Integral moments

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(with Diaconu and Goldfeld)

• **Idea**: *integral moments* over *families* of L-functions arise as *coefficients* in automorphic spectral decompositions.

• **Example**: in several cases, extract *subconvex bounds*

• **Note**: in some applications a subconvex bound replaces Lindelöf

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Sample of moments and bounds: Euler-Riemann zeta

$$\begin{cases} (\text{RH} \Rightarrow)\textbf{Lindelöf} & \zeta(\tfrac{1}{2}+it) \ll t^{\varepsilon} \quad \forall \varepsilon>0 \quad \text{(unknown)} \\ \text{convexity} & \zeta(\tfrac{1}{2}+it) \ll t^{\frac{1}{4}+\varepsilon} \quad \forall \varepsilon>0 \quad \text{(true, easy)} \end{cases}$$

$$2k^{th} \textbf{ moment } = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt$$

Example of **integral moments**

$$Z_k(w) = \int_1^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} \; t^{-w} \; dt$$

which has natural boundary for $k \geq 3$ (Diaconu-PG-Goldfeld).

$$\text{Lindelöf} \iff \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \ll T^{1+\varepsilon} \qquad \text{(for all } k,\ \varepsilon > 0)$$

Suitable moment estimates yield subconvex bounds:

$$\begin{cases} \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\, dt \ll T^{1+\varepsilon} & \not\Longrightarrow \text{ subconvex bd} \\ \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\, dt \ll T^{1+\varepsilon} & \not\Longrightarrow \text{ subconvex bd} \\ \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\, dt = T\,P(\log T) + O(T^{1-\delta}) & \Longrightarrow \text{ subconvex bd} \\ \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \ll T^{1+\varepsilon} \text{ with } 2k \geq 6 & \Longrightarrow \text{ subconvex bd} \end{cases}$$

1918 Hardy-Littlewood: $\int_0^T |\zeta(\frac{1}{2}+it)|^2\,dt \;\sim\; T\log T$ ($\not\Longrightarrow$ subconvex bound)

1921 Weyl (*not* by moments): $|\zeta(\frac{1}{2}+it)| \;\ll\; (1+|t|)^{\frac{1}{6}+\varepsilon}$ (= subconvex)

1926 Ingham: $\int_0^T |\zeta(\frac{1}{2}+it)|^4\,dt \;\sim\; \frac{1}{2\pi^2}\,T(\log T)^4$ ($\not\Longrightarrow$ subconvex bound)

1979 Heath-Brown:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \;\sim\; T\cdot P(\log T) + O(T^{\frac{7}{8}+\varepsilon}) \quad (\Longrightarrow \text{ subconvex bound})$$

Our prototype: **1982, A. Good:** holo cuspform f

$$\int_0^T |L(\tfrac{1}{2}+it,f)|^2\,dt \;=\; aT(\log T+b) + O((T\log T)^{2/3}) \;\Longrightarrow\; \text{subconvex}$$

Proof mechanism:
(1) automorphic spectral decomposition
(2) L-functions as decomposition coefficients.

Examples of decomposition coefficients and L-functions:

$$\begin{cases} \int y^{s-\frac{1}{2}} f\begin{pmatrix} y & \\ & 1\end{pmatrix} & = \quad \Lambda(s,f) \qquad (f \text{ on } GL_2) \\ \langle fg\,,\; E_s\rangle & = \quad \Lambda(s,f\otimes g) \quad (f,g \text{ on } GL_2)\end{cases}$$

Decomposition example: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

$$\Phi \;\sim\; \sum_{F \text{ cfm}} \frac{\langle \Phi, F\rangle}{\langle F,F\rangle}\cdot F \;+\; \frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \langle \Phi, E_s\rangle\cdot E_s\,ds \;+\; \frac{\langle \Phi, 1\rangle}{\langle 1,1\rangle}\cdot 1$$

Over a number field:

$$\Phi \;\sim\; \sum_{F \text{ cfm}} \frac{\langle \Phi, F\rangle}{\langle F,F\rangle}\cdot F \;+\; \frac{1}{4\pi i\kappa}\sum_{\chi}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \langle \Phi, E_{s,\chi}\rangle\cdot E_{s,\chi}\,ds$$

$$+\;\sum_{\chi^2=1} \frac{\langle \Phi, \chi\circ\det\rangle}{\|\chi\circ\det\|}\cdot(\chi\circ\det)$$

See *sum* over χ, *integral* over t.

Create integral kernel/Poincaré series $\mathfrak{P}^{\alpha,w}$ such that:

Integral moment of cuspform f produced by integral

$$\int \mathfrak{P}^{\alpha,w} \cdot |f|^2 \;=\; \frac{1}{2\pi} \int L(\tfrac{1}{2}+it+\alpha, f) \cdot L(\tfrac{1}{2}-it, \overline{f}) \cdot \big[\sim t^{-w}\big]\; dt$$

Spectral expansion of $\mathfrak{P}^{\alpha,w}$

$$\mathfrak{P}^{\alpha,w} \;=\; \frac{\pi^{\frac{1-w}{2}}\Gamma(\frac{w-1}{2})}{\pi^{-\frac{w}{2}}\Gamma(\frac{w}{2})} \cdot E_{1+\alpha} \;+\; \frac{1}{2} \sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2}+\alpha, \overline{F})}{\langle F, F\rangle} \cdot \mathcal{G}(\tfrac{1}{2}-it_F, \alpha, w) \cdot F$$

$$+\frac{1}{4\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\zeta(\alpha+s)\,\zeta(\alpha+1-s)}{\xi(2-2s)}\, \mathcal{G}(1-s, \alpha, w) \cdot E_s \, ds$$

with

$$\mathcal{G}(s,\alpha,w) = \pi^{-(\alpha+\frac{w}{2})} \frac{\Gamma(\frac{\alpha+1-s}{2})\,\Gamma(\frac{\alpha+s}{2})\,\Gamma(\frac{\alpha-s+w}{2})\,\Gamma(\frac{\alpha+s-1+w}{2})}{\Gamma(\alpha+\frac{w}{2})}$$

Continuous part cancels pole of leading term at $\alpha = 0$. Evaluated at $\alpha = 0$:

$$\int \mathfrak{P}^{0,w} \cdot |f|^2 \;=\; \frac{1}{2\pi} \int \Big| L(\tfrac{1}{2}+it, f) \Big|^2 \cdot \big[\sim t^{-w}\big]\; dt$$

and

$$\mathfrak{P}^{0,w} \;=\; \Big(\text{pole at } w\!=\!1\Big) \;+\; \frac{1}{2} \sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2}, \overline{F})}{\langle F, F\rangle} \cdot \mathcal{G}(\tfrac{1}{2}-it_F, 0, w) \cdot F$$

$$+\;\frac{1}{4\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\xi(s)\,\xi(1-s)}{\xi(2-2s)}\, \frac{\Gamma(\frac{w-s}{2})\,\Gamma(\frac{w-1+s}{2})}{\Gamma(\frac{w}{2})} \cdot E_s \; ds$$

Over $\mathbb{Q}(i)$ *grossencharacters* appear (Diaconu-Goldfeld 2006)

Similarly over number fields.
With suitable data,

Theorem: t-aspect subconvexity for GL_2 over number fields (Diaconu-PG 2006)

Proof ingredients: First, over $\mathbb{Q}$:

Spectral expansion of $\mathfrak{P}^{\alpha,w}$ gives meromorphic continuation

Obtain meromorphic continuation of generating function $Z(w)$

Leading pole at $w = 1$.

Trailing poles at $\mu = \mu_f$ with $\mu(\mu - 1)$ eigenvalue for waveform f.

Need polynomial vertical growth

Need **spectral gap** (Kim-Shahidi): separate cuspidal poles from leading pole

Need asymptotics of **triple integrals**

$$\int F \cdot |f|^2$$

of eigenfunctions (Sarnak, Bernstein-Reznikoff, Krötz-Stanton). Namely, *exponentially* decreasing, not merely *rapidly*.

See also Goldfeld-Hoffstein-Lockhart, Hoffstein-Ramakrishnan for asymptotics of

$$\frac{L(\frac{1}{2}, F)}{\langle F, F \rangle} \qquad (F \text{ cuspform})$$

Use **positivity** of moment sum-and-integral ... Landau's lemma

Complications over **number field**:

Freeze parameters w at all but one place... thus, breaking t-aspect convexity *at a single place*, not hybrid

Poles at eigenvalues of *one* Laplacian presumably *accumulate*, ... requiring finesse proving polynomial vertical growth

The diagonal/positivity property:

The diagonal form

$$\sum_{\chi} \int |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \ldots$$

rather than a smeared-out form

$$\sum_{\chi_1, \chi_2} \int \int L(\tfrac{1}{2} + it_1, f \otimes \chi_1) \cdot L(\tfrac{1}{2} - it_2, \overline{f} \otimes \chi_2) \ldots$$

is essential, or at least extremely convenient.

Arises as *deformation* of *diagonal distribution.*

Simpler example: Distribution u on $S^1 \times S^1$ integrating along the diagonal

$$u(f \otimes g) \;=\; \int_{S^1} f \cdot g$$

Has diagonal Fourier expansion

$$u \;=\; \sum_n e^{2\pi i n x} \otimes e^{-2\pi i n y}$$

since, by Plancherel,

$$\int_{S^1} f \cdot g \;=\; \sum_n \widehat{f}(n) \widehat{g}(-n) \;=\; \sum_{m,n} \widehat{f \otimes g}(m,n) \cdot \widehat{u}(m,n)$$

Similarly for *any* diagonal integral.

Clean-but-harsh pure diagonal distributions often usefully *deformed* to nearly-diagonal more-classical function.

Something appealing that doesn't work: to obtain $2k^{th}$ moments for GL_2 cuspforms?

Corresponding idea in more tangible situation of classical Fourier series: let u be the distribution on

$$H = \underbrace{S^1 \times \ldots \times S^1}_{2k}$$

given by integration along the subgroup

$$\Theta = \{(x_1, \ldots, x_k, y_1, \ldots, y_k) \;:\; \frac{x_1 \ldots x_k}{y_1 \ldots y_k} = 1\} \subset H$$

The Fourier expansion is concentrated along a diagonal line:

$$u = \sum_n e^{2\pi i n(x_1 + \ldots + x_k - y_1 - \ldots - y_k)}$$

For automorphic forms: let

$$H = \left\{ \begin{pmatrix} x_1 & \\ & 1 \end{pmatrix} \times \ldots \times \begin{pmatrix} x_k & \\ & 1 \end{pmatrix} \times \begin{pmatrix} y_1 & \\ & 1 \end{pmatrix} \times \ldots \times \begin{pmatrix} y_k & \\ & 1 \end{pmatrix} \right\} \approx GL_1^{2k}$$

and let u be integration along

$$\Theta = \Big\{ \text{elements in } H \text{ with } \;:\; \frac{x_1 \ldots x_k}{y_1 \ldots y_k} = 1 \Big\}$$

For cuspform f on GL_2, restrict

$$F = \underbrace{f \otimes \ldots \otimes f \otimes \overline{f} \otimes \ldots \overline{f}}_{2k}$$

to $H_k \backslash H_{\mathbb{A}}$ and evaluate u on it. Over $\mathbb{Q}$, produces

$$\int_{-\infty}^{\infty} \Big| \Lambda(\tfrac{1}{2} + it, f) \Big|^{2k} \, dt$$

Obstacle: *spectral* interpretation/expansion for $2k > 2$? Can *deform* to soften exponential decay of gamma factors, but...

Technicality: the weight in the integral moment

At $\alpha = 0$, *finite*-prime factors are literal Hecke-Jacquet-Langlands Mellin transforms of local Whittaker functions

$$L_v(s, f \otimes \chi) \;=\; \int_{k_v^\times} |y|^{s-\frac{1}{2}}\, \chi(y)\, W_{f,v}\begin{pmatrix} y & \\ & 1 \end{pmatrix}$$

and

$$\Big|L_v(\tfrac{1}{2}+it, f)\Big|^2 \;=\; \int\!\!\int |y/y'|^{it} \cdot W_{f,v}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \overline{W}_{f,v}\begin{pmatrix} y' & \\ & 1 \end{pmatrix}$$

Deformation at archimedean places entangles this with integration over unipotent radical: with additive character ψ and

$$h = \begin{pmatrix} y & \\ & 1 \end{pmatrix} \qquad h' = \begin{pmatrix} y' & \\ & 1 \end{pmatrix} \qquad n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

$|L_v(\frac{1}{2}+it, f)|^2$ is deformed into

$$\Big(\text{local factor at archimedean } v\Big)$$

$$= \int\!\!\int\!\!\int \varphi_v(n)\, |y'/y|_v^{s-\frac{1}{2}} \cdot W_{f,v}(hn) \cdot \overline{W}_{f,v}(h'n) \cdot$$

$$= \int\!\!\int\!\!\int \varphi_v(n)\, \psi\big((y-y')x\big) \cdot W_{f,v}(h) \cdot \overline{W}_{f,v}(h') \cdot |y'/y|_v^{s-\frac{1}{2}}$$

$$= \int\!\!\int \widehat{\varphi}_v(y-y') \cdot W_{f,v}(h) \cdot \overline{W}_{f,v}(h') \cdot |y'/y|_v^{s-\frac{1}{2}}$$

Except for holomorphic discrete series, these seem not usefully expressible by classical special functions.

The map from data φ_v to these integrals is definitely not *surjective* to an elementary space of functions.

Nevertheless, **asymptotically** t^{-w} for data $\varphi_v \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = (1+x^2)^{-w/2}$

Other features/possibilities for GL_2

Over quadratic extensions of $\mathbb{Q}$, *should* break convexity in χ-**aspect**, with same proof.

Full χ-aspect over number field subtler, like **hybrid** bounds.

Allow deformation at *finite* place, *should* break convexity in **depth**: unlimited ramification of χ's at *fixed* place. (Delia Letang, work-in-progress)

Replace cuspform by (packet of) Eisenstein series:

... *should* break t-aspect convexity for **Dedekind zetas of number fields**, via fourth moments.

... and *should* break t-aspect convexity for Hecke L-functions with grossencharacters, via fourth moments.

... and *should* break χ-aspect convexity for quadratic fields.

... and *should* break convexity in **depth** by deforming *finite prime* data.

Higher rank example (Diaconu-PG-Goldfeld, 2006)

Create $\mathfrak{P}^{\alpha,w}$ to produce **moment expansion** for $GL_3(\mathbb{Q})$ cuspform f

$$\int \mathfrak{P}^{0,w} \cdot |f|^2 \;=\; \sum_{F \text{ on } GL_2} \int \frac{|L(\frac{1}{2}+it, f\otimes F)|^2}{\langle F,F\rangle}\, M_{{}_F}\, dt$$

$$+\int\int \left| \frac{L(\frac{1}{2}+it_1-it_2, f)\cdot L(\frac{1}{2}+it_1+it_2, f)}{\zeta(1-2it_2)} \right|^2 M_{{}_E}\, dt_1\, dt_2$$

More-continuous part is sort of integral moment of *standard* $L(s,f)$ of f.

Spectral expansion of GL_3 Poincaré series is induced from GL_2

$$\mathfrak{P}^{\alpha,w} \quad = \quad (\infty\!-\!\text{part})\cdot E^{2,1}_{\alpha+1}$$

$$+ \quad \sum_{F \text{ on } GL_2} (\infty\!-\!\text{part})\cdot \frac{L(\frac{3\alpha+1}{2}+\frac{1}{2}, \overline{F})}{\langle F,F\rangle} \cdot E^{1,2}_{\frac{\alpha+1}{2},F}$$

$$+ \quad \int_{\text{Re}\,(s)=\frac{1}{2}} (\infty\!-\!\text{part})\cdot \frac{\zeta(\frac{3\alpha+1}{2}+1-s)\cdot\zeta(\frac{3\alpha+1}{2}+s)}{\zeta(2-2s)}$$

$$\times \quad E^{1,1,1}_{\alpha+1,\, s-\frac{\alpha+1}{2},\, -s-\frac{\alpha+1}{2}} \;\; ds$$

Only GL_2 cuspforms appear in spectral expansion of $\mathfrak{P}$! No GL_3 cuspforms!

It is *good* that no cuspidal data beyond GL_2 appears.

But this is also *confusing*: obscures easiest heuristics for computing and understanding *spectral* expansion of $\mathfrak{P}$.

In fact, $\mathfrak{P}^{\alpha,w}$ is a *residue* of **overlying Poincaré series** $\mathfrak{Q}^{\alpha,\beta,w}$ with more balanced *spectral* expansion, and retaining meaningful *moment* expansion. More on this later.

$GL_n(\mathbb{Q})$ more generally

Moment expansion:

$$\int \mathfrak{P}^{0,w} \cdot |f|^2 \;=\; \sum_{F \text{ on } GL_{n-1}} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{|L(s, f\otimes F)|^2}{\langle F, F\rangle} M_{_F}(s)\, ds \;+\; \ldots$$

Spectral expansion of $\mathfrak{P}^{\alpha,w}$

$$\mathfrak{P}^{\alpha,w} \quad = \quad (\infty\text{−part}) \;\cdot\; E^{n-1,1}_{1+\alpha}$$

$$+ \sum_{F \text{ on } GL_2} (\infty\text{−part}) \cdot \frac{L(\frac{n\alpha+n-2}{2}+\frac{1}{2}, F)}{\langle F, F\rangle} \cdot E^{n-2,2}_{\frac{1+\alpha}{2}, F}$$

$$+ \quad \int_{\mathrm{Re}\,(s)=\frac{1}{2}} (\infty\text{−part}) \times \;\; \frac{L(\frac{n\alpha+n-2}{2}+1-s, \overline{\chi}) \cdot L(\frac{n\alpha+n-2}{2}+s, \chi)}{L(2-2s, \overline{\chi}^2)}$$

$$\times \;\; E^{n-2,1,1}_{\alpha+1,\; s-(n-2)\frac{\alpha+1}{2},\; -s-(n-2)\frac{\alpha+1}{2}} \quad ds$$

Only GL_2 cuspforms appear in spectral expansion of $\mathfrak{P}$!
No GL_3, GL_4, GL_5, ... cuspforms!

Construction of kernel $\mathfrak{P}^{\alpha,w}$

$$U = \begin{pmatrix} 1_{n-1} & * \\ 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} (n-1) & 0 \\ 0 & 1 \end{pmatrix}$$

$Z =$ center G, $\;K_v$ maximal compact in G_v. Let $\varphi = \bigotimes_v \varphi_v$ with

$$\varphi_v(\begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \cdot K_v) = \Big|(\det A)/d^{n-1}\Big|_v^{\alpha} \qquad\qquad (\text{for } v \text{ finite})$$

Extend by 0 off $H_v Z_v K_v$. For $v|\infty$ require the same right left equivariance and K_v-invariance, with φ_v determined by values on U_v. For example, take

$$\varphi_v \begin{pmatrix} 1_{n-1} & x \\ 0 & 1 \end{pmatrix} = (1 + |x_1|^2 + \ldots + |x_{n-1}|^2)^{-w/2}$$

Kernel is

$$\mathfrak{P}^{\alpha,w}(g) = \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g)$$

More-continuous terms: higher integral moments

Most-continuous part of moment expansion for GL_n

$$\int \int_\Lambda |L(\tfrac{1}{2}+it, f \otimes E^{\min}_{\frac{1}{2}+i\widetilde{t}})|^2 \, M \; dt \, d\widetilde{t}$$

$$= \int \int_\Lambda \left| \frac{\Pi_{1 \le \ell \le n-1} \; L(\frac{1}{2}+it+it_\ell, f)}{\Pi_{1 \le j < \ell < n} \; \zeta(1-it_j+it_\ell)} \right|^2 \; M \; dt \, d\widetilde{t}$$

where

$$\Lambda = \{\widetilde{t} \in \mathbb{R}^{n-1} \; : \; t_1 + \ldots + t_{n-1} = 0\}$$

More generally, let $n-1 = m \cdot k$. For F on GL_m, let

$$F^\Delta = F \otimes \ldots \otimes F \quad \text{on } \underbrace{GL_m \times \ldots \times GL_m}_{k}$$

In moment expansion have

$$\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \int_\Lambda |L(s, f \otimes E_{F^\Delta, \; \frac{1}{2}+it})|^2 \, M_{F,t,s} \; ds \, dt$$

$$= \int \int \left| \frac{\Pi_{1 \le \ell \le k} \; L(s+it_\ell, f \otimes F)}{\Pi_{1 \le j < \ell \le k} \; L(1-it_j+it_\ell, F \otimes F^\vee)} \right|^2 M \; ds \, dt$$

$$\sim \; \text{a kind of } \textit{higher} \text{ moment of } L(\tfrac{1}{2}+it, f \otimes F)$$

Wave-packets of Eisenstein series

Replace cuspform f on GL_n by (*packet* of) minimal-parabolic Eisenstein series E_β with $\beta \in \mathbb{C}^{n-1}$.

Most-continuous part of the moment expansion

$$\int \int_\Lambda \left| \frac{\prod_{1\le\mu\le n,\ 1\le\ell\le n-1} \ \zeta(\beta_\mu + \frac{1}{2} + it + it_\ell)}{\prod_{1\le j<\ell<n} \ \zeta(1 - it_j + it_\ell)} \right|^2 dt\, d\widetilde{t}$$

where, again,

$$\Lambda = \{\widetilde{t} \in \mathbb{R}^{n-1} \ : \ t_1 + \ldots + t_{n-1} = 0\}$$

At $\beta = 0 \in \mathbb{C}^{n-1}$ (ignore vanishing of E_β!)

$$\int \int_\Lambda \left| \frac{\prod_{1\le\ell\le n-1} \ \zeta(\frac{1}{2} + it + it_\ell)^n}{\prod_{1\le j<\ell<n} \ \zeta(1 - it_j + it_\ell)} \right|^2 M\, dt\, d\widetilde{t}$$

$\Longrightarrow$ *GL_n produces high moments of ζ.*

Example: for GL_3, where $\Lambda = \{(t_1, -t_1)\} \approx \mathbb{R}$,

$$\int \int_\mathbb{R} \left| \frac{\zeta(\frac{1}{2} + it + it_1)^3 \cdot \zeta(\frac{1}{2} + it - it_1)^3}{\zeta(1 - 2it_1)} \right|^2 M\ dt\, dt_1$$

Example: for GL_4

$$\int \int_\Lambda \left| \frac{\zeta(\frac{1}{2} + it + it_1)^4 \cdot \zeta(\frac{1}{2} + it + it_2)^4 \cdot \zeta(\frac{1}{2} + it + it_3)^4}{\zeta(1 - it_1 + it_2)\, \zeta(1 - it_1 + it_3)\, \zeta(1 - it_2 + it_3)} \right|^2 M\, dt\, d\widetilde{t}$$

Actually, these should be *integrated* as in corresponding *packets*.

Caution: not all *moments* results bear on *subconvexity*:

Assume Lindelöf, Weyl's Law to determine asymptotic-with-error for *moment*, see implied pointwise estimate.

Failure to match convexity ... does not imply corresponding asymptotic-with-error is *useless*.

Outcome depends upon the parameters varied in the L-functions... upon the *family* averaged-over.

Much more difficult to understand *joint* asymptotics-with-error. Do *not* expect to prove *hybrid* subconvexity. Essentially no result of this type is known! Thus, be cautious about reasonable forms of joint asymptotics-with-error, though Iwaniec-Sarnak formulation makes sense for hybrid estimates.

Of special interest are *second* moments, most easily produced by *spectral identities*.

Simplest failure to match convexity:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \;=\; T\,P(\log T) + O(T^{1-\text{small}}) \qquad \text{(known)}$$

implies, by standard methods,

$$|\zeta(\tfrac{1}{2}+it)|^2 \;\ll\; (1+|t|)^{1-\text{small}}$$

and then

$$|\zeta(\tfrac{1}{2}+it)| \;\ll\; (1+|t|)^{\frac{1}{2}-\text{small}}$$

But *convexity* is

$$\zeta(\tfrac{1}{2}+it) \;\ll\; (1+|t|)^{\frac{1}{4}+\varepsilon}$$

Small shift in exponent is *very* small, so cannot help get the exponent below 1/4. *Failed* to match convexity.

Success:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \;=\; T\,P(\log T) + O(T^{1-\text{small}}) \qquad \text{(known)}$$

does break convexity.

Example: second moments of GL_2 L-functions

Convexity bound in terms of *analytic conductor* (Iwaniec-Sarnak)

$$|L(\tfrac{1}{2}+it,f)| \ \ll_\varepsilon \ (1+|t+\mu_f|)^{\frac{1}{4}+\varepsilon}\,(1+|t-\mu_f|)^{\frac{1}{4}+\varepsilon}$$

Success: t**-aspect** (as above). Fix f.

$$|L(\tfrac{1}{2}+it,f)| \ \ll_{\varepsilon,f} \ (1+|t|)^{\frac{1}{2}+2\varepsilon} \qquad \text{(convexity)}$$

From

$$\int_0^T |L(\tfrac{1}{2}+it,f)|^2\,dt \ = \ T\,P(\log T)+O(T^{1-\text{small}}) \qquad \text{(fixed } f\text{)}$$

Get pointwise

$$|L(\tfrac{1}{2}+it,f)| \ \ll_{\varepsilon,f} \ (1+|t|)^{\frac{1}{2}-\text{small}}$$

breaking convexity in the t-aspect.

Failure: eigenvalue-aspect Fix t.

$$|L(\tfrac{1}{2}+it,f)| \ \ll_{\varepsilon,t} \ (1+|\mu_f|)^{\frac{1}{2}+\varepsilon} \qquad \text{(convexity)}$$

From

$$\sum_{|\mu|\le T} |L(\tfrac{1}{2}+it_o,f)|^2 \ = \ T^2\,P(\log T)+O(T^{2-\text{small}})$$

since (Weyl) number of f with $|\mu_f| \ll T$ is of the order of T^2, would have

$$|L(\tfrac{1}{2}+it,f)| \ \ll_{\varepsilon,t} \ (1+|\mu_f|)^{1-\text{small}}$$

failing to equalize convexity.

Example: Rankin-Selberg convolutions

Convexity bound in terms of *analytic conductor*

$$|L(\tfrac{1}{2}+it, f\otimes g)| \ \ll_\varepsilon \ \Big(\prod_{signs}(1 \ + \ |t \pm \mu_f \pm \mu_g|)\Big)^{\frac{1}{4}+\varepsilon}$$

t**-aspect?** Fix f, g.

$$|L(\tfrac{1}{2}+it, f\otimes g)| \ \ll_{\varepsilon,f,g} \ (1+|t|)^{1+\varepsilon} \qquad \text{(convexity)}$$

From

$$\int_0^T |L(\tfrac{1}{2}+it, f\otimes g)|^2\,dt \ = \ T\,P(\log T) + O(T^{1-\text{small}}) \qquad \text{(fixed } f\text{)}$$

get pointwise

$$|L(\tfrac{1}{2}+it, f\otimes g)| \ \ll_{\varepsilon,f} \ (1+|t|)^{\frac{1}{2}-\text{small}}$$

breaking convexity in t-aspect.

Too good for existence of corresponding spectral family.

g**-aspect** Fix t, f.

$$|L(\tfrac{1}{2}+it, f\otimes g)| \ \ll_{\varepsilon,t,f} \ (1+|\mu_g|)^{1+\varepsilon} \qquad \text{(convexity)}$$

From

$$\sum_{|\mu_g|\le T} |L(\tfrac{1}{2}+it, f\otimes g)|^2 \ = \ T^2\,P(\log T) + O(T^{2-\text{small}})$$

would have

$$|L(\tfrac{1}{2}+it, f\otimes g)| \ \ll_{\varepsilon,t,f} \ (1+|\mu_g|)^{1-\text{small}}$$

breaking convexity.

Plausible spectral family.

Spectral identity, Rankin-Selberg for GL_2

Avoiding conductor dropping considerations...

Can produce spectral identity

$$\int\int \mathfrak{P}^{0,w} \cdot |f \otimes E_{\frac{1}{2}+it}|^2 \;=\; \sum_{g \text{ cfm on } GL_2} |L(\tfrac{1}{2}+it, f\otimes g)|^2 \cdot \mathrm{wt}(g) + \ldots$$

with positivity and diagonal properties, weight depends upon archimedean data of g (and of fixed t and f).

$\mathfrak{P}^{0,w}$ has good spectral expansion on $GL_2 \times GL_2$.

Apparently:

Leading pole of $\mathfrak{P}$ at $w = 2$.

Trailing poles of $\mathfrak{P}$ at $\mu = \mu_F \in \mathbb{C}$ with cuspform F eigenvalue $\mu(\mu - 1)$.

Known **spectral gap** should allow subconvex bound.

Over *number field*, separate parameters at archimedean places.

Deformation at finite set of finite places should allow averaging g in *depth*.

Rankin-Selberg for GL_3?

Avoiding conductor dropping considerations...

A spectral identity exists, producing g**-aspect moment**

$$\int \mathfrak{P} \cdot |f \otimes E_s|^2 \;=\; \sum_{g \text{ cfm on } GL_3} |L(s, f \otimes g)|^2 \cdot \mathrm{wt}(g) + \ldots$$

with positivity and diagonal properties, and weight function depending only upon archimedean data of g (and t and f).

$\mathfrak{P}$ itself has good spectral expansion on $GL_3 \times GL_3$.

Convexity bound:

$$|L(\tfrac{1}{2} + it, f \otimes g)| \;\ll_{\varepsilon,t,f}\; \Big((1 + |\mu_1|)(1 + |\mu_2|)(1 + |\mu_3|)\Big)^{\frac{9}{4} + \varepsilon}$$

Asymptotic with *very good* error

$$\sum_{|\mu_j| \le T} |L(s, f \otimes g)|^2 \;=\; T^5\, P(\log T) + O(T^{\frac{9}{2} - \text{small}})$$

would break convexity in g-aspect.