 (locally). Since elements of the adele group are in K_v almost everywhere, this property is not changed by multiplication by $g \in GL(n, \mathbb{A})$. That is, any primitive vector x has the property that at almost all v the components of x are locally integral and have (local) greatest common divisor 1.

For primitive $x \in V_\mathbb{A}$ define the **global height**

$$\eta(x) = \prod_v \eta_v(x_v)$$

Since x is primitive, at almost all finite primes the local height is 1, so this product has only finitely many non-1 factors.

- For $t \in \mathbf{J}$ and primitive $x \in \mathbb{A}^n$, $\eta(tx) = |t|\eta(x)$, where $|t|$ is the idele norm.
- If a sequence of vectors in $\mathbb{A}^n$ goes to 0, then their heights go to zero also.
- If the heights of some (primitive) vectors x_i go to zero, then there are scalars $t_i \in \mathbb{Q}^\times$ so that $t_i x_i$ goes to 0 in $\mathbb{A}^n$.

• For $g \in GL(n, \mathbb{A})$ and $c > 0$, the set of non-zero vectors $x \in \mathbb{Q}^n$ so that $\eta(xg) < c$ is finite modulo $\mathbb{Q}^\times$. In particular, the infimum of $\{\eta(xg) : x \in \mathbb{Q}^n - 0\}$ is positive, and is assumed.
• For a compact subset E of $GL(n, \mathbb{A})$ there are constants $c, c' > 0$ so that for all primitive vectors x and for all $g \in E$

$$c\,\eta(x) \leq \eta(xg) \leq c'\,\eta(x)$$

Proof: The first assertion follows from the product formula.

For the *second* assertion: if a sequence of vectors x_i goes to 0, then for every large $N > 0$ and small $\varepsilon > 0$ there is i_o so that $i \geq i_o$ implies $\eta_v(x_v) < \varepsilon$ at archimedean primes, and $x_v \in N\mathbb{Z}_v^n$ for every finite v. Then $\eta(x) \leq \varepsilon^\ell / N$ where ℓ is the number of archimedean primes. So the heights go to zero.

For the *third* assertion: suppose that $\eta(x_i)$ goes to 0, for some primitive vectors x_i. At almost all finite v the vector x_i is in $\mathbb{Z}_v^n$ and the entries have local gcd 1. Since $\mathbb{Z}$ is a principal ideal domain, we can choose $s_i \in \mathbb{Q}$ to that at *every* finite prime v the components of $s_i x_i$ are locally integral and have greatest common divisor 1. Then the local contribution to the height function from *all* finite primes is 1. Therefore, the archimedean height of $s_i x_i$, Euclidean distance, goes to 0. Finally, we need some choice of trick to make the vectors go to 0 in $\mathbb{A}^n$. For example, for each index i let N_i be the greatest integer so that

$$\eta_\infty(s_i x_i) < \frac{1}{(N_i!)^2}$$

Let $t_i = s_i \cdot N_i!$. Then $t_i x_i$ goes to 0 in $\mathbb{A}^n$.

For the *fourth* assertion: fix $g \in GL(n, \mathbb{A})$. Since K preserves heights, via the Iwasawa decomposition we may suppose that g is in the group $P_\mathbb{A}$ of upper triangular matrices in $GL(n, \mathbb{A})$. Let g_{ij} be the $(i, j)^{th}$ entry of g. Choose representatives $x = (x_1, \ldots, x_n)$ for non-zero vectors in $\mathbb{Q}^n$ modulo $\mathbb{Q}^\times$ such that, letting μ be the first index with $x_\mu \neq 0$, then $x_\mu = 1$. That is, x is of the form

$$x = (0, , \ldots, 0, 1, x_{\mu+1}, \ldots, x_n)$$

To illustrate the idea of the argument with a light notation, first consider $n = 2$, let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $x = (1, y)$. Thus,

$$x \cdot g \;=\; (1, y) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;=\; (a, b + yd)$$

From the definition of the local heights, at each place v of k

$$\max(|a|_v, |b + yd|_v) \;\leq\; h_v(xg)$$

from which

$$|b + yd|_v \prod_{w \neq v} |a|_w \;\leq\; \prod_{\text{all } w} h_w(xg) \;=\; h(xg)$$

Since g is fixed, a is fixed, and at almost all places $|a|_w = 1$. Thus, for $h(xg) < c$ there is a uniform c' such that

$$|b + yd|_v \;\leq\; c' \qquad \text{(for all } v)$$

Since for almost all v the residue class field cardinality q_v is strictly greater than c

$$|b + yd|_v \;\leq\; 1 \qquad \text{(for almost all } v)$$

Therefore, $b + yd$ lies in a compact subset C of $\mathbb{A}$. Since b, d are fixed, and since $\mathbb{Q}$ is discrete and closed in $\mathbb{A}$, the collection of images $\{b + dy : y \in k\}$ is discrete in $\mathbb{A}$. Thus, the collection of y such that $b + dy$ lies in C is finite.

Now consider general n and $x \in \mathbb{Q}^n$ such that $h(xg) < c$. Let $\mu - 1$ be the least index such that $x_\mu \neq 0$. Adjuts by $k^\times$ such that $x_\mu = 1$. For each v, from $h(xg) < c$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \ \leq\ h(gx) < c$$

For almost all places v we have $|g_{\mu-1,\mu-1}|_v = 1$, so there is a uniform c' such that

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \ <\ c' \qquad\qquad (\text{for all } v)$$

For almost all v the residue field cardinality q_v is strictly greater than c', so for almost all v

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \ \leq\ 1$$

Therefore, $g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}$ lies in a compact subset C of $\mathbb{A}$. Since $\mathbb{Q}$ is discrete, the collection of x_μ is finite.

Continuing similarly, there are only finitely many choices for the other entries of x. Inductively, suppose $x_i = 0$ for $i < \mu - 1$, and $x_\mu, \ldots, x_{\nu-1}$ fixed, and show that x_ν has only finitely many possibilities. Looking at the ν^{th} component $(xg)_\nu$ of xg,

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \leq h(xg) \ \leq\ c$$

For almost all places v we have $|g_{\mu-1,\mu-1}|_w = 1$, so there is a uniform c' such that for all v

$$|(xg)_\nu|_v = |g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \ <\ c'$$

For almost all v the residue field cardinality q_v is strictly greater than c', so for almost all v

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \ \leq\ 1$$

Therefore,

$$g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}$$

lies in the intersection of a compact subset C of $\mathbb{A}$ with a closed discrete set, so lies in a finite set. Thus, the number of possibilities for x_ν is finite. By induction we obtain the finiteness.

For the *last* assertion: let E be a compact subset of $GL(n, \mathbb{A})$, and let $K = \prod_v K_v$. Then $K \cdot E \cdot K$ is compact, being the continuous image of a compact set. So without loss of generality E is left and right K-stable. By Cartan decompositions the compact set E of $GL(n, \mathbb{A})$ is contained in a set

$$K \, \Delta \, K$$

where Δ is a compact set of diagonal matrices in $GL(n, \mathbb{A})$. Let $g = \theta_1 \delta \theta_2$ with $\theta_i \in K$, and x a primitive vector. By the K-invariance of the height,

$$\frac{\eta(xg)}{\eta(x)} \ =\ \frac{\eta(x\theta_1\delta\theta_2)}{\eta(x)} \ =\ \frac{\eta(x\theta_1\delta)}{\theta(x)} \ =\ \frac{\eta((x\theta_1)\delta)}{\eta((x\theta))}$$

Thus, the set of ratios $\eta(xg)/\eta(x)$ for g in a compact set and x ranging over primitive vectors is exactly the set of values $\eta(x\delta)/\eta(x)$ where δ ranges over a compact set and x varies over primitives. With diagonal entries δ_i of δ,

$$0 \ <\ \inf_{\delta \in \Delta} \inf_i |\delta_i| \ \leq\ \eta(x\delta)/\eta(x) \ \leq\ \sup_{\delta \in \Delta} \sup_i |\delta_i| < \infty$$

by compactness of Δ. ///

2. *Minkowski reduction*

The previous preparations set things up to prove the basic reduction-theory result for non-compact quotients: we prove that there is a nice **approximate fundamental domain** for the action of $GL(n,\mathbb{Q})$ on $GL(n,\mathbb{A})$.

[2.0.1] **Theorem:** *(Adelic form of Minkowski reduction)* Given $g \in GL(n,\mathbb{A})$, there are $\gamma \in GL(n,\mathbb{Q})$ and $\theta \in K$ so that

$$p = \gamma g \theta$$

is **upper-triangular** and so that the diagonal entries p_{ii} of p satisfy the **inequalities**

$$\Big|\frac{p_{ii}}{p_{i+1\,i+1}}\Big| \geq \frac{\sqrt{3}}{2} \quad \text{(idele norm)}$$

Further, for $i < j$, the entry p_{ij} of p can be arranged to lie in any specified set of representatives in $\mathbb{A}$ for the quotient $p_{ii}\mathbb{Q}\backslash\mathbb{A}$, such as $\mathbb{R}/\mathbb{Z} \times \widehat{\mathbb{Z}}$.

[2.0.2] **Remark:** Combined with Strong Approximation for $SL(n)$, this recovers classical Minkowski reduction for $SL(n,\mathbb{Z})$ on $SL(n,\mathbb{R})$. More importantly, it begins the general fundamental domain results, in terms of **Siegel sets**.

From above, given $g \in GL(n,\mathbb{A})$ there is $x \in \mathbb{Q}^n - 0$ such that $\eta(xg) > 0$ is minimal among values $\eta(x'g)$ with $x \in \mathbb{Q}^n - 0$. Take $\gamma \in GL(n,\mathbb{Q})$ so that $e_n\gamma = x$, where $\{e_i\}$ is the standard basis for $\mathbb{Q}^n$. By Iwasawa, there is $\theta \in K$ such that $p = \gamma g \theta$ is upper-triangular. Then

$$\eta(\gamma g \theta) \;=\; |p_{nn}| \qquad\qquad (p_{ij} \text{ is } ij^{th} \text{ entry of } p)$$

Let H be the subgroup of $GL(n,\mathbb{A})$ fixing e_n and stabilizing the subspace spanned by $e_1,\ldots,e_{n-1}$. Then $H \approx GL(n-1,\mathbb{A})$, and by induction we can suppose that $|p_i/p_{i+1,i+1}| \geq \frac{\sqrt{3}}{2}$ already for $i < n-1$. Looking at just the lower-right two-by-two block inside these n-by-n matrices, it suffices to consider $n = 2$.

Repeating: given $g \in GL(2,\mathbb{A})$ there is $x \in \mathbb{Q}^2 - 0$ such that $\eta(xg)$ is positive and minimal among all the values $\eta(x'g)$ with $x \in \mathbb{Q}^2 - 0$. Take $\gamma \in GL(2,\mathbb{Q})$ such that $(0\ 1)\gamma = x$. By Iwasawa there is θ in the standard compact subgroup K of $GL(2,\mathbb{A})$ such that $p = \gamma g\theta$ is upper-triangular, say

$$p \;=\; \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

We wish to see that the minimality of $\eta(xg) = \eta((0\ 1)p)$ gives $|a/d| \geq \frac{\sqrt{3}}{2}$. Let $x' = (1,t) \in \mathbb{Q}^2$. The inequality

$$\eta((0\ 1)p) \;\leq\; \eta(x'p)$$

gives

$$|d| \leq \eta(a, b+dt) \qquad\qquad (\text{for all } t \in \mathbb{Q})$$

For brevity, let $r = a/d$ and $s = b/d$. Dividing through by d gives, by elementary properties of the height,

$$1 \;\leq\; \eta(r, s+t)$$

Changing $(r, s+t)$ by an element of $\mathbb{Q}^\times$, the idele r is a local unit at all finite primes of $\mathbb{Q}$. By right-multiplying by suitable

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard compact subgroups at finite primes, the idele r is 1 at all finite primes.

Given $s \in \mathbb{A}$, choose $t \in \mathbb{Q}$ so that $s+t$ is integral at all finite primes and $|s+t|_\infty \le \frac{1}{2}$. With this $t \in \mathbb{Q}$, the height of $(r, s+t)$ is

$$\eta(r, s+t) \;=\; \eta_\infty(r, s+t) \;=\; \sqrt{|r|_\infty^2 + |s+t|_\infty^2} \;\le\; \sqrt{|r|_\infty^2 + \frac{1}{4}}$$

From $1 \le \eta(r, s+t)$

$$1 \;\le\; |r|_\infty^2 + \frac{1}{4}$$

which gives

$$\frac{\sqrt{3}}{2} \;\le\; |r|_\infty$$

Since r was a local unit at all finite primes,

$$\frac{\sqrt{3}}{2} \;\le\; |r|$$

Since $|r| = |a/d|$,

$$\frac{\sqrt{3}}{2} \;\le\; |\frac{a}{d}|$$

This proves the theorem. ///

[2.0.3] **Remark:** This proof of Minkowski reduction uses the Euclidean-ness of $\mathbb{Q}$, and does not generalize simply to general situations. Rather, a relatively complicated argument *reduces* the general case to this. The general *conclusion* is analogous but the proof is different.

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3. *Imbeddings of arithmetic quotients*

Let k be a number field. Let $Q = \langle , \rangle$ be a non-degenerate quadratic form on a k-vectorspace V, and $G = O(Q)$ the corresponding orthogonal group. We have the natural imbedding $G \to GL(V)$.

[3.0.1] **Proposition:** The inclusion $G_k \to GL(V)_k$ induces an inclusion

$$G_k \backslash G_\mathbb{A} \to GL(V)_k \backslash GL(V)_\mathbb{A}$$

with closed image.

A general topological lemma is necessary.

[3.0.2] **Lemma:** Let X, Y be locally compact Hausdorff topological spaces. Further, X has a countable open cover $\{U_i\}$ such that every U_i has compact closure. Let G be a group acting continuously on X and Y, *transitively* on X. Let $f : X \to Y$ be a continuous *injective* G-set map whose image is a closed subset of Y. Then f is a *homeomorphism* of X to its image in Y.

Proof: This is a version of the Baire Category argument. Since $f(X)$ is closed in Y the image $f(X)$ is itself (with the subset topology) a locally compact Hausdorff space. Therefore, without loss of generality, f is *surjective*. Let C_i be the closure of U_i. The images $f(C_i)$ of the C_i are compact, hence closed, by Hausdorff-ness. We claim that some $f(C_i)$ must have non-empty interior. If not, we do the usual Baire argument: fix a non-empty open set V_1 in Y with compact closure. Since $f(C_1)$ contains no non-empty open set, V_1 is not contained in $f(C_1)$, so there is a non-empty open set V_2 whose closure is compact and whose closure is contained in $V_1 - f(C_1)$. Since $f(C_2)$ cannot contain V_2, there is a non-empty open set V_3 whose closure is compact and whose closure is contained in $V_2 - f(C_2)$. A descending chain of non-empty open sets is produced:

$$V_1 \supset \text{clos}(V_2) \supset V_2 \supset \text{clos}(V_2) \supset V_3 \supset \ldots$$

By construction, the intersection of the chain of compact sets $\mathrm{clos}(V_i)$ is disjoint from all the sets $f(C_i)$. Yet the intersection of a descending chain of compact sets is non-empty. Contradiction. Therefore, some $f(C_i)$ has non-empty interior. In particular, for y_o in the interior of $f(C_i)$, the map f is **open** at $x_o = f^{-1}(y_o)$.

Now use the G-equivariance of f. For an open U_o containing x_o such that $f(U_o)$ is open in Y, for any $g \in G$ the set gU_o is open containing gx_o. By the G-equivariance,

$$f(gU_o) \;=\; gf(U_o) \;=\; \text{continuous image of open set} \;=\; \text{open}$$

Therefore, since G is transitive on X, f is open at all points of X. ///

Proof: By definition of the quotient topologies, $GL(V)_k G_{\mathbb{A}}$ must be shown closed in $GL(V)_A$.

Let X be the k-vectorspace of k-valued quadratic forms on V. We have a linear action ρ of $g \in GL(V)_k$ on $q \in X$ by

$$\rho(g)q\,(v,v) \;=\; q(g^{-1}v, g^{-1}v)$$

(with inverses for associativity). This extends to give a continuous group action of $GL(V)_{\mathbb{A}}$ on $X_{\mathbb{A}} = X \otimes \mathbb{A}$. Note that G_k is the subgroup of $GL(V)_k$ fixing the point $Q \in X$, essentially by definition.

Let Y be the set of images of Q under $GL(V)_k$. Then

$$GL(V)_k G_{\mathbb{A}} = \{g \in GL(V)_{\mathbb{A}} : g(Q) \in Y\}$$

That is, $GL(V)_k G_{\mathbb{A}}$ is the inverse image of Y. By the continuity of the group action, to prove that $GL(V)_k G_{\mathbb{A}}$ is closed in $GL(V)_{\mathbb{A}}$ it suffices to prove that the orbit

$$Y = GL(V)_k G_{\mathbb{A}}\,(Q)$$

is closed in $X_{\mathbb{A}}$. Indeed, Y is a subset of $X \subset X_{\mathbb{A}}$, which is a (closed) discrete subset of $X_{\mathbb{A}}$. This proves the proposition, invoking the previous lemma. ///

If the global base field is not $\mathbb{Q}$, we need more preparation:

[3.0.3] **Proposition:** Let k be a number field and K a finite extension of k. Let V be K^n viewed as a k-vectorspace. Let $H = GL(n, K)$ viewed as a k-group, and $G = GL_k(V)$. Then the natural inclusion

$$i \;:\; GL_K(K^n) \;=\; H \to G = GL_k(V)$$

gives a homeomorphism of $H_k\backslash H_{\mathbb{A}}$ to its image in $G_k\backslash G_{\mathbb{A}}$, and this image is closed.

Proof: (This resembles the argument for the previous lemma. More will be said in the next version of these notes.)

[3.0.4] **Theorem:** *Mahler's criterion for compactness:* Let G be an orthogonal group attached to an n-dimensional non-degenerate k-valued quadratic form. For a subset X of $G_{\mathbb{A}} \subset GL(n, \mathbb{A})$ to be compact left modulo G_k, it is necessary and sufficient that, given $x_i \in X$ and $v_i \in k^n$ such that $x_i v_i \to 0$ in $\mathbb{A}^n$, $v_i = 0$ for sufficiently large i.

Proof: The propositions above the problem to proving an analogue for $G = GL(n, k)$ with $k = \mathbb{Q}$. In particular, for $GL(n)$ suppose there are positive constants c' and c'' such that

$$X \subset \{g \in GL(n, \mathbb{A}) : c' \le |\det g| \le c''\}$$

The serious direction of implication is to show that, if the condition is satisfied, then X is compact modulo G_k. Let η be the affine height function on k^n. Then $\eta(xv) \ge c_1$ for some c_1 for any non-zero $v \in k^n$. By the Iwasawa decomposition, can write $x = p\theta$ with $\theta \in GL(n, \mathfrak{o}_k)$ and p upper-triangular, where $\mathfrak{o}_k$ is the ring of integers in k. Further, since we consider x modulo G_k, and using the fact that actually $k = \mathbb{Q}$, the Minkowski reduction allows us to suppose that the diagonal entries p_i of p satisfy $|p_i/p_{i+1}| \geq c$ for some $c > 0$. Therefore, letting e_i be the usual basis vectors in k^n, $c_1 \leq |p_i| = \eta(xe_1)$. And our extra hypothesis gives us

$$c' \leq |p_1 \ldots p_n| \leq c''$$

Thus, (by Fujisaki's lemma, for example) the diagonal entries of elements p coming from elements of X lie inside some compact subset of $\mathbf{J}/k^\times$.

Certainly the superdiagonal entries, left-modulo k-rational upper-triangular matrices, can be put into a compact set.

Therefore, X is compact left modulo $GL(n, k)$, for $k = \mathbb{Q}$. But, as remarked at the outset, the propositions above about imbeddings of arithmetic quotients reduce the general case and the orthogonal group case to this. ///

[3.0.5] **Theorem:** Let G be the orthgonal group of a non-degenerate quadratic form $Q = \langle , \rangle$ on a vectorspace $V \approx k^n$ over a number field k. Then $G_k \backslash G_\mathbb{A}$ is compact if and only if Q is k-anisotropic.

Proof: On one hand, suppose Q is k-anisotropic. If $g_n v_n \to 0$ in $\mathbb{A}^n$ with $g_n \in G_\mathbb{A}$ and $v_n \in \mathbb{A}^n$, then $Q(v_n g_n)$ also goes to $Q(0) = 0$, by the continuity of Q. But $Q(g_n v_n) = Q(v_n)$, because $G_\mathbb{A}$ preserves values of Q. Since Q has no non-zero k-rational isotropic vectors and k^n is discrete in $\mathbb{A}^n$, this means that eventually $v_n = 0$. By Mahler's criterion this implies that the quotient is compact.

On the other hand, suppose that Q is isotropic. Then there is a non-zero isotropic vector $v \in k^n$. Let H be the subgroup of $G_\mathbb{A}$ fixing v. For all indices i let $v_i = v$. So certainly v_i does not go to 0. Now we'll need to exploit the fact that the topology on $\mathbf{J}$ is *not* simply the subspace topology from $\mathbb{A}$, but is inherited from the imbedding $\alpha \to (\alpha, \alpha^{-1})$ of $\mathbf{J} \to \mathbb{A} \times \mathbb{A}$: we can find a sequence t_i of ideles which go to 0 in the $\mathbb{A}$-topology (but certainly not in the $\mathbf{J}$-topology). Then $t_i v_i \to 0$. And certainly still $Q(t_i v_i) = 0$, so by Witt's theorem there is $g_i \in G_\mathbb{A}$ so that $g_i v_i = t_i v_i$. Thus, $g_i v_i \to 0$, but certainly v_i does not do so. Thus, Mahler's criterion says that the quotient is not compact. ///