(*July 28, 2014*)

Discrete spectrum of pseudo-cuspforms on GL_n

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We prove that the space of square-integrable functions on $PGL_r(\mathbb{Z})\backslash PGL_r(\mathbb{R})/O(n,\mathbb{R})$ with all constant terms vanishing beyond fixed heights has purely discrete spectrum with respect to the Friedrichs extension of the restriction of the invariant Laplacian to (smooth functions in) this space.

This implies the discrete decomposition of the space of cuspforms, and sets up an instance of H. Jacquet's extension of Y. Colin de Verdière's proof of meromorphic continuation of Eisenstein series with cuspidal data. This argument is simpler than the Selberg-Bernstein approach, and gives a stronger conclusion.

The proof will show that the resolvent of (the Friedrichs extension of a restriction of) the invariant Laplacian is *compact*. Let L^2_η be the subspace of $L^2 = L^2(PGL_r(\mathbb{Z})\backslash PGL_r(\mathbb{R})/O(n,\mathbb{R}))$ with all constant terms vanishing above given fixed heights (specified by a real-valued function η on simple positive roots, described precisely below). By its construction, the resolvent of the Friedrichs extension maps continuously from L^2 to the automorphic Sobolev space $H^1 = H^1(PGL_r(\mathbb{Z})\backslash PGL_r(\mathbb{R})/O(n,\mathbb{R}))$ with its finer topology. Letting $H^1_\eta = H^1 \cap L^2_\eta$ with the topology of H^1, it suffices to show that the injection $H^1_\eta \to L^2_\eta$ is compact.

To prove this compactness, we will show that the image of the unit ball of H^1_η is totally bounded in L^2_η.

Let A be the standard maximal torus consisting of diagonal elements of $G = GL_r$, Z the center of G, and $K = O(n,\mathbb{R})$. Let A^+ be the subgroup of $A_\mathbb{R}$ with positive diagonal entries, and $Z^+ = Z_\mathbb{R} \cap A^+$. A standard choice of positive simple roots is

$$\Phi \;=\; \{\alpha_i(a) \;=\; \frac{a_i}{a_{i+1}} \;:\; i = 1,\ldots,r-1\} \qquad\qquad (\text{with } a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix})$$

Let $N^{\min}$ be the unipotent radical of the standard minimal parabolic $P^{\min}$ consisting of upper-triangular elements of G. For $g \in G_\mathbb{R}$, let $g = n_g a_g k_g$ be the corresponding Iwasawa decomposition with respect to $P^{\min}$.

From basic reduction theory, the quotient $Z_\mathbb{R} G_\mathbb{Z} \backslash G_\mathbb{R}$ is covered by the Siegel set

$$\mathfrak{S} \;=\; N^{\min}_\mathbb{Z} \backslash N^{\min}_\mathbb{R} \cdot Z^+ \backslash A^+_o \cdot K \;=\; Z^+ N^{\min}_\mathbb{Z} \Big\backslash \Big\{ g \in G : \alpha(a_g) \ge \tfrac{\sqrt{3}}{2},\ \text{for all } \alpha \in \Phi \Big\}$$

Further, there is an absolute constant so that

$$\int_\mathfrak{S} |f| \;\ll\; \int_{Z_\mathbb{R} G_\mathbb{Z} \backslash G_\mathbb{R}} |f| \qquad\qquad (\text{for all } f)$$

For a non-negative real-valued function η on the set of simple roots, let

$$X^\alpha_\eta \;=\; \{g \in \mathfrak{S} \;:\; \alpha(a_g) \ge \eta(\alpha)\} \qquad\qquad (\text{for } \alpha \in \Phi)$$

and

$$C_\eta \;=\; \{g \in \mathfrak{S} \;:\; \alpha(a_g) \le \eta(\alpha) \text{ for all } \alpha \in \Phi\}$$

The latter is compact. Certainly

$$\mathfrak{S} \quad=\quad C_\eta \;\cup\; \bigcup_{\alpha \in \Phi} X^\alpha_\eta$$

For $\alpha \in \Phi$, let P^α be the standard maximal proper parabolic whose unipotent radical N^α has Lie algebra $\mathfrak{n}^\alpha$ including the α^{th} root space. That is, for $\alpha(a) = a_i/a_{i+1}$, the Levi component M^α of P^α is $GL_i \times GL_{r-i}$.

The constant term c^α along a parabolic P of a function f on $G_\mathbb{Z}\backslash G_\mathbb{R}$ is

$$(c^P f)(g) \;=\; \int_{N_\mathbb{Z}\backslash N_\mathbb{R}} f(ng)\,dn \qquad\qquad (\text{with } N \text{ the unipotent radical of } P)$$

For $P = P^\alpha$, write $c^\alpha = c^P$. For a non-negative real-valued function η on the set of simple roots, the space of square-integrable functions with constant terms vanishing above heights η is

$$L^2_\eta \;=\; \{f \in L^2(Z_\mathbb{R} G_\mathbb{Z}\backslash G_\mathbb{R}/K) \;:\; c^\alpha f(g) = 0 \text{ for } \alpha(a_g) \ge \eta(\alpha), \text{ for all } \alpha \in \Phi\}$$

Vanishing is meant in a distributional sense. The global automorphic Sobolev space H^1 is the completion of $C^\infty_c(Z_\mathbb{R} G_\mathbb{Z}\backslash G_\mathbb{R})^K$ with respect to the H^1 Sobolev norm

$$|f|^2_{H^1} \;=\; \int_{Z_\mathbb{R} G_\mathbb{Z}\backslash G_\mathbb{R}} (1-\Delta)f \cdot \overline{f}$$

where Δ is the invariant Laplacian descended from the Casimir operator Ω. Put $H^1_\eta = H^1 \cap L^2_\eta$.

[0.0.1] Theorem: The Friedrichs self-adjoint extension $\widetilde{\Delta}_\eta$ of the restriction of the symmetric operator Δ to (test functions in) L^2_η has *compact resolvent*, thus has purely *discrete spectrum.*

Proof: Let

$$A^+_o \;=\; \{a \in A \;:\; \alpha(a) \ge \frac{\sqrt{3}}{2} \;:\; \text{for all } \alpha \in \Phi\}$$

We grant ourselves that we can control smooth cut-off functions:

[0.0.2] Lemma: Fix a positive simple root α. Given $\mu \ge \eta(\alpha)+1$, there are smooth functions φ^α_μ for $\alpha \in \Phi$ and φ^o_μ such that: all these functions are real-valued, taking values between 0 and 1, φ^o is supported in $C_{\mu+1}$ and $\varphi^\alpha\mu$ is supported in X^α_μ, and $\varphi^o_\mu + \sum_\alpha \varphi^\alpha_\mu = 1$. Further, there is a bound C *uniform* in $\mu \ge \eta(\alpha)+1$, such that $|f \cdot \varphi^o_\mu|_{H^1} \;\le\; C \cdot |f|_{H^1}$ and

$$|f \cdot \varphi^\alpha_\mu|_{H^1} \;\le\; C \cdot |f|_{H^1} \qquad\qquad (\text{for all } \mu \ge \eta(\alpha)+1)$$

Then the key point is

[0.0.3] Claim: For $\alpha \in \Phi$,

$$\lim_{\mu \to +\infty} \Big(\sup_{f \in H^1_\eta \text{ and } \mathrm{spt} f \subset X^\alpha_\mu} \frac{|f|_{L^2}}{|f|_{H^1}} \Big) \;=\; 0$$

Temporarily grant the claim. To prove total boundedness of $H^1_\eta \to L^2_\eta$, given $\varepsilon > 0$, take $\mu \ge \eta(\alpha)+1$ for all $\alpha \in \Phi$, large enough so that $|f \cdot \varphi^\alpha_\mu|_{L^2} < \varepsilon$ for all $\alpha \in \Phi$, for all $f \in H^1_\eta$ with $|f|_{H^1} \le 1$. This covers the images $\{f \cdot \varphi^\alpha_\mu : f \in H^1_\eta\}$ with $\alpha \in \Phi$ with $\mathrm{card}(\Phi)$ open balls in L^2 of radius ε.

The remaining part $\{f \cdot \varphi^o_\mu : f \in H^1_\eta\}$ consists of smooth functions supported on the compact C_μ. The latter can be covered by finitely-many coordinate patches $\psi_i : U_i \to \mathbb{R}^d$. Take smooth cut-off functions φ_i for this covering. The functions $(f \cdot \varphi_i) \circ \psi_i^{-1}$ on $\mathbb{R}^d$ have support strictly inside a Euclidean box, whose opposite faces can be identified to form a flat d-torus $\mathbb{T}^d$. The flat Laplacian and the Laplacian inherited from G admit uniform comparison on each $\psi(U_i)$, so the $H^1(\mathbb{T}^d)$-norm of $(f \cdot \varphi_i) \circ \psi_i^{-1}$ is uniformly bounded by the H^1-norm. The classical Rellich lemma asserts compactness of $H^1(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$. By restriction, this gives the compactness of each $H^1 \cdot \varphi_i \to L^2$. A finite sum of compact maps is compact, so $H^1 \cdot \varphi^o_\mu \to L^2$ is compact. In particular, the image of the unit ball from H^1 admits a cover by finitely-many ε-balls for any $\varepsilon > 0$.

Combining these finitely-many ε-balls with the card$\,(\Phi)$ balls covers the image of H^1_η in L^2 by finitely-many ε-balls, proving that $H^1_\eta \to L^2$ is compact.

It remains to prove the claim. Fix $\alpha = \alpha_i \in \Phi$, and $f \in H^1_\eta$ with support inside X^α_μ for $\mu \gg \eta(\alpha)$. Let $N = N^\alpha$, $P = P^\alpha$, and let $M = M^\alpha$ be the standard Levi component of P. Use exponential coordinates coordinates

$$n_x \;=\; \begin{pmatrix} 1_i & x \\ 0 & 1_{r-i} \end{pmatrix}$$

In effect, the coordinate x is in the Lie algebra $\mathfrak{n}$ of $N_\mathbb{R}$. Let $\Lambda \subset \mathfrak{n}$ be the lattice which exponentiates to $N_\mathbb{Z}$. Give $\mathfrak{n}$ the natural inner product $\langle,\rangle$ invariant under the (Adjoint) action of $M_\mathbb{R} \cap K$ that makes root spaces mutually orthogonal. Fix a non-trivial character ψ on $\mathbb{R}/\mathbb{Z}$. We have the Fourier expansion

$$f(n_x m) \;=\; \sum_{\xi \in \Lambda'} \psi\langle x, \xi\rangle \; \widehat{f}_\xi(m) \qquad\qquad (\text{with } n \in N_\mathbb{R} \text{ and } m \in M_\mathbb{R})$$

where Λ' is the dual lattice to Λ in $\mathfrak{n}$ with respect to $\langle,\rangle$, and

$$\widehat{f}_\xi(m) \;=\; \int_{\mathfrak{n}/\Lambda} \overline{\psi}\langle x, \xi\rangle \; f(n_x m)\, dx$$

Let $\Delta^\mathfrak{n}$ be the flat Laplacian on $\mathfrak{n}$ associated to the inner product $\langle,\rangle$, normalized so that

$$\Delta^\mathfrak{n} \psi\langle x, \xi\rangle \;=\; -\langle \xi, \xi\rangle \cdot \psi\langle x, \xi\rangle$$

Let $U = M \cap N^{\min}$. Abbreviating $A_u = \mathrm{Ad}\, u$,

$$|f|^2_{L^2} \;\le\; \int_\mathfrak{S} |f|^2 \;=\; \int_{Z^+\backslash A^+_o} \int_{U_\mathbb{Z}\backslash U_\mathbb{R}} \int_{A_u^{-1}\Lambda\backslash\mathfrak{n}} |f(un_x a)|^2 \; dx\; du\; \frac{da}{\delta(a)}$$

with Haar measures dx, du, da, where δ is the modular function of $P_\mathbb{R}$. Using the Fourier expansion,

$$f(un_x a) \;=\; f(un_x u^{-1} \cdot ua) \;=\; \sum_{\xi \in \Lambda'} \psi\langle A_u x, \xi\rangle \cdot \widehat{f}_\xi(ua) \;=\; \sum_{\xi \in \Lambda'} \psi\langle x, A_u^* \xi\rangle \cdot \widehat{f}_\xi(ua)$$

Then

$$-\Delta^\mathfrak{n} f(un_x a) \;=\; \sum_{\xi \in \Lambda'} \langle A^*_u \xi, A^*_u \xi\rangle \cdot \psi\langle x, A^*_u \xi\rangle \cdot \widehat{f}_\xi(ua)$$

The compact quotient $U_\mathbb{Z}\backslash U_\mathbb{R}$ has a compact set R of representatives in $U_\mathbb{R}$, so there is a *uniform* lower bound for $0 \neq \xi \in \Lambda'$:

$$0 \;<\; b \;\le\; \inf_{u \in R}\; \inf_{0 \neq \xi \in \Lambda'} \langle A^*_u \xi, A^*_u \xi\rangle$$

By Plancherel applied to the Fourier expansion in x, using the hypothesis that $\widehat{f}_0 = 0$ in X^α_μ,

$$\int_{A_u^{-1}\Lambda\backslash\mathfrak{n}} |f(un_x a)|^2 \; dx \;=\; \int_{A_u^{-1}\Lambda\backslash\mathfrak{n}} |f(un_x u^{-1} \cdot ua)|^2 \; dx \;=\; \sum_{\xi \in \Lambda'} |\widehat{f}_\xi(ua)|^2$$

$$\le\; b^{-1} \sum_{\xi \in \Lambda'} \langle A^*_u \xi, A^*_u \xi\rangle \cdot |\widehat{f}_\xi(ua)|^2 \;=\; \sum_{\xi \in \Lambda'} -\widehat{\Delta^\mathfrak{n} f}_\xi(ua) \cdot \overline{\widehat{f}}\,(ua)$$

$$= \int_{u^{-1}\Lambda u\backslash\mathfrak{n}} -\Delta^{\mathfrak{n}} f(un_x u^{-1}\cdot ua)\cdot \overline{f}(un_x u^{-1}\cdot ua)\, dx \;=\; \int_{A_u^{-1}\Lambda\backslash\mathfrak{n}} -\Delta^{\mathfrak{n}} f(un_x a)\cdot \overline{f}(un_x a)\, dx$$

Thus, for f with $\widehat{f}(0)=0$ on $\alpha(g)\geq\eta$,

$$|f|^2_{L^2} \;\ll\; \int_{Z^+\backslash A_o^+}\int_{U_{\mathbb{Z}}\backslash U_{\mathbb{R}}}\int_{A_u^{-1}\Lambda\backslash\mathfrak{n}} -\Delta^{\mathfrak{n}} f(un_x a)\cdot\overline{f}(un_x a)\,dx\,du\,\frac{da}{\delta(a)}$$

Next, we compare $\Delta^{\mathfrak{n}}$ to the invariant Laplacian Δ. Let $\mathfrak{g}$ be the Lie algebra of $G_{\mathbb{R}}$, with non-degenerate invariant pairing $\langle u,v\rangle = \mathrm{tr}(uv)$. The Cartan involution $v\to v^\theta = -v^\top$ has $+1$ eigenspace the Lie algebra $\mathfrak{k}$ of K, and -1 eigenspace $\mathfrak{s}$, the space of symmetric matrices.

Let Φ^N be the set of positive roots β whose root-space $\mathfrak{g}_\beta$ appears in $\mathfrak{n}$. For each $\beta\in\Phi^N$, take $x_\beta\in\mathfrak{g}_\beta$ such that $x_\beta+x_\beta^\theta\in\mathfrak{s}$, $x_\beta-x_\beta^\theta\in\mathfrak{k}$, and $\langle x_\beta, x_\beta^\theta\rangle = 1$: for $\beta(a)=a_i/a_j$ with $i<j$, x_β has a single non-zero entry, at the ij^{th} place. Let

$$\Omega' \;=\; \sum_{\beta\in\Phi^N}(x_\beta x_\beta^\theta + x_\beta^\theta x_\beta) \qquad\qquad \text{(in the universal enveloping algebra } U\mathfrak{g})$$

Let $\Omega''\in U\mathfrak{g}$ be the Casimir element for the Lie algebra $\mathfrak{m}$ of $M_{\mathbb{R}}$, normalized so that Casimir Ω for $\mathfrak{g}$ is the sum $\Omega=\Omega'+\Omega''$. We rewrite Ω' to fit the Iwasawa coordinates: for each β,

$$x_\beta x_\beta^\theta + x_\beta^\theta x_\beta \;=\; 2x_\beta x_\beta^\theta + [x_\beta^\theta, x_\beta] \;=\; 2x_\beta^2 - 2x_\beta(x_\beta - x_\beta^\theta) + [x_\beta^\theta, x_\beta] \;\in\; 2x_\beta^2 + [x_\beta^\theta,x_\beta] + \mathfrak{k}$$

Thus,

$$\Omega' \;=\; \sum_{\beta\in\Phi^N} 2x_\beta^2 + [x_\beta^\theta, x_\beta] \qquad\qquad (\text{modulo } \mathfrak{k})$$

The commutators $[x_\beta^\theta, x_\beta]$ are in $\mathfrak{m}$. In the coordinates $un_x a$ with $U\mathfrak{g}$ acting on the right, $x_\beta\in\mathfrak{n}$ is acted on by a before translating x, by

$$un_x a\cdot e^{tx_\beta} \;=\; un_x\cdot e^{t\beta(a)\cdot x_\beta}\cdot a \;=\; un_{x+\beta(a)x_\beta}a$$

That is, x_β acts by $\beta(a)\cdot\frac{\partial}{\partial x_\beta}$.

For two symmetric operators S,T on a not-necessarily-complete inner product space V, write $S\leq T$ when

$$\langle Sv,v\rangle \;\leq\; \langle Tv,v\rangle \qquad\qquad (\text{for all } v\in V)$$

Say a symmetric operator T is *non-negative* when $0\leq T$. Since $a\in A_o^+$, there is an absolute constant so that $\alpha(a)\geq\mu$ implies $\beta(a)\gg\mu$. Thus,

$$-\Delta^{\mathfrak{n}} \;=\; -\sum_{\beta\in\Phi^N}\frac{\partial^2}{\partial x_\beta^2} \;\ll\; \frac{1}{\mu^2}\cdot\Big(-\sum_{\beta\in\Phi^N} x_\beta^2\Big) \qquad\qquad (\text{operators on } C_c^\infty(X_\mu^\alpha)^K)$$

where $C_c^\infty(X_\mu^\alpha)^K$ has the L^2 inner product. We claim that

$$-\sum_{\beta\in\Phi^N}[x_\beta^\theta, x_\beta] - \Omega'' \;\geq\; 0 \qquad\qquad (\text{operators on } C_c^\infty(X_\mu^\alpha)^K))$$

From this, it would follow that

$$-\Delta^{\mathfrak{n}} \;\ll\; \frac{1}{\mu^2}\cdot\Big(-\sum_{\beta\in\Phi^N}x_\beta^2\Big) \;\leq\; \frac{1}{\mu^2}\cdot\Big(-\sum_{\beta\in\Phi^N}x_\beta^2 - \sum_{\beta\in\Phi^N}[x_\beta^\theta,x_\beta] - \Omega''\Big) \;=\; \frac{1}{\mu^2}\cdot(-\Delta)$$

Then for $f \in H^1_\eta$ with support in X^α_μ we would have

$$|f|^2_{L^2} \ll \int_{\mathfrak{S}} -\Delta^{\mathfrak{n}} f \cdot \overline{f} \ll \frac{1}{\mu^2} \int_{\mathfrak{S}} -\Delta f \cdot \overline{f} \ll \frac{1}{\mu^2} \int_{Z_{\mathbb{R}} G_{\mathbb{Z}} \backslash G_{\mathbb{R}}} -\Delta f \cdot \overline{f} \ll \frac{1}{\mu^2} \cdot |f|^2_{H^1}$$

Taking μ large makes this small. Since we can do the smooth cutting-off to affect the H^1 norm only up to a *uniform* constant, this would complete the proof of total boundedness of the image in L^2 of the unit ball from H^1_η.

To prove the claimed non-negativity of $T = -\sum_{\beta \in \Phi^N} [x^\theta_\beta, x_\beta] - \Omega''$, exploit the Fourier expansion along N and the fact that $x \in \mathfrak{n}$ does not appear in T: noting that the order of coordinates $n_x u$ differs from that above,

$$\int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \int_{\Lambda \backslash \mathfrak{n}} T f(n_x ua)\, \overline{f}(n_x ua)\, dx\, du\, \frac{da}{\delta(a)}$$

$$= \int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \int_{\Lambda \backslash \mathfrak{n}} T\Big(\sum_\xi \psi\langle x, \xi\rangle\, \widehat{f}_\xi(ua) \Big) \sum_{\xi'} \overline{\psi}\langle x, \xi'\rangle\, \overline{\widehat{f}}_\xi(ua)\, dx\, du\, \frac{da}{\delta(a)}$$

Only the diagonal summands survive the integration in $x \in \mathfrak{n}$, and the exponentials cancel, so this is

$$\int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_\xi T \widehat{f}_\xi(ua) \cdot \overline{\widehat{f}}_\xi(ua)\, du\, \frac{da}{\delta(a)}$$

Let F_ξ be a left-$N_{\mathbb{R}}$-invariant function taking the same values as $\widehat{f}_\xi$ on $U_{\mathbb{R}} A^+ K$, defined by

$$F_\xi(n_x uak) = \widehat{f}_\xi(uak) \qquad (\text{for } n_x \in N,\ u \in U,\ a \in A^+,\ k \in K)$$

Since T does not involve $\mathfrak{n}$, and since F_ξ is left $N_{\mathbb{R}}$-invariant,

$$T\widehat{f}_\xi(ua) = TF_\xi(n_x ua) = -\Delta F_\xi(n_x ua)$$

and then

$$\int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_\xi T \widehat{f}_\xi(ua) \cdot \overline{\widehat{f}}_\xi(ua)\, du\, \frac{da}{\delta(a)} = \int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_\xi -\Delta F_\xi(ua) \cdot \overline{F}_\xi(ua)\, du\, \frac{da}{\delta(a)}$$

The individual summands are not left-$U_{\mathbb{Z}}$-invariant. Since $\widehat{f}_\xi(\gamma g) = \widehat{f}_{A^*_\gamma \xi}(g)$ for γ normalizing $\mathfrak{n}$, we can group $\xi \in \Lambda'$ by $U_{\mathbb{Z}}$ orbits to obtain $U_{\mathbb{Z}}$ subsums, and then *unwind*. Pick a representative ω for each orbit $[\omega]$, and let U_ω be the isotropy subgroup of ω in $U_{\mathbb{Z}}$, so

$$\int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_\xi -\Delta F_\xi(ua) \cdot \overline{F}_\xi(ua)\, du = \sum_{[\omega]} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_{\xi \in [\omega]} -\Delta F_\xi(ua) \cdot \overline{F}_\xi(ua)\, du$$

$$= \sum_\omega \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_{\gamma \in U_\omega \backslash U_{\mathbb{Z}}} -\Delta F_{A^*_\gamma \omega}(ua) \cdot \overline{F}_{A^*_\gamma \omega}(ua)\, du = \sum_\omega \int_{U_\omega \backslash U_{\mathbb{R}}} -\Delta F_\omega(ua) \cdot \overline{F}_\omega(ua)\, du$$

Then

$$\int_{Z^+ \backslash A^+_o} \int_{U_{\mathbb{Z}} \backslash U_{\mathbb{R}}} \sum_\xi -\Delta F_\xi(ua) \cdot \overline{F}_\xi(ua)\, du = \sum_\omega \int_{Z^+ \backslash A^+_o} \int_{U_\omega \backslash U_{\mathbb{R}}} -\Delta F_\omega(ua) \cdot \overline{F}_\omega(ua)\, du\, \frac{da}{\delta(a)}$$

Since $-\Delta$ is a non-negative operator on functions on every quotient $Z^+ N_{\mathbb{R}} U_\omega \backslash G_{\mathbb{R}}/K$ of $G_{\mathbb{R}}/K$, each double integral is non-negative, proving T is non-negative.

This completes the proof that $H^1_\eta \to L^2$ is compact, and, thus, that the Friedrichs extension of the restriction of Δ to (test functions in) L^2_η has purely discrete spectrum. ///

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