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Euler factorization of global integrals

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We consider integrals of cuspforms f on reductive groups G defined over numberfields k against restrictions ι^*E of Eisenstein series E on "larger" reductive groups $\tilde{G}$ over k via imbeddings $\iota : G \to \tilde{G}$. We give hypotheses sufficient to assure that such global integrals have Euler products. At good primes, the local factors are shown to be rational functions in the corresponding parameters q^{-s} from the Eisenstein series and in the Satake parameters q^{-s_i} coming from the spherical representations locally generated by the cuspform. The *denominators* of the Euler factors at good primes are estimated in terms of "anomalous" intertwining operators, computable via orbit filtrations on test functions. The standard intertwining operators (attached to elements of the (spherical) Weyl group) among these unramified principal series yield symmetries among the anomalous intertwining operators, thereby both sharpening the orbit-filtration estimate on the denominator and implying corresponding symmetry in it. Finally, we note a very simple dimension-counting heuristic for fulfillment of our hypotheses, thereby giving a simple test to *exclude* configurations $\iota : G \to \widetilde{G}$, E, as candidates for Euler product factorization. Some simple examples illustrate the application of these ideas.

Introduction

Application of the *analytic* properties of Eisenstein series to L-functions occurs in a variety of settings wherein L-functions are expressed as integrals of cuspforms against restrictions of Eisenstein series (and similar devices). At the very least, the analytic continuation and functional equation of an Eisenstein series translate immediately into an analytic continuation and functional equation for whatever it is that arises as an integral against the Eisenstein series.

The *arithmetic* of L-functions is subtler, and of more recent origin. The idea that special values of L-functions could inherit their *arithmetic* properties from those of Eisenstein series was made clear in [Shimura 1975a] and in the many succeeding papers of Shimura and a few others. For that matter, in lectures 1975-77 at Princeton, Shimura gave a compelling exposition of the results of [Klingen 1962] wherein one employs Hilbert modular Eisenstein series to prove special values results for certain zeta functions (and L-functions) of totally real number fields.

As in [Shimura 1975a] and [Shimura 1975b] and many other of his papers, Shimura consistently made the general arithmetic of holomorphic automorphic forms appear as a corollary of his results on canonical models [Shimura 1970]. In fact, even the arithmetic of Eisenstein series can be made to be a consequence of canonical models, as in [Harris 1981], [Harris 1984]. Although this viewpoint seems to be the broadest in scope (see [Harris 1985] and [Harris 1986]), it is also possible to prove many foundational arithmetic results directly, as done in [Garrett 1990] and [Garrett 1992], by using properties of Eisenstein series.

It is not a trivial matter to obtain an integral representation of an L-function, whether for analytic or arithmetic ends. In the first place, it is *not* the case that one chooses an L-function and then hunts for an integral representation of it. Rather, one "searches" among integrals of cuspforms against Eisenstein series (etc.) for those integrals which admit Euler product factorizations. Traditionally, it is only after the local integrals are computed that one discovers what L-function has been obtained. (Possible exceptions are [Langlands 1967], [Moreno-Shahidi 1985], [Shahidi 1989], and other papers of these authors which obtain L-functions as constant terms of suitable Eisenstein series, in a less mysterious fashion than Rankin-Selberg integrals).

Certainly few among integrals of cuspforms against Eisenstein series have any sort of Euler product expansion at all, and indeed may fail to be Dirichlet series entirely (with or without tolerance for trouble with the archimedean integrals).

And even when an integral of cuspform against Eisenstein series can be shown to factor over primes in a meaningful way, there can remain an onerous computation to determine the local integrals. If a too-classical viewpoint is taken, it may be impossible to anticipate the ultimate success or failure of such a venture until the last moment. Surveys such as [Gelbart-Shahidi 1988] give an idea of the variety of phenomena which appear.

Given the haphazard nature of hunting for integral representations, it is striking to this author that, whenever a global integral *does* factor over primes, the local factors are not only *rational* functions in the respective p^{-s}'s, but, quite magically, are also *automorphic*, as opposed to being meaningless Dirichlet series capriciously attached to a cuspform. In part, the present paper is this author's attempt to explain why integral representations should work as well as they do.

Now we give an overview of this paper using terminology explained only later.

The first third of the paper addresses Euler factorization of Rankin-Selberg type global integrals involving Eisenstein series and cuspforms.

In section 1.1 we do the by-now-usual unwinding of the Eisenstein series occurring in the global integral. Although it is not strictly necessary, we assume for simplicity that there is a single *cuspidal representative.*

In section 1.2 we rearrange the unwound integral to isolate all really global information in what we call the *inner integral*, which does not involve the parameter "s" from the Eisenstein series. It is observed that with a certain multiplicity-one hypothesis locally almost everywhere, the inner integral factors over primes, up to a global constant which is reasonably viewed as a *period* of the cuspform. In this case, it then follows that the *whole* integral factors over primes, with the local factors depending only upon local data, except for the single global object, the period. Of course, an important element here is that we view all the integrals involved as *values of intertwining operators.*

Section 1.3 makes a trivial but important observation that vanishing of global integrals can be due to local features of the situation.

In section 1.4 we use orbit filtrations on the relevant induced representations (from 3.2) to give an easily verifiable (when true) condition which suffices implies that the inner integral factors. To do so, we effectively replace spherical representations by unramified principal series. The natural intertwining operators among unramified principal series, together with the generic irreducibility of the unramified principal series, offer a further refinement of this multiplicity estimate.

Section 1.5 gives a heuristic criterion for success or failure of this approach in a given situation, in terms of a simple dimension-count.

The middle third of this paper addresses computation of a class local integrals including the so-called *outer integrals* which are the Euler factors of the global integrals in 1.2.

Section 2.1 sets up the notion of *parametrized family of representations*, and *parametrized family of intertwining operators.* For our purposes, the parameter spaces are affine algebraic varieties. Some simple natural examples are given, mostly referring to unramified principal series and degenerate principal series, which are all we need here.

In section 2.2 the *Rationality Lemma* is stated. This is the key device in proving the rationality of the local factors. We immediately invoke the Rationality Lemma to prove two theorems asserting the *rationality of certain types of local integrals.*

Section 2.3 uses the results on rationality of local integrals to give an analytic continuation result for the natural integral expressions for the intertwining operators arising in the *inner integral* in the Euler factors, under finite-orbit and generic multiplicity-one hypotheses.

Section 2.4 again invokes the rationality of local integrals to prove that, under finite-orbit and *generic* multiplicity-one hypotheses, the Euler factors at almost all primes are *rational functions* in the corresponding parameter q^{-s}, and in the Satake parameters q^{-s_i} of the cuspform. We also anticipate the later result which gives a limitation on the possible hypersurfaces along which these rational functions may have poles.

The newest item here is the study of the *denominators* of the local factors. In preparation for this, section 2.5 states the theorem on *strong meromorphy* of families of intertwining operators under assumptions of generic multiplicity one. As a corollary of this theorem on strong meromorphy, section 2.6 shows how to associate an *anomalous intertwining operator* to a hypersurface along which a parametrized intertwining operator has a pole by taking the residue (or, generally, leading term in a Laurent expansion). These anomalous intertwining operators can be estimated by orbit-filtration methods. Thus, as a corollary, we have a computational device to estimate the denominators of the local factors. And the intertwining operators among unramified principal series (attached to elements of the Weyl group) imply *symmetry* in this estimate.

Section 2.7 illustrates the application of these ideas to some simple situations where for other reasons we know "the answer" for comparison: Tate's thesis [Tate 1950], and an instance of [Godement-Jacquet 1972].

The last third of the paper completes the proofs. In section 3.1 we prove the Rationality Lemma. This is essentially elementary, but a little complicated.

Section 3.2 settles necessary details concerning the orbit filtrations. Section 3.3 proves the exactness of Jacquet functors (and of the functors which project to K-fixed vectors, for compact open subgroups K) in the context of parametrized families of representations. Similarly, section 3.4 proves Frobenius Reciprocity for parametrized families.

Finally, section 3.5 proves the orbit criterion for strong meromorphy, thus completing all the arguments.

The idea of viewing integrals as intertwining operators is well known, as is the idea of using multiplicity-free-ness of induced representations. [Piatetskii-Shapiro 1975] and [Jacquet-Langlands 1972] are two of the earliest examples of this.

There is the more technical theme of understanding the dependence of local integrals upon parameters. The *Rationality Lemma* here is a variation and development of a natural idea evidently originating already in [Bruhat 1961], and systematically exploited in [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977]. The fullest treatment of the idea of parametrization of families of representations seems to be [Bernstein 1984]. In our context, we use a Rationality Lemma both to prove that various local integrals are rational expressions in parameters q^{-s} (and in Satake parameters q^{-s_i}), and (thereby) also to give *analytic continuations* of intertwining operators. The present issues are in part different from those in [Bernstein 1984], seeking as we do to *assert* an algebraic structure on intertwining operators given by integrals. But after the algebraicization is accomplished, the present issues are an amplification of a special case of [Bernstein 1984], aimed simply at unramified principal series, refined a bit as preparation for more delicate discussion of *the hypersurfaces along which natural intertwining operators have "poles"*.

It should be noted that throughout we systematically replace (admissible) spherical representations by the unramified principal series which map surjectively to them, invoking [Matsumoto 1977], [Borel 1976], or [Casselman 1980]. In this optic, the Satake parameters can be read from the data for the corresponding unramified principal series. Most importantly, modelling spherical representations as (images of) induced representations allows application of a variety of more "physical" arguments, including not only Frobenius Reciprocity, but also *orbit filtrations* on spaces of test functions, as instigated in [Bruhat 1961] and used extensively in [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977], [Bernstein 1984]. Thus, *generically*, the surjection of unramified principal series to a spherical representation is an *isomorphism.*

One particular sharpening of the ideas about parametrized families is necessary for detailed consideration of the *residue* (or, generally, *leading term* in a Laurent expansion) at a pole of a meromorphic family of intertwining operators. Such residues are demonstrably intertwining operators of a rather special sort. Coming from the other side, attempting to define an *anomalous* intertwining operator as one which "does not exist for *generic* parameter values", we can *estimate* the poles of the family of intertwining operators by estimating the possible anomalous intertwining operators. Orbit-filtration methods are very effective here, as well. But, of course, the phrase "does not exist for generic parameter values" must be made precise.

There is a further complication in discussion of residues of meromorphic families of intertwining operators, namely verification that the Laurent expansion along a hypersurface has *uniformly* finite order. This is a *strong meromorphy* condition, which we see follows from a natural Noetherian-ness condition. In the present context, the relevant Noetherian-ness for unramified principal series is not hard to prove. A much more general Noetherian-ness result is proven in [Bernstein 1984].

Discussion of residues of families of intertwining operators along hypersurfaces, studied as anomalous intertwining operators, also requires that we tolerate representations over the fields of rational functions on the hypersurfaces, which typically are of positive transcendence degree over the natural base field (e.g., $\mathbf{C}$). This complication arises because in addition to the ubiquitous q^{-s} parameter we might also want to track the Satake parameters q^{-s_i} of spherical representations, leading naturally to many-variable situations.

Incidental to the above, we prove some lemmas about filtrations of various test-function representation spaces by orbits. These are the parametrized-family version of results going back to [Bruhat 1961]. Parametrized-family versions of facts about Jacquet functors and related matters are also necessary. It must be noted that these extensions of otherwise standard representation-theoretic facts are not entirely trivial, since many properties of representations on *vectorspaces* certainly do not extend to representations on *modules*. See [Bernstein 1984] for similar extensions of some standard results.

A small but meaningful point is the choice of category in which to do representation theory: for present purposes, we operate almost entirely in the category of *smooth* representations (of totally disconnected groups, or of Lie groups, or of adele groups). Thus, there are technical but significant differences in comparison to the somewhat more popular category of *unitary* representations. See [Gross 1991] for some illustrations of effective methods in the unitary category. For effective discussion of *parametrized families*, it appears necessary to give up unitariness in order to gain the possibility of having the parameter space be an (affine) algebraic variety. This viewpoint is corroborated by [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977], [Bernstein 1984].

1. Factorization over Primes

1.1The usual unwinding trick

In this section, we set up family of a class of global integrals, in which a cuspform is integrated against a restriction of an Eisenstein series. The only thing we accomplish is the usual *unwinding of Eisenstein series*, and explain the technical hypothesis that there is a *unique cuspidal orbit.*

Let $G \subset \tilde{G}$ be two reductive linear groups defined over a number field k. Let $\tilde{P}$ be k-rational parabolic subgroup of $\tilde{G}$. Let

$$\alpha : \tilde{P} \to \mathbf{G}_m$$

be an algebraic character on $\tilde{P}$, and for a complex parameter s let

$$\tilde{\chi} : \tilde{P}_{\mathbf{A}} \to \mathbf{C}^\times$$

be defined by

$$\tilde{\chi}(\tilde{p}) = |\alpha(\tilde{p})|^{-s}$$

where $|\,|$ is the idele norm. This character factors over primes, and we write

$$\tilde{\chi} = \Pi'_v \, \tilde{\chi}_v$$

where $\tilde{\chi}_v$ is a continuous (unramified) character of the v-adic points $\tilde{P}_v$ of $\tilde{P}$. Choose a vector ε of the special form

$$\varepsilon = \bigotimes_v \varepsilon_v$$

in the induced representation

$$\text{c-Ind}_{\tilde{P}_{\mathbf{A}}}^{\tilde{G}_{\mathbf{A}}} \tilde{\chi} \approx \bigotimes{}'_v \text{c-Ind}_{\tilde{P}_v}^{\tilde{G}_v} \tilde{\chi}_v$$

where each ε_v is in the induced representation $\text{c-Ind}_{\tilde{P}_v}^{\tilde{G}_v} \tilde{\chi}_v$ and form an Eisenstein series

$$E(\tilde{g}) = E_\varepsilon(\tilde{g}) = \sum_{\gamma \in \tilde{P}_k \backslash \tilde{G}_k} \varepsilon(\gamma \tilde{g})$$

(The adelic compact-induced representation is defined to be the restricted product of all the local compact-induced representations).

Of course, a similar construction with several algebraic characters α_i and several complex parameters gives a more general class of Eisenstein series. One can involve a cuspform on a Levi component of $\tilde{P}$, as well.

Let G be a reductive group over k, with $G \subset \tilde{G}$. Let the center of G be Z, and let $\tilde{Z}$ be the center of $\tilde{G}$. There is a minor but unavoidable technical issue here, which is sometimes overlooked: we would wish to assume that the center Z of G is central in $\tilde{G}$, but this is far from true, generally, and is needlessly restrictive. Rather, we require that

$$(\tilde{Z}_{\mathbf{A}} \cdot Z_k)\backslash Z_{\mathbf{A}} \qquad \text{is compact}$$

This is the case if the k-ranks of Z and $\tilde{Z}$ are the same (see [Godement 1963]).

Let f be a cuspform on G. Consider **global integrals**

$$\zeta(f, \tilde{\chi}) = \zeta(f, s) = \int_{\tilde{Z}_{\mathbf{A}} G_k \backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg$$

The complex parameter s enters via the Eisenstein series.

Regarding convergence of this integral: from [Godement 1966], smooth square-integrable cuspforms are of rapid decay in Siegel sets. On the other hand, Eisenstein series are of moderate growth in Siegel sets, even when analytically continued, by [Langlands 1964] or [Moeglin-Waldspurger 1995]. A comparison of Siegel sets in G versus those in $\tilde{G}$ is necessary to be sure that the moderate growth of the Eisenstein series is inherited by its restriction (see [Godement 1963]).

The first fundamental trick is the by-now-standard *unwinding*

$$\int_{\tilde{Z}_{\mathbf{A}} G_k \backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg = \sum_{\xi \in \tilde{P}_k \backslash \tilde{G}_k / G_k} \int_{\tilde{Z}_{\mathbf{A}} G_k \backslash G_{\mathbf{A}}} \sum_{\gamma \in \Theta_k^\xi \backslash G_k} \varepsilon(\xi \gamma g)\, f(g)\, dg$$

$$= \sum_{\xi \in \tilde{P}_k \backslash \tilde{G}_k / G_k} \int_{\tilde{Z}_{\mathbf{A}} \Theta_k^\xi \backslash G_{\mathbf{A}}} \varepsilon(\xi g)\, f(g)\, dg$$

where

$$\Theta^\xi = \xi^{-1} \tilde{P} \xi \cap G$$

which is valid for s in the range so that the Eisenstein series converges.

A representative ξ or orbit $\tilde{P}\xi G$ is **non-cuspidal** (or *negligible*) if Θ^ξ has a normal subgroup which is the unipotent radical of some k-parabolic of G. So say that ξ is **cuspidal** if Θ^ξ does *not* have a normal subgroup which is a unipotent radical. If there *is* such a normal subgroup, then by the Gelfand definition of cuspform the corresponding integral is 0. Thus,

$$\int_{\tilde{Z}_{\mathbf{A}} G_k \backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg = \sum_{\xi \text{ cuspidal}} \int_{\tilde{Z}_{\mathbf{A}} \Theta_k^\xi \backslash G_{\mathbf{A}}} \varepsilon(\xi g)\, f(g)\, dg$$

Remark: It is traditional to hypothesize that there is just one cuspidal double-coset representative $\xi \in \tilde{P}_k \backslash \tilde{G}_k / G_k$, but this may be needlessly restrictive.

Remark: Later we will take the hypothesis that the v-adic double-coset space $\tilde{P}_v \backslash \tilde{G}_v / P_v$ is *finite* for a minimal parabolic P_v of G (over the v-adic completion k_v of k). In practice, it seems that this finiteness assumption "nearly" assures the finiteness of $\tilde{P}_k \backslash \tilde{G}_k / G_k$, but this author knows of no simple general assertion in this direction. Certainly from a dimension-counting viewpoint the global double-coset space $\tilde{P}_k \backslash \tilde{G}_k / G_k$ is "smaller"; however, *rationality* issues enter strongly in the case of $\tilde{P}_k \backslash \tilde{G}_k / G_k$. It is often necessary to make adjustments, such as replacing isometry groups by the corresponding similitude groups.

1.2Multiplicity-one criterion for Euler factorization

In this section we rewrite the integral in a form so that the global features are isolated in what we call the **inner integral**. The (apocryphal?) theorem of this section asserts roughly that *if the inner integral factors over primes, then the whole integral factors over primes.* We clarify what should be meant by "factors over primes".

In subsequent sections this will be refined, so as to anticipate more precisely the nature of the local factors, proving *a priori* that the p^{th} factor is a *rational function* in the usual q^{-s} *and* in the Satake parameters $\alpha_i = q^{-s_i}$ attached to the cuspform. Further, we will indicate how to estimate the set of parameter values at which the denominator of this rational function *vanishes.* In particular, we will give a criterion to assure that the denominator is a product of terms exclusively of the form

$$1 - c\, q^{-m_1 s_1} \ldots q^{-m_n s_n}\, q^{-s}$$

where the m_i are integers and c is a constant independent of the s, s_i. And we indicate computationally effective means to verify the hypotheses.

First we recall an integration identity (see [Weil 1965], chapter II). In temporary notation, let G be a topological group (locally compact, Hausdorff, with a countable basis). Let H be a closed subgroup. Let dg and dh denote right Haar measures on G and H. Let δ_G and δ_H be the modular functions on G and H. As usual, let $C_c^o(G)$ denote the collection of continuous compactly-supported (complex-valued) functions on G. Let $C_c^o(H\backslash G, \delta_H \delta_G^{-1})$ denote the collection of continuous functions f on G so that

$$f(hg) = \frac{\delta_H(h)}{\delta_G(h)} \cdot f(g)$$

for all $h \in H$ and $g \in G$, and whose support is compact left-modulo H. Then, given the Haar measures on G and H, there is a unique functional on $C_c^o(H\backslash G, \delta_H \delta_G^{-1})$, denoted

$$F \to \int_{H\backslash G} F(\overline{g})\, d\overline{g}$$

so that for any $f \in C_c^o(G)$ we have the identity

$$\int_G f(g)\, dg = \int_{H\backslash G} \left(\int_H f(h\overline{g}) \, \frac{\delta_G(h)}{\delta_H(h)} \, dh \right) d\overline{g}$$

It is elementary that the **averaging map**

$$\text{avg} : f \to \left(g \to \int_H f(h\overline{g}) \, \frac{\delta_G(h)}{\delta_H(h)} \, dh \right)$$

is a surjection to $C_c^o(H\backslash G, \delta_H \delta_G^{-1})$. Note that unless

$$\delta_H = \delta_G \,|_H$$

it is not the case that the functional

$$F \to \int_{H\backslash G} F(\overline{g})\, d\overline{g}$$

is literally an *integral.* As a further abuse of notation, we will omit the overbar, thus writing the identity as

$$\int_G f(g)\, dg = \int_{H\backslash G} \left(\int_H f(hg) \, \frac{\delta_G(h)}{\delta_H(h)} \, dh \right) dg$$

Of course, the Haar integral $f \to \int_G f(g)\, dg$ extends by continuity to a continuous functional on $L^1(G)$ (the latter with right Haar measure). The proof of Fubini's theorem (e.g., by monotone families) still works in this case, at least with base field $\tilde{k} = \mathbf{R}$ or $\mathbf{C}$, proving that for $f \in L^1(G)$ and for almost all $g \in G$

$$\int_H |f(hg)| \frac{\delta_G(h)}{\delta_H(h)} \, dh < +\infty$$

This allows an extension of the "integral" on $H\backslash G$, and the identity above still holds.

In particular, we can define **"integrable"** functions in a generalized sense on $H\backslash G$ as being the images of elements of $L^1(G)$ under the extension of the averaging map. We can also talk about **convergence** of such generalized integrals by similar devices.

Now return to the global integral. Continue with notation and terminology from above. Consider a *cuspidal* ξ. For reductive G, the adele group $G_{\mathbf{A}}$ is unimodular, so no δ_G will appear. Further, for any linear algebraic group H defined over a number field k, the modular function $\delta_{H_{\mathbf{A}}}$ is trivial on the group of k-rational points

H_k of H. (See [Weil 1961]: this is essentially the *product formula*). Thus, using the identity above twice, we can rewrite the global integral as

$$\int_{\tilde{Z}_{\mathbf{A}} G_k\backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg = \int_{\tilde{Z}_{\mathbf{A}}\Theta^\xi_{\mathbf{A}}\backslash G_{\mathbf{A}}} \int_{Z'_{\mathbf{A}}\Theta^\xi_k\backslash\Theta^\xi_{\mathbf{A}}} \varepsilon(\xi\theta g)\, f(\theta g)\, \delta_{\Theta_{\mathbf{A}}}(\theta)^{-1}\, d\theta\, dg$$

where

$$Z' = \tilde{Z} \cap \Theta^\xi$$

A crucial hypothesis, which is elementary to prove or disprove in any particular situation, is that

$$\xi\Theta^\xi_{\mathbf{A}}\xi^{-1} \subset \ker(\tilde{\chi} \text{ on } \tilde{P}_{\mathbf{A}})$$

Granting this, we have

$$\int_{\tilde{Z}_{\mathbf{A}} G_k\backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg = \int_{\tilde{Z}_{\mathbf{A}}\Theta^\xi_{\mathbf{A}}\backslash G_{\mathbf{A}}} \varepsilon(\xi g) \int_{Z'_{\mathbf{A}}\Theta^\xi_k\backslash\Theta^\xi_{\mathbf{A}}} f(\theta g)\, \delta^{-1}_{\Theta_{\mathbf{A}}}(\theta)\, d\theta\, dg$$

Refer to the integral

$$\int_{Z'_{\mathbf{A}}\Theta^\xi_k\backslash\Theta^\xi_{\mathbf{A}}} f(\theta g)\, \delta^{-1}_{\Theta_{\mathbf{A}}}(\theta)\, d\theta$$

as **the inner integral**.

The hypothesis just mentioned, namely that

$$\xi\Theta^\xi_{\mathbf{A}}\xi^{-1} \subset \ker(\tilde{\chi} \text{ on } \tilde{P}_{\mathbf{A}})$$

assures that the function $\varepsilon(\xi g)$ acting as kernel for formation of the Eisenstein series can be moved outside the inner integral. Thus, this hypothesis is that **the Eisenstein kernel escapes from the inner integral**.

The intuitive point is that *if the inner integral factors over primes, then the whole integral factors over primes.* Indeed, by its definition, the kernel $\varepsilon(\xi g)$ for the Eisenstein series is at worst a finite sum of products over primes, so the only serious obstacle to such factorization is the *inner integral.*

And, of course, we must clarify what **factorization over primes** must mean. First, and most elementarily, a function $F(g)$ on an adele group would be said to *factor over primes* v if it can be rewritten as

$$F(g) = \Pi_v\, F_v(g_v)$$

where g_v is the v^{th} component of g in the adele group. But, when F depends additionally upon a parameter such as a cuspform f, more must be said. Indeed, by hypothesis, (and using the strong notion of "cuspform") the representation π_f of the adele group $G_{\mathbf{A}}$ generated by f under the right regular representation is *irreducible* (and unitarizable), so factors over primes as (the completion of) a restricted product

$$\pi_f \approx \bigotimes{}'_v\, \pi_v$$

where π_v is an irreducible unitarizable representation of the v-adic points G_v of the group G. The representation π_v is the **local data** of f at v. For almost all v π_v is *spherical.*

We will say that **the inner integral factors over primes** if

$$\int_{Z'_{\mathbf{A}}\Theta^\xi_k\backslash\Theta^\xi_{\mathbf{A}}} f(\theta g)\, \delta^{-1}_{\Theta_{\mathbf{A}}}(\theta)\, d\theta = \Pi_v\, F_v(g_v)$$

where the function F_v on G_v depends only upon the local data of f at v. This notion may be modified to exclude a finite set S of 'bad' primes, saying that

$$\int_{Z'_{\mathbf{A}}\Theta^\xi_k\backslash\Theta^\xi_{\mathbf{A}}} f(\theta g)\, \delta^{-1}_{\Theta_{\mathbf{A}}}(h)\, d\theta = \Pi_{v\not\in S}\, F_v(g_v) \times F_{\text{bad}}(g_{\text{bad}})$$

where for $v \notin S$ the function F_v depends only upon the local data of f at v, and $g_{\rm bad}$ varies on the product

$$G_{\rm bad} = \Pi_{v \in S}\, G_v$$

In all cases known to this author, if the whole integral factors over primes, then it is an L-function or product of L-functions. Thus, the immediate issue is to *give a criterion for the inner integral to factor over primes* in the sense above. The role that representation theory plays in such factorization has been addressed many times, going back to [Piatetski-Shapiro 1972] and [Jacquet-Langlands 1970]. By contrast, the idea of [Godement-Jacquet 1972] (extending [Tate 1950] in a natural way) directly presents a product of local factors, whose *analytic continuation* is less patent.

Theorem *(Apocryphal?)***:** We assume that there is a single cuspidal orbit, and that the Eisenstein kernel escapes from the corresponding inner integral (in the sense above). If for all primes v outside a finite set S we have the *multiplicity-one* property

$$\dim_{\mathbf{C}} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v^\xi}^{G_v}\, \delta_H) = 1$$

then the inner integral factors over primes (in the sense above). And, as a consequence, in that event *the whole integral factors over primes.*

Proof: For simplicity, we suppress all the index ξ. Write

$$I(f,g) = \int_{Z'_{\mathbf{A}}\Theta_k\backslash\Theta_{\mathbf{A}}} f(\theta g)\, \delta_{\Theta_v}(\theta)\; d\theta$$

And we may as well enlarge the finite set S to include all primes v so that π_v is *not* spherical, and to contain all archimedean primes.

Let $\pi_f = \bigotimes'_v \pi_v$ be the restricted tensor product of irreducible smooth representations π_v generated by f under the right regular representation of $G_{\mathbf{A}}$, with basepoint vector e_v in π_v in the restricted product. (For all but finitely-many v, the e_v must be a spherical vector in π_v, which must be a spherical representation). Fix an isomorphism

$$\iota : \bigotimes{}'_v\; \pi_v \to \pi_f$$

At all the primes v where π_v is spherical, and where

$$\dim_{\mathbf{C}} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v}\, \delta_{\Theta_v}) = 1$$

normalize an intertwining

$$\Phi_v : \pi_v \to \mathrm{Ind}_{\Theta_v}^{G_v}\, \delta_{\Theta_v}$$

by

$$\Phi_v(e_v)(1_{G_v}) = 1$$

Let T be a finite set of primes disjoint from the set S. Let

$$\varphi = \left(\bigotimes_{v \in T} \varphi_v\right) \otimes \left(\bigotimes_{v \notin T} e_v\right)$$

Let g_T be an element of $G_{\mathbf{A}}$ with components just 1 unless $v \in T$, and g'_T an element of $G_{\mathbf{A}}$ with components 1 at $v \in T$. Then the function

$$g_T \to I(\iota\varphi, g_T)$$

is an element of $\mathrm{Ind}_{\Theta_T}^{G_T}\, \delta_{\Theta_T}$, so

$$\varphi \to (g_T \to I(\iota\varphi, g_T))$$

is an element of

$$\mathrm{Hom}_{G_T}(\bigotimes_{v\in T} \pi_v, \mathrm{Ind}_{\Theta_T}^{G_T}\, \delta_{\Theta_T}^{-1})$$

where for an algebraic subgroup H of G, H_T denotes $\Pi_{v\in T} H_v$. Since T is *finite*, and since we are operating in the relatively algebraic category of *smooth* representations, it is elementary that there is a natural isomorphism

$$\mathrm{Hom}_{G_T}(\bigotimes_{v\in T} \pi_v, \mathrm{Ind}_{\Theta_T}^{G_T}\, \delta_{\Theta_T}^{-1}) \approx \bigotimes_{v\in T} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v}\, \delta_{\Theta_T}^{-1})$$

Therefore, by the multiplicity-one hypothesis,

$$I(\iota\varphi, g_T) = c_T \cdot \Pi_{v\in T}\, \Phi_v(\varphi_v)$$

where c_T is a constant (not depending upon g_T).

By definition (of restricted tensor product), the whole restricted direct product comprising π_f is merely a colimit (direct limit) of the finite products

$$\bigotimes_{v\in T} \pi_v \otimes \bigotimes_{v\notin T} \{e_v\}$$

Let

$$g = g_{\mathrm{bad}} \cdot g_{\mathrm{good}}$$

where g_{bad} is the aggregate of components of g at $v \in S$, and g_{good} is the aggregate of the components of g not in S. We see that for each fixed g_{bad} the map

$$\varphi \to (g_{\mathrm{good}} \to I(\iota\varphi, g))$$

gives an intertwining operator

$$i_{g_{\mathrm{bad}}} : \bigotimes{}'_{v\notin S}\, \pi_v \to \bigotimes{}'_{v\notin S}\, \mathrm{Ind}_{\Theta_v}^{G_v}\, \delta_{\Theta_T}^{-1}$$

Therefore, by the multiplicity-one hypothesis, with a constant $c_{\mathrm{bad}}(g_{\mathrm{bad}})$ not depending upon g_{good}, and with intertwining $\Phi_v : \pi_v \to \mathrm{Ind}_{\Theta_v}^{G_v}\, \mathbf{C}$ normalized by $\Phi(e_v)(1) = 1$, we have

$$i_{g_{\mathrm{bad}}} = c_{\mathrm{bad}} \cdot \bigotimes_{v\notin S} \Phi_v$$

Thus, granting the multiplicity-one hypothesis, the global integral factors as

$$\zeta(f, s) = \int_{\tilde{Z}_{\mathbf{A}} G_k \backslash G_{\mathbf{A}}} E(g)\, f(g)\, dg$$

$$= c_{\mathrm{bad}} \cdot \Pi_{v\notin S} \int_{\tilde{Z}_v \Theta_v \backslash G_v} \varepsilon_v(\xi g) \Phi_v(e_v)(g_v)\, dg_v$$

(The elementary theory of spherical representations shows that a spherical function cannot vanish at 1_G).

Thus, the whole integral is a constant multiple (depending on bad-prime data) multiple of a product of integrals over quotients $\tilde{Z}_v \Theta_v \backslash G_v$ of integrands respectively depending only upon the local data π_v of f at v. *Done*.Thus, the issues are

- Criteria for dim $\mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v}\, \delta_{\Theta_T}^{-1}) \leq 1$.
- Evaluation of the good-prime **outer (local) integrals**

$$\int_{\tilde{Z}_v \Theta_v \backslash G_v} \varepsilon_v(\xi g) \Phi_v(e_v)(g_v)\, dg_v$$

whose value is the v^{th} Euler factor.

1.3Trivial local criterion for period-vanishing

It can happen that the *inner integral* above vanishes identically for "most" cuspforms, for purely local reasons. The explanation is a simple application of the idea of viewing integrals as values of intertwining operators. Here we give this general but essentially trivial *local* condition for vanishing of certain *global* constants attached to cuspforms. Given the definition of these global constants (below) it is reasonable to call them **periods**.

And, again, the computational aspects of this discussion are clearest when (admissible) spherical representations are rewritten as images of unramified principal series, as in the next section.

Let f be a cuspform on a reductive group G defined over a numberfield k. Let $\tilde{Z}$ be a k-subgroup of the center Z of G so that

$$(\tilde{Z}_{\mathbf{A}} \cdot Z_k)\backslash Z_{\mathbf{A}} \qquad \text{is compact}$$

Let H be a subgroup of G defined over k. The **period** Π_f of f over H is

$$\Pi_f = \int_{Z'_{\mathbf{A}}\, H_k\backslash H_{\mathbf{A}}} f(h)\, \delta^{-1}_{H_{\mathbf{A}}}(h)\; dh$$

where $Z' = \tilde{Z} \cap H$. Let π_f be the irreducible representation of the adele group $G_{\mathcal{A}}$ generated by f, and let π_v be the v^{th} local factor of π_f (when expressed as restricted tensor product of local representations). We will not worry about convergence here.

Proposition: *(Apocryphal?)* If for any prime v

$$\mathrm{Hom}_{G_v}(\pi_v,\ \mathrm{Ind}_{H_v}^{G_v}\ \delta_{H_v}) = \{0\}$$

then the period vanishes:

$$\Pi_f = 0$$

Proof: Let $g_v \in G_v$. The function

$$g_v \to \int_{Z'_{\mathbf{A}}\, H_k\backslash G_{\mathbf{A}}} f(hg_v)\, \delta^{-1}_{H_{\mathbf{A}}}(h)\; dh$$

is a function in the induced representation $\mathrm{Ind}_{H_v}^{G_v}\ \delta_{H_v}$. Thus, the map

$$\varphi \to \left(g_v \to \int_{Z'_{\mathbf{A}}\, H_k\backslash H_{\mathbf{A}}} \varphi(hg_v)\, \delta^{-1}_{H_{\mathbf{A}}}(h)\; dh \right)$$

gives a G_v-intertwining from π_v to $\mathrm{Ind}_{H_v}^{G_v}\ \delta_{H_v}$. If there is no such intertwining operator (other than 0), then it must be that the integral is 0. *Done.*

Remarks: Generally, after Jacquet, we would call π_f **distinguished** with respect to the subgroup H if the period of φ over H is non-zero for some vector in the representation π_f generated by f. It seems that in many cases such *distinguished* representations are in correspondence with representations on another group (and perhaps distinguished with respect to a subgroup of that other group). Such results are non-trivial. For example, in [Jacquet-Rallis 1992] and [Jacquet-Rallis 1992b] it is shown that cuspforms on $GL(2n)$ have no non-zero periods for $Sp(n)$, and in [Kumanduri 1997] $U(n,n)$ periods inside $O^*(4n)$ are considered. One necessary ingredient is Jacquet's relative trace formula, which appears in [Jacquet-Lai 1985], [Jacquet 1986], [Jacquet 1987]. [Jacquet-Lai-Rallis 1993] gives one general formalism to address such problems.

1.4Orbit-filtration estimate of multiplicity

We continue with the notations and conventions from above. At good primes, we obtain computational criteria sufficient to assure

$$\dim \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v} \delta_{H_v}) \leq 1$$

which (from section 1.2) would assure factorization of the *inner integral* over primes. (The case that this dimension is 0 is the case of *trivial vanishing* of the period for local reasons, treated in section 1.3).

First, we rewrite spherical representations as images of unramified principal series, thereby making available Frobenius Reciprocity and other standard elementary devices applicable to induced representations. In particular, we use orbit filtrations on test functions to estimate spaces of intertwining operators on filtered spaces of test functions via the intertwining operators on the graded pieces of the filtration. (See [Bruhat 1961], [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977]). It is at this point that we require the *finiteness* of a double-coset space, denoted $P_v\backslash G_v/\Theta_v$ below.

This estimate on intertwining operators can be refined somewhat by noting the symmetry implicitly required by the intertwining operators among the unramified principal series whose characters (suitably normalized) differ only by an element of the spherical Weyl group.

Thus, we make use of the non-trivial results of [Matsumoto 1977], [Borel 1976], [Casselman 1980] concerning the relations between *unramified principal series* and admissible representations with Iwahori-fixed vectors: an irreducible admissible representation with an Iwahori-fixed vector imbeds into an unramified principal series. Further, any subrepresentation or quotient of an unramified principal series is generated by its Iwahori-fixed vectors. Thus, an (admissible) spherical representation imbeds into an unramified principal series, so, by dualization, is also an *image* of an unramified principal series.

Thus, for a spherical representation π_v of a p-adic reductive group G_v, there is at least one surjection

$$\text{c-Ind}_{P_v}^{G_v} \chi_v \delta_{P_v}^{1/2} \to \pi_v$$

where P is a minimal parabolic of G (defined over the v^{th} completion k_v of k), χ_v is an unramified character, and δ_{P_v} is the modular function on the (v-adic points of) P. (The latter is introduced as a convenient normalization for discussion of Weyl-group symmetry, among other things.) That is, χ_v is trivial on the unipotent radical of P_v, and is trivial on the unique maximal compact subgroup M_o of a Levi component M_v of P_v. Here c-Ind is compactly-supported smooth induction.

Thus, to prove that

$$\dim \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v} \delta_{\Theta_v}) \leq 1$$

it certainly suffices to prove

$$\dim \mathrm{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v} \chi_v \delta_{P_v}^{1/2}, \mathrm{Ind}_{\Theta_v}^{G_v}\ \delta_{\Theta_v}) \leq 1$$

for the appropriate χ_v. By Frobenius Reciprocity,

$$\mathrm{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v} \chi_v \delta_{P_v}^{1/2}, \mathrm{Ind}_{\Theta_v}^{G_v}\ \delta_{\Theta_v}) \approx \mathrm{Hom}_{\Theta_v}(\mathrm{Res}_{\Theta_v}^{G_v} \text{c-Ind}_{P_v}^{G_v} \chi_v \delta_{P_v}^{1/2},\ \delta_{\Theta_v})$$

From section 3.2 below on *orbit filtrations of test functions*, we decompose G_v by $P_v \times \Theta_v$-orbits, with P_v acting on the left and Θ_v on the right, thereby obtaining an orbit filtration of $\mathrm{Res}_{\Theta_v}^{G_v} \left(\text{c-Ind}_{P_v}^{G_v} \chi_v \delta_{P_v}^{1/2}\right)$, with *graded pieces*

$$\text{c-Ind}_{H_\alpha}^{\Theta_v} \chi^\alpha$$

where $\alpha \in P_v\backslash G_v/\Theta_v$,

$$H_\alpha = \alpha^{-1} P_v \alpha_v \cap \Theta_v$$

and

$$\chi^\alpha(\theta) = \chi_v \delta_{P_v}^{1/2}(\alpha\theta\alpha^{-1})$$

for $\theta \in H_\alpha$. The orbit filtration requires that $P_v\backslash G_v/\Theta_v$ be *finite*.

Therefore, *assuming the finiteness of* $P_v\backslash G_v/\Theta_v$, to show that the dimension of the space of intertwining operators is less than or equal 1 it suffices to show that

$$\sum_\alpha \mathrm{Hom}_{\Theta_v}(\text{c-Ind}_{H_\alpha}^{\Theta_v}\chi^\alpha,\, \delta_{\Theta_v}) \le 1$$

This condition can be simplified: dualizing and using Frobenius Reciprocity again, this is

$$\sum_\alpha \dim \mathrm{Hom}_{\Theta_v}(\mathbf{C},\, \delta_{\Theta_v} \otimes \mathrm{Ind}_{H_\alpha}^{\Theta_v} \check{\chi}^\alpha \frac{\delta_{H_\alpha})}{\delta_{\Theta_v}} = \sum_\alpha \dim \mathrm{Hom}_{H_\alpha}(\mathbf{C}, \delta_{\Theta_v}\check{\chi}^\alpha \frac{\delta_{H_\alpha})}{\delta_{\Theta_v}})$$

$$= \sum_\alpha \dim \mathrm{Hom}_{H_\alpha}(\chi^\alpha, \delta_{H_\alpha})$$

where $\check{\chi}^\alpha$ is the smooth dual of χ^α.

That is, we have proven
Theorem: Assume that for almost all places v of k the double coset space $\tilde{P}_v\backslash\tilde{G}_v/G_v$ is finite. The dimension of the space of intertwining operators from π_v to $\mathrm{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v}$ satisfies

$$\dim(\pi_v, \mathrm{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v}) \le \sum_\alpha \dim \mathrm{Hom}_{H_\alpha}(\chi^\alpha,\, \delta_{H_\alpha})$$

Thus, if this sum is ≤ 1, then the inner integral factors over primes, and the global integral has an Euler product.

Remarks: In principle, for any particular choices of the characters, the simplified version of the condition on intertwining operators is effectively provable or disprovable since it involves simply questions of equality of restrictions to a common subtorus of two characters on two subgroups.

We can refine this Theorem somewhat by use of the intertwining operators among unramified principal series, as studied in [Bruhat 1961] and [Casselman 1980]. For an element w of the spherical Weyl group W attached to a maximal split torus A inside the minimal parabolic P_v of G_v, and for $a \in A$, define another character χ_v^w on A by

$$\chi_v^w(a) = \chi_v(w\,a\,w^{-1})$$

Say that χ_v is *regular* if $\chi_v \ne \chi_v^w$ for $w \ne 1$.

With the unramified principal series $\text{c-Ind}_{P_v}^{G_v}\chi_v\delta_{P_v}^{1/2}$ normalized by the inclusion of the square root $\delta_{P_v}^{1/2}$ of the modular function δ_{P_v} of P_v, for regular χ_v and for $w \in W$

$$\dim\ \mathrm{Hom}(\text{c-Ind}_{P_v}^{G_v}\chi_v\delta_{P_v}^{1/2}, \text{c-Ind}_{P_v}^{G_v}\chi_v^w\delta_{P_v}^{1/2}) = 1$$

Further, [Casselman 1980] determines finitely-many essentially elementary conditions on χ_v so that, except when these conditions are met, the unramified principal series $\text{c-Ind}_{P_v}^{G_v}\chi_v\delta_{P_v}^{1/2}$ is *irreducible*. That is, for "generic" unramified principal series, the essentially unique intertwining operators among the unramified principal series with characters differing by elements of the Weyl group W are all isomorphisms.

Thus, for given regular χ_v so that $\text{c-Ind}_{P_v}^{G_v}\chi_v\delta_{P_v}^{1/2}$ is *irreducible*, suppose that the α^{th} graded piece in the orbit filtration had a non-zero intertwining operator. Above, we saw that this is equivalent to

$$\delta_{H_\alpha} = \chi^\alpha|_{H_\alpha}$$

If the intertwining operator on the graded piece were to *extend* to an intertwining operator for the whole representation $\mathrm{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v}$, then the isomorphism of c-$\mathrm{Ind}_{P_v}^{G_v}\,\chi_v\delta_{P_v}^{1/2}$ with c-$\mathrm{Ind}_{P_v}^{G_v}\,\chi_v^w\delta_{P_v}^{1/2}$ would imply that also

$$\delta_{H_\alpha} = (\chi^w)^\alpha|_{H_\alpha}$$

That is, we have proven

Theorem: Assume that for almost all places v of k the double coset space $\tilde{P}_v\backslash\tilde{G}_v/G_v$ is finite. Let χ_v be a *regular* character so that the unramified principal series c-$\mathrm{Ind}_{P_v}^{G_v}\,\chi_v\delta_{P_v}^{1/2}$ is *irreducible.* The dimension of the space of intertwining operators from π_v to $\mathrm{Ind}_{\Theta_v}^{G_v}\,\delta_{\Theta_v}$ satisfies

$$\dim \mathrm{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v}\,\chi_v\delta_{P_v}^{1/2}, \mathrm{Ind}_{\Theta_v}^{G_v}\,\delta_{\Theta_v})$$

$$\leq \min{}_{w\in W} \sum_{\alpha\in P_v\backslash G_v/\Theta_v} \dim\,\mathrm{Hom}_{H_\alpha}((\chi^w)^\alpha,\,\delta_{H_\alpha})$$

Remarks: The chief difficulty in reaching clear conclusions in this manner is that one knows nearly nothing about *which* unramified characters χ_v can occur in such manner associated to a cuspform, apart from a few things such as square-integrability of the representation π_v. Therefore, one is apparently in the position of having to prove the above estimate on intertwinings of the graded pieces in the orbit filtration *for all unramified principal series.* But it is seldom the case that the estimate on intertwining operators of the graded pieces really holds for *all* χ_v.

Nevertheless, it does happen very often that for **generic** χ_v the estimate above holds, assuring the Euler factorization. The sense of 'generic' here is neither the colloquial one, nor is it the condition 'has a Whittaker model'. Rather, the relevant sense is with regard to the position of the complex parameters specifying the character, as in section 2.1 below.

In some examples, invocation of the *unitariness* of the representations π_v occurring in association with a square-integrable cuspform is sufficient to dispatch the non-generic cases (in the above sense) in which the sum of the dimensions of the intertwining operators on the graded pieces is greater than 1. That is, it is often possible to obtain the desired multiplicity-one conclusion simply by using the orbit filtration and the unitariness. Otherwise, one can pretend to be invoking some sort of generalized Ramanujan hypothesis to ensure that the unramified principal series are sufficiently well-behaved. On the other hand, such "pseudo-Ramanujan" hypotheses may inadvertently imply identical vanishing of the *inner integral* for trivial reasons (as in section 1.3).

1.5Simple heuristic criterion

Now we give a simple heuristic criterion, in terms of dimension-counting, for Euler factorization of a global integral.

We do *not* assert that *failing* the dimension-count *proves* that there is no Euler product, nor that *passing* the dimension-count test proves that there *is* an Euler product. Nevertheless, it appears to be rather difficult to arrange cases in which dimension-count criterion is not met and yet there is an Euler product. And, on the other hand, in practice it seems to be that *meeting* the criterion nearly guarantees that factorization into an Euler product.

The papers [Brion 1987] and [Kasai 1996] systematically address related dimension-counting issues.

Again, let G and $\tilde{G}$ be two reductive groups defined over a number field, with G a subgroup of $\tilde{G}$. Let $\tilde{Z}$ be the center of $\tilde{G}$, and Z the center of G, and assume that

$$\tilde{Z}_{\mathbf{A}}\cdot Z_k\backslash Z_{\mathbf{A}} \qquad \text{is compact}$$

Let $\tilde{P}$ be a parabolic subgroup of $\tilde{G}$, and P a minimal parabolic subgroup of G. Let E be an Eisenstein series attached to a one-dimensional representation of $\tilde{P}$, and let f be a cuspform on G. As above, the global integral which we would *like* to have an Euler product is

$$Z(f) = \int_{\tilde{Z}_{\mathbf{A}}\, G_k \backslash G_{\mathbf{A}}} E(g)\; f(g)\; dg$$

Observation: If

$$\dim\, \tilde{P} + \dim\, P < \dim\, \tilde{G}$$

then our present approach will most likely fail to demonstrate that the global integral has an Euler product expansion.

Support for the observation: The unwinding trick (see section 1.1) expresses such a global integral as a sum over $\tilde{P}_k\backslash\tilde{G}_k/G_k$ of global integrals. For each $\xi \in \tilde{P}_k\backslash\tilde{G}_k/G_k$, let

$$\Theta = \Theta_\xi = G \cap \xi^{-1}\tilde{P}\xi$$

Then the associated *inner integral* (as in section 1.2) will factor over primes if

$$\dim\, \mathrm{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v}\, \chi_v,\; \mathrm{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v}) \le 1$$

for almost all primes v (with suitable characters χ_v).

The orbit filtration method (3.2) to estimate this (co-)multiplicity requires that the double-coset space $P_v\backslash G_v/\Theta_v$ be *finite*. In the discussion of the orbit filtration it was shown that this implies (already at the level of the Baire category theorem) that there is an orbit $\Theta_v x P_v$ which is *open* in G_v. For this to be the case, it must be that

$$\dim\, \Theta + \dim\, P \ge \dim\, G$$

(where dimension is that of algebraic varieties).

We may reasonably call ξ *generic* if

$$\dim\, \tilde{G} - \dim\, \xi^{-1}\tilde{P}\xi = \dim\, G - \dim\,(G \cap \xi^{-1}\tilde{P}\xi)$$

Certainly

$$\dim\, \tilde{G} - \dim\, \tilde{P} = \dim\, \tilde{G} - \dim\, \xi^{-1}\tilde{P}\xi$$

for all ξ. When the double coset space $\tilde{P}_k\backslash\tilde{G}_k/G_k$ is *finite* there must exist at least one generic ξ (assuming that $\tilde{G}$ is not anisotropic over k, i.e., assuming that $\tilde{P}$ is a *proper* parabolic.)

For generic ξ, we can rewrite the dimension condition

$$\dim\, \Theta + \dim\, P \ge \dim\, G$$

as

$$\dim\, P \ge \dim\, G - \dim\, \Theta = \dim\, \tilde{G} - \dim\, \tilde{P}$$

That is, we expect that this inequality is a *necessary* condition for success of an orbit filtration argument that the inner integral factors over primes. Thus, its negation (as in the *Observation*) would make it unlikely that we would be able to detect Euler factorization by the methods presented here. *Done.*

2. Computation of Euler factors

2.1Parametrized families

This section sets up an essentially algebraic notion of 'family' of smooth representations of p-adic and other totally disconnected groups. For our present purposes, it suffices to consider only the case that the parametrizing spaces are affine algebraic varieties. Compare the more analytical version (due to Bernstein) sketched in [Gelbart, Piatetski-Shapiro, Rallis 1987], as well as the general purely algebraic set-up in [Bernstein-Zelevinski 1976], [Bernstein-Zelevinski 1977], [Bernstein 1984].

We will discuss *meromorphic* families of intertwining operators among families of representations. The sense of "meromorphic" here is (a weak vector-valued form of) the algebraic one, designed for application to the families of representations above, parametrized by affine algebraic varieties in an algebraic manner. The notion of *locally strong meromorphy* of vector-valued function at the end of this section is important for later applications.

For a commutative ring k with a unit $1 = 1_k$, a **k-algebra** is a ring $\mathcal{O}$ with a k-module structure so that 1_k acts as the identity upon $\mathcal{O}$, and so that for $x \in k$ and $\alpha, \beta \in \mathcal{O}$

$$x(\alpha\beta) = (x\alpha)\beta = \alpha(x\beta)$$

Let k be a field. Let $\mathcal{O}$ be a commutative k-algebra with unit, and suppose that $\mathcal{O}$ is an integral domain. Let $\mathcal{M}$ be the field of fractions of $\mathcal{O}$, and let

$$X = \text{Spec } \mathcal{O}$$

be the prime spectrum of $\mathcal{O}$. For a prime ideal $x \in X$ let S_x be the multiplicative subset of $\mathcal{O}$ which is the set complement of x in $\mathcal{O}$, and define the **local ring at** x to be $\mathcal{O}_x = S_x^{-1}\mathcal{O}$. The **residue field at** x is the quotient

$$k_x = \mathcal{O}_x / x\mathcal{O}_x$$

The residue field k_x is naturally a k-algebra, an $\mathcal{O}$-algebra, and an $\mathcal{O}_x$-algebra. Since $\mathcal{O}$ is a domain, the zero ideal is prime, and we denote its residue field $\mathcal{M}$ by k_o.

As usual, a *height one prime* in $\mathcal{O}$ is a prime ideal y so that if x is another prime ideal and $x \subset y$ then either $x = 0$ or $x = y$. The **irreducible hypersurface** $\eta = \eta_y$ attached to a height-one prime is defined to be

$$\eta = \eta_y = \{x \supset y : x \in X\}$$

In the spirit of the Baire category theorem, A **meager** subset Y of X is a subset contained in a countable union of irreducible hypersurfaces. A **non-meager** subset of X is simply one which is not *meager*.

Let

$$q_x : \mathcal{O}_x \to k_x$$

be the natural **quotient** or **evaluation** map, and also write

$$q_x : V \to V \otimes_{\mathcal{O}} k_x$$

for any $\mathcal{O}$-module V, and further

$$q_x : V \to V \otimes_{\mathcal{O}_x} k_x$$

for any $\mathcal{O}_x$-module V.

Let G be a (locally compact, Hausdorff) totally disconnected group. Let V be an $\mathcal{O}$-module, and

$$\pi : G \to \text{Hom}_{\mathcal{O}}(V, V)$$

a group homomorphism. Suppose that π is (uniformly) **smooth** in the sense that for every $v \in V$ the isotropy subgroup

$$\{g \in G : gv = v\}$$

of v is *open.* Say that π is a **$\mathcal{O}$-parametrized family** of smooth representations of G.

For $x \in X$ we have the **pointwise representation**

$$q_x\pi = \pi_x : G \to \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, V \otimes_{\mathcal{O}} k_x)$$

of G on the k_x-vectorspace $V \otimes_{\mathcal{O}} k_x$.

When $\mathcal{O}$ is Noetherian and k is algebraically closed and x is maximal, by the Nullstellensatz the residue field $q_x\mathcal{O} = k_x$ is just k itself. Thus, for k algebraically closed, the pointwise representations at maximal ideals (geometrically closed points) x form a *family* $\{q_x\pi = \pi_x : x \in X\}$ of k-linear representations in a more colloquial sense.

Remarks: Of course, such a parametrized family is really just a single $\mathcal{O}$-linear representation of G. However, we cannot reasonably expect to develop representation theory over general commutative rings $\mathcal{O}$. Fortunately, for the most part we can skirt such issues, simply viewing the $\mathcal{O}$-linear representation as glueing together the various point-wise representations as just defined. Compare [Bernstein-Zelevinski 1976], [Bernstein-Zelevinski 1977], and [Bernstein 1984].

The **generic representation** in the parametrized family π is $\pi_o = \pi_{\{0\}}$, the pointwise representation corresponding to the *generic point* $\{0\}$ in the prime spectrum. The generic representation is most naturally an $\mathcal{M}$-linear representation, and typically $k_o = \mathcal{M}$ has positive transcendence degree over the base field k.

The **trivial** $\mathcal{O}$-parametrized family of smooth representations of G is the $\mathcal{O}$-module $\mathcal{O}$ itself, with trivial action of G upon it. The trivial representation of G on a residue field k_x will also be denoted simply k_x.

Let $\tilde{k}$ be an extension field of k so that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain, with $\mathcal{M} \otimes_k \tilde{k}$ the field of fractions of $\mathcal{O} \otimes_k \tilde{k}$. Given an $\mathcal{O}$-parametrized family π of smooth representations of G with underlying $\mathcal{O}$-module V, the $\mathcal{O} \otimes_k \tilde{k}$-parametrized family $\pi \otimes_k \tilde{k}$ of representations obtained by **extending scalars** from k to $\tilde{k}$ has underlying $\mathcal{O} \otimes_k \tilde{k}$-module

$$V \otimes_k \tilde{k} \approx V \otimes_{\mathcal{O}} (\mathcal{O} \otimes_k \tilde{k})$$

and is given by the natural formula

$$(\pi \otimes_k \tilde{k})(g)(v \otimes \alpha) = \pi(g)(v) \otimes \alpha$$

On the other hand, an $\mathcal{O} \otimes_k \tilde{k}$-parametrized family $\tilde{\pi}$ is **defined over** k or k-**rational** if there is an $\mathcal{O}$-parametrized family π so that

$$\tilde{\pi} \approx \pi \otimes_k \tilde{k}$$

A trivial example of a parametrized family is that of a **constant** (along $\mathcal{O}$) family, by which we mean the following. Let ρ be a k-linear representation of G, and put

$$\pi = \rho \otimes_k \mathcal{O}$$

Then π is an $\mathcal{O}$-parametrized family which is **constant** along $\mathcal{O}$, in the sense that for every $x \in X$

$$q_x\pi = \rho \otimes_k k_x$$

The simplest meaningful example of a parametrized family of smooth representations arises from the $\mathbf{C}$-linear representations

$$\alpha \to |\alpha|^s$$

on the multiplicative group $F^\times$ of an ultrametric local field F, where the norm $|\ |$ is normalized (e.g.) so that

$$|\varpi| = \frac{1}{q}$$

where ϖ is a local parameter and q is the cardinality of the residue field. These are parametrized in a colloquial sense by $s \in \mathbf{C}$, or by $s \in \mathbf{C}/2\pi i \log q$. Then write

$$\pi(\alpha) = z^{\mathrm{ord}\,\alpha}$$

where z is an indeterminate (thought of as being q^{-s}) and *ord* is the usual ord-function. In this example, we take $k = \mathbf{C}$, let $\mathcal{O} = \mathbf{C}[z, z^{-1}]$, and let V be a copy of $\mathcal{O}$ with $\mathcal{O}$ acting by multiplication. Then we have the family

$$\pi : F^\times \to \mathrm{Hom}_{\mathbf{C}[z,z^{-1}]}(\mathbf{C}[z, z^{-1}], \mathbf{C}[z, z^{-1}])$$

by

$$\pi(\alpha)v = z^{\mathrm{ord}\,\alpha} \cdot v$$

In this example, for a maximal ideal x of $\mathcal{O} = \mathbf{C}[z, z^{-1}]$ generated by $z - q^{-s}$ for a complex number s, the pointwise representation $q_x\pi = \pi_x$ is the one-dimensional representation

$$\alpha \to |\alpha|^s$$

On the other hand, the *generic* representation π_o is $\mathbf{C}(z)$-linear, on the one-dimensional $\mathbf{C}(z)$-vectorspace $\mathbf{C}(z)$ itself, by

$$\pi_o(\alpha)v = z^{\mathrm{ord}\,\alpha} \cdot v$$

A little more generally, let M be a product of n copies of $F^\times$ for a local field F, and define

$$\pi(m_1, \ldots, m_n) = z_1^{\mathrm{ord}\ m_1} z_2^{\mathrm{ord}\ m_2} \ldots z_n^{\mathrm{ord}\ m_n}$$

where $z_1, z_2, \ldots, z_n$ are indeterminates. Here we can take

$$\mathcal{O} = \mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

and take V to be a copy of $\mathcal{O}$ with module structure just given by left multiplication. Then for choice of n non-zero complex numbers $s_1, s_2, \ldots, s_n$, with $z_{1o} = q^{-s_1}, z_{2o} = q^{-s_2}, \ldots, z_{no} = q^{-s_n}$, letting x be the ideal

$$x = \mathcal{O} \cdot (z_1 - z_{1o}) + \mathcal{O} \cdot (z_2 - z_{2o}) + \ldots + \mathcal{O} \cdot (z_n - z_{no})$$

the pointwise representation π_x is

$$\pi_x(m_1, m_2, \ldots, m_n) = |m_1|^{s_1}\, |m_2|^{s_2} \ldots |m_n|^{s_n}$$

Here the *generic* representation is simply the $\mathbf{C}(z_1, z_2, \ldots, z_n)$-linear representation

$$\pi_o : (m_1, m_2, \ldots, m_n) \to \text{ (multiplication by) } z_1^{\mathrm{ord}\ m_1} z_2^{\mathrm{ord}\ m_2} \ldots z_n^{\mathrm{ord}\ m_n}$$

on a one-dimensional $\mathbf{C}(z_1, z_2, \ldots, z_n)$-vectorspace. Here there are non-maximal non-zero prime ideals in the parameter space for $n > 1$.

More generally, let M be a Levi component of a minimal F-parabolic in a reductive linear group G of F-rank n, defined over the local field F. Because of the minimality, there is a unique maximal compact subgroup M_o of M, and M_o is *normal*, and M/M_o is *abelian*. In fact, $M/M_o \approx \mathbf{Z}^n$ for some n. A one-dimensional representation of M is **unramified** if M_o lies inside its kernel. Let $\varpi_1, \varpi_2, \ldots, \varpi_n$ be elements in M so that their images in $M/M_o \approx \mathbf{Z}^n$ are a $\mathbf{Z}$-basis for the latter. Let $z_1, z_2, \ldots, z_n$ be indeterminates over $\mathbf{C}$. Then put

$$\pi(\varpi_1^{m_1} \varpi_2^{m_2} \ldots \varpi_n^{m_n}) = z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$$

Here again we take

$$\mathcal{O} = \mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

and take V to be a copy of $\mathcal{O}$ with module structure just given by left multiplication. With maximal primes x, we recover all the unramified representations of M. At the other extreme, the *generic representation* π_o in this family is the $\mathbf{C}(z_1, z_2, \ldots, z_n)$-linear representation

$$\pi_o(\varpi_1^{m_1}\varpi_2^{m_2}\ldots\varpi_n^{m_n}) = z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}$$

on any one-dimensional $\mathbf{C}(z_1, z_2, \ldots, z_n)$-vectorspace. And there are non-maximal non-zero primes, as well.

Generally, let P be a parabolic subgroup of a reductive p-adic group G over an ultrametric local field F, with unipotent radical N, and let $M \approx P/N$ be a copy of a Levi component. Let σ be a supercuspidal representation of M, and let π be a parametrized family of one-dimensional representations of M, as above. Then $\sigma \otimes \pi$ is certainly a parametrized family of smooth representations. We do not need such families here, but see [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977], [Bernstein 1984] for applications.

As expected, for an $\mathcal{O}$-parametrized family σ of smooth representations of a closed subgroup H of G, the **compactly-induced** representation $\text{c-Ind}_H^G\,\sigma$ is the collection of σ-valued functions f on G which are compactly supported left modulo H, are locally constant, and so that

$$f(hg) = \sigma(h)\, f(g)$$

for $h \in H$ and $g \in G$. Likewise, the (not-necessarily compactly-) **induced** representation $\text{Ind}_H^G\,\sigma$ is the collection of σ-valued functions f on G which are *uniformly* locally constant, and so that

$$f(hg) = \sigma(h)\, f(g)$$

It is elementary but important to note that for $x \in X$ and for any $\mathcal{O}$-algebra $\mathcal{A}$ we have

$$(\text{c-Ind}_H^G\,\sigma) \otimes_{\mathcal{O}} \mathcal{A} \approx \text{c-Ind}_H^G\,(\sigma \otimes_{\mathcal{O}} \mathcal{A})$$

(This is because compactly-supported induction itself is a tensor product, and tensor products are associative). That is, *the usual construction does yield an $\mathcal{O}$-parametrized family whose pointwise representations are the compactly-induced representations from the pointwise representations σ_x.*

By contrast,

$$(\text{Ind}_H^G\,\sigma) \otimes_{\mathcal{O}} \mathcal{A} \subset\subset \text{Ind}_H^G\,(\sigma \otimes_{\mathcal{O}} \mathcal{A})$$

with strict inclusion in general unless G/H is actually compact. Thus, in general, the usual construction of not-necessarily-compact induction does *not* yield a parametrized family whose pointwise representations are the induced representations of the pointwise representations.

The notion of **smooth dual** of an $\mathcal{O}$-parametrized family σ has similar difficulties. First, for any $\mathcal{O}$-module V upon which G acts, the collection of **smooth vectors** V^∞ in V is

$$V^\infty = \{v \in V :\ v \text{ has open isotropy group }\}$$

For any $x \in X$ let π_x be the pointwise representation at x. Naively, one would want an $\mathcal{O}$-parametrized family $\check{\pi}$ of smooth representations of G so that for every $x \in X$

$$\text{Hom}_{k_x}(\pi_x, k_x)^\infty = (\check{\pi})_x$$

But

$$\text{Hom}_{k_x}(\pi_x, k_x) \approx \text{Hom}_{k_x}(\pi \otimes_{\mathcal{O}} k_x, k_x) \approx \text{Hom}_{\mathcal{O}}(\pi, k_x)$$

There is the obvious natural map

$$\text{Hom}_{\mathcal{O}}(\pi, \mathcal{O}) \to \text{Hom}_{\mathcal{O}}(\pi, k_x)$$

but this is not guaranteed to be a *surjection* unless (for example) π is *projective* as an $\mathcal{O}$-module.

The simplest scenario in which this projectivity holds is the trivial *constant family* case. Less trivially, the commutative ring $\mathcal{O}$ itself (with left multiplication) is a free module over itself, hence is projective (since it has an *identity*). Thus, any $\mathcal{O}$-parametrized family χ of representations of G on $\mathcal{O}$ itself *is* projective, so has a smooth dual $\mathcal{O}$-parametrized family of representations

$$\check{\chi} = \mathrm{Hom}_{\mathcal{O}}(\chi, \mathcal{O})^{\infty}$$

where G acts on the dual by the usual contragredient action

$$\check{\chi}(g)(\lambda)(v) = \lambda(\chi(g^{-1})v)$$

However, the assertion

$$(\text{c-Ind}_H^G \sigma)\check{} \approx \mathrm{Ind}_H^G \, \check{\sigma} \frac{\delta_H}{\delta_G}$$

(which is true for representations on *vectorspaces*) is *false* for $\mathcal{O}$-parametrized families in general, since the not-necessarily-compact induction does not commute with tensor products, as already noted above.

Let σ, τ be two $\mathcal{O}$-parametrized families of smooth representations of G. An $\mathcal{M}$-linear map

$$\Phi : \sigma \otimes_{\mathcal{O}} \mathcal{M} \to \tau \otimes_{\mathcal{O}} \mathcal{M}$$

is an **$\mathcal{O}$-parametrized (meromorphic) family of intertwining operators** from σ to τ if for all $s \in \sigma$ we have

$$\Phi(\sigma(g)s) = \tau(g)\Phi(s)$$

for all $g \in G$. That is, Φ is an $\mathcal{M}$-vectorspace homomorphism, and is a G-module homomorphism. The family Φ is **holomorphic** at $x \in X$ if

$$\Phi(\sigma) \subset \tau \otimes_{\mathcal{O}} \mathcal{O}_x$$

The following is immediate:

Lemma: Let σ, τ be two $\mathcal{O}$-parametrized families of smooth representations of G. If σ is a *countably-generated* $\mathcal{O}$-module, then a meromorphic family of intertwining operators

$$\Phi : \sigma \to \tau$$

fails to be holomorphic on at worst a *meager* set, i.e., fails to be holomorphic on a subset of some *countable union of hypersurfaces*. *Done*.For $x \in X$ at which π is holomorphic, there is a natural **pointwise intertwining operator**

$$\Phi_x : \sigma \otimes_{\mathcal{O}} k_x \to \tau \otimes_{\mathcal{O}} k_x$$

Taking the *generic point* $x = 0$ in X, we have the **generic intertwining operator** in the family, simply the $\mathcal{M}$-linear map

$$\Phi : \sigma \otimes_{\mathcal{O}} \mathcal{M} \to \tau \otimes_{\mathcal{O}} \mathcal{M}$$

Let $\tilde{k}$ be an extension field of k so that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain. Let $\tilde{\mathcal{M}}$ be the fraction field of $\mathcal{O} \otimes_k \tilde{k}$. Then a meromorphic family of intertwining operators

$$\tilde{\Phi} : \sigma \otimes_{\mathcal{O}} \tilde{\mathcal{M}} \to \tau \otimes_{\mathcal{O}} \tilde{\mathcal{M}}$$

is **rational over** k if there is a meromorphic family of intertwining operators

$$\Phi : \sigma \otimes_{\mathcal{O}} \mathcal{M} \to \tau \otimes_{\mathcal{O}} \mathcal{M}$$

so that $\tilde{\Phi}$ is the natural $\tilde{k}$-linear extension of Φ.

2.2Rationality of local integrals

In the *Rationality Lemma* just below, the last of the conclusions is seemingly the most banal. However, this conclusion has corollaries which are considerably less obvious, such as that various local integrals are rational functions of q^{-s} or of several quantities $q^{-s}, q^{-s_1}, \ldots, q^{-s_n}$. As a very special case, in the section 2.4 we obtain the rationality of the Euler factors of global integrals (under an additional convergence hypothesis for integrals involved.)

Postponing the proof of the Rationality Lemma to 3.1, we use it to prove that certain integrals on totally disconnected groups are rational functions in naturally-occurring parameters. In practice, these parameters will be the familiar q^{-s}, or several such parameters $q^{-s}, q^{-s_1}, \ldots, q^{-s_n}$. Further, we have a corresponding rationality result concerning the 'scalars' involved. We give two versions of the result.

We view the latter results as a *qualitative* computation of various classes of local integrals, under hypotheses which are very frequently met in practice. The obvious follow-up is closer determination of the local integrals, without *ad hoc* computation. The latter goal is pursued in the succeeding sections.

We recall the context for discussion of parametrized families, from the last section. Fix a field k of characteristic zero. Let G be a (locally-compact Haudorff) totally disconnected group with a *countable dense subset.* Let $\mathcal{O}$ be a commutative k-algebra with unit, without zero divisors, with prime spectrum X. Let $\tilde{k}$ be a field extension of k, so that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain. Let $X \otimes_k \tilde{k}$ be the prime spectrum of $\mathcal{O} \otimes_k \tilde{k}$.

Let π be an $\mathcal{O}$-parametrized family of smooth representations of G on an $\mathcal{O}$-module V, so that the underlying $\mathcal{O}$-module V is *countably-generated.* Let $\tilde{\pi} = \pi \otimes_k \tilde{k}$ be the parametrized family obtained by *extending scalars* from k to $\tilde{k}$.

Recall that a *meager* set is one contained in a countable union of hypersurfaces, while a *non-meager* set is simply not *meager.* And recall the notation that k_x is the trivial one-dimensional k_x-linear representation.

Rationality Lemma:

• Suppose that for x on a non-meager subset Y of $X \otimes_k \tilde{k}$ (on which $\pi \otimes_k \tilde{k}$ is holomorphic)

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G \times \tilde{k}_x}(\tilde{\pi}_x, \tilde{k}_x) \le 1$$

• Suppose that there is $\mu_o \in V \otimes_{\mathcal{O}} \mathcal{M}$ so that for each x in Y there is an intertwining operator $\psi_{(x)} : \tilde{\pi}_x \to \tilde{k}_x$ with the normalization

$$\psi_{(x)}(\mu_o) = 1 \in \tilde{k}_x$$

Then

• There is an $\mathcal{O} \otimes_k \tilde{k}$-parametrized meromorphic family Φ of intertwining operators from $\tilde{\pi}$ to the trivial representation $\mathcal{O}$, *rational over* k, so that for x in a non-meager set $Y' \subset Y$ (on which Φ is holomorphic) the pointwise intertwining operators Φ_x arising from Φ are equal to the respective intertwining operators $\psi_{(x)}$:

$$\Phi_x = \psi_{(x)} \quad \text{for } \ x \in Y'$$

• *Off* a meager subset of $X \otimes_k \tilde{k}$,

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G \times \tilde{k}_x}(\tilde{\pi}_x, \tilde{k}_x) = 1$$

• For any $v \in V \otimes_{\mathcal{O}} \mathcal{M}$,

$$\Phi(v) \in \mathcal{M}$$

(Proof given below in section 3.1, after examples of applications).

Remarks: The hypothesis

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G\times\tilde{k}_x}(\pi_x, \tilde{k}_x) \le 1$$

for x in a non-meager set is the **generic multiplicity one** condition, and the element $\mu_o \in V \otimes_{\mathcal{O}} \mathcal{M}$ is the **good test vector**. And note that the very last conclusion contains two kinds of **rationality assertions**: first there is the assertion that the 'function' $v \to \Phi(v)$ lies in the *field of rational functions* $\mathcal{M} \otimes_k \tilde{k}$, but there is also the *rationality of scalars* assertion, that $\Phi(v)$ actually lies in the smaller field $\mathcal{M}$ itself, without the need to extend scalars from k to $\tilde{k}$.

And, last, it is important to realize that for k *countable* (e.g., $\mathbf{Q}$), it may well be that the whole of X is *meager*. Thus, it may well be necessary to consider the larger $X \otimes_k \tilde{k}$, by extending scalars to an *uncountable* field $\tilde{k}$ (e.g., $\mathbf{C}$ or $\mathbf{Q}_p$) in order to apply this result.

Now we will consider integrals (and somewhat more general functionals) whose integrands are $\tilde{k}_x$-valued for varying maximal ideals x in $\mathcal{O} \otimes_k \tilde{k}$, so we want to assume that $\tilde{k}$ and finite extensions $\tilde{k}_x$ of it are *complete*. Thus, taking $\tilde{k}$ to be $\mathbf{R}$ or $\mathbf{C}$ or finite extensions of $\mathbf{Q}_p$ suffices. (By the Nullstellensatz, the residue fields $\tilde{k}_x$ of $\mathcal{O} \otimes_k \tilde{k}$ for maximal ideals x are finite extensions of $\tilde{k}$).

Let A, B be closed subgroups of a totally disconnected, locally compact, Hausdorff, countably-based topological group G. For our applications, we can suppose that G is *unimodular*. We assume that there is some $\xi \in G$ so that the set

$$A\,\xi\,B \subset G$$

is an **open** subset of G. That is, considering the group action of $A \times B$ upon G by

$$(a \times b)(g) = a \cdot g \cdot b^{-1}$$

there is an **open orbit**. As catch-phrase, this is the **open-orbit condition**.

Let σ, τ be $\mathcal{O}$-parametrized families of *one-dimensional* smooth representations of A, B, respectively. Thus, we may suppose that these are simply group homomorphisms

$$\sigma : A \to \mathcal{O}^\times \qquad \tau : B \to \mathcal{O}^\times$$

For x a *maximal* ideal in the prime spectrum of $\mathcal{O} \otimes_k \tilde{k}$ consider the ($\tilde{k}_x$-valued) **local integral**

$$I(f,\varphi)(x) = \int_{(A\cap\xi^{-1}B\xi)\backslash G} q_x f(g)\, q_x\varphi(\xi^g)\; dg$$

where f is in c-$\mathrm{Ind}_A^G\sigma$, φ is in c-$\mathrm{Ind}_B^G\tau$, q_x denotes the pointwise evaluation.

Remarks: Without loss of generality ξ can be taken to be just 1, by conjugating B. However, if that simplification is made at the outset it becomes too easy to overlook the fact that the hypothesis is as general as it really is.

According to the notational convention of 1.2, an "integral" on a quotient $H\backslash G$ of G by a closed subgroup H is only a literal integral when for $h \in H$

$$\delta_G(h) = \delta_H(h)$$

In that case, since the groups in question are totally disconnected, we can suppose that there is a right G-invariant $\mathbf{Q}$-valued measure (denoted by dg) on the quotient $H\backslash G$. In general, we *can* always assume that there is a $\mathbf{Q}$-valued right G-invariant measure on G, and on H. We have the identity

$$\int_{H\backslash G} \int_H f(hg)\,\frac{\delta_G(h)}{\delta_H(h)}\; dh\; dg$$

and the surjectivity of the averaging map

$$\mathrm{C}_c^\infty(G) \to \mathrm{C}_c^\infty(H\backslash G, \frac{\delta_H}{\delta_G})$$

given by

$$f \to \Big(g \to \int_H f(hg)\, \frac{\delta_G(h)}{\delta_H(h)}\, dh \Big)$$

(See 3.2). Because of the total-disconnected-ness of H, we also know that the modular functions δ_G, δ_H are $\mathbf{Q}^\times$-valued. Therefore, we may assume quite generally that **the "integral"**

$$\int_{H\backslash G} F(g)\, dg$$

is Q-valued, in the sense that for $\mathbf{Q}$-valued $f \in \mathrm{C}_c^\infty(G)$ and for

$$F(g) = \int_H f(hg)\, \frac{\delta_G(h)}{\delta_H(h)}\, dh$$

we have

$$\int_{H\backslash G} F(g)\, dg \in \mathbf{Q}$$

(And, further, this function F is $\mathbf{Q}$-valued.)

Remark: It is important to realize that even though we can assume without loss of generality that the "measure" is $\mathbf{Q}$-valued, there is no assurance that the integrands relevant to global integrals are compactly supported modulo $A \cap B$ for general f and φ. Thus, such an integral is a genuinely analytic object rather than a purely algebraic one, although there seems no obstacle to its being *p-adic* rather than real or complex.

We **assume** that for maximal ideals x in some non-meager subset of $X \otimes_k \tilde{k}$ the "integral" *converges* absolutely for *all* $f_x \in \text{c-Ind}_A^G q_x(\sigma \otimes_k \tilde{k})$ and for *all* $\varphi_x \in \text{c-Ind}_B^G\, q_x(\tau \otimes_k \tilde{k})$.

We also **assume** the **generic multiplicity-one** condition: for x in a non-meager subset of $X \otimes_k \tilde{k}$ we have the *multiplicity-one* condition

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G \times \tilde{k}_x}(\text{c-Ind}_A^G q_x(\sigma \otimes_k \tilde{k}) \otimes \text{c-Ind}_B^G q_x(\tau \otimes_k \tilde{k}), \tilde{k}_x) \le 1$$

Then we have the following corollary of the Rationality Lemma, which includes rationality assertions concerning both rationality as a function of the parameter $x \in X \otimes_k \tilde{k}$, and also rationality of scalars:

Theorem: With the assumptions that there is an open orbit $H\,\xi\,G$, that the "integral"

$$I(f, \varphi)(x) = \int_{(A \cap \xi^{-1} B \xi)\backslash G} q_x f(g)\, q_x \phi(\xi^g)\, dg$$

converges on a non-meager set in $X \otimes_k \tilde{k}$, and that

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G \times \tilde{k}_x}(\text{c-Ind}_A^G q_x(\sigma \otimes_k \tilde{k}) \otimes \text{c-Ind}_B^G q_x(\tau \otimes_k \tilde{k}), \tilde{k}_x) \le 1$$

for x in a non-meager set in $X \otimes_k \tilde{k}$, we conclude that the map

$$f \otimes \varphi \to \int_{(A \cap \xi^{-1} B \xi)\backslash G} q_x f(g)\, q_x \phi(g)\, dg$$

extends to an $\mathcal{O} \otimes_k \tilde{k}$-parametrized meromorphic family of intertwining operators

$$\Phi : \text{c-Ind}_A^G(\sigma \otimes_k \tilde{k}) \otimes \text{c-Ind}_B^G(\tau \otimes_k \tilde{k}) \to \mathcal{O} \otimes_k \tilde{k}$$

which is *rational over* k. In particular, for each fixed $f \in \text{c-Ind}_A^G \sigma$ and $\varphi \in \text{c-Ind}_B^G \tau$, the function $\Phi(f \otimes \varphi)$ is in the field $\mathcal{M}$.

Remarks: Thus, in a more mundane sense, as a 'function' of the parameter x, the integral $I(f,\varphi)(x)$ is a *rational function.* Further, again in a more mundane sense, the 'coefficients' of that rational 'function' lie in k, not merely in the extension $\tilde{k}$, for f,φ in the k-rational points c-$\mathrm{Ind}_A^G\sigma$ and c-$\mathrm{Ind}_B^G\tau$ of the representations.

Proof: The hypotheses above include two of the three hypotheses needed to invoke the Rationality Lemma. The crux of the present argument is proof that there is a *good test vector*, thus verifying the third of the necessary hypotheses.

For most of the proof, we may as well suppose that $k=\tilde{k}$ to simplify notation. And, throughout, there is no loss of generality in assuming that $\xi=1$. This simplifies the notation considerably.

For maximal x so that the integral converges, the map

$$q_x f\otimes q_x\varphi\to I(f,\varphi)(x)$$

is an intertwining operator

$$\text{c-Ind}_A^G q_x\sigma\otimes\text{c-Ind}_B^G q_x\tau\to k_x$$

Thus, we have a pointwise intertwining on a non-meager subset of the parameter space, extending scalars from k to $\tilde{k}$ if necessary, which is *one* hypothesis of the Rationality Lemma.

The third hypothesis of the Rationality Lemma, which we have *not* simply assumed away, is the presence of a *good test vector*, which here must be of the form

$$f_o\otimes\varphi_o\in\text{c-Ind}_A^G\sigma\otimes\text{c-Ind}_B^G\tau$$

which should have the property

$$I(f_o,\varphi_o)(x)=1$$

for all x in a non-meager subset of $X\otimes_k\tilde{k}$. Before addressing this, we need:

Lemma: Let A,B be closed subgroups of a *separable* locally compact Hausdorff totally disconnected topological group G. Suppose that $A\cdot B$ is *open* in G. Then, for arbitrarily small compact open subgroups A_o,B_o of A,B, respectively, the product $A_o\cdot B_o$ is open in G.

Proof (of Lemma): We have

$$A\cdot B=\bigcup_{\alpha,\beta}\alpha\cdot A_oB_o\cdot\beta$$

where $\alpha\in A/A_o$ and $\beta\in B_o\backslash B$. The latter collections are *countable.* Thus, by the Baire category theorem, one (hence, *all*) of the (mutually homeomorphic) sets $\alpha\cdot A_oB_o\cdot\beta$ has non-empty interior. Let $a_o\cdot b_o\in U$ with U an open set inside $A_o\cdot B_o$, with $a_o\in A_o$ and $b_o\in B_o$. Then for any $a_1\in A$ and $b_1\in B$, as usual we have

$$a_1b_1=(a_1a_o^{-1})a_o\,b_o(b_o^{-1}b_1)\in(a_1a_o^{-1})U(b_o^{-1}b_1)\subset A_oUB_o$$
$$\subset A_oA_oB_oB_o=A_oB_o$$

That is, every point is in the interior, so the set is open. *Done.*Returning to construction of the good test vector and proof of the theorem: Take a small-enough compact-open subgroup K_o of G so that σ is trivial on $A\cap K_o$ and τ is trivial on $B\cap K_o$. Shrink K_o if necessary so that

$$K_o\subset A\cdot B$$

where we now make use of the fact that $A\cdot B$ is *open* in G. Put $A_o=A\cap K_o$ and $B_o=B\cap K_o$. Finally, take a small-enough compact open subgroup K inside K_o and so that

$$K\subset A_oB_o\cap B_oA_o$$

Consider f_o,φ_o in c-$\mathrm{Ind}_A^G\sigma$, c-$\mathrm{Ind}_B^G\tau$, respectively, defined by

$$f_o(a\theta)=\begin{cases}=\;\;\sigma(a) & \text{for }\; a\in A,\,\theta\in K\\ =\;\;0 & \text{off }\; A\cdot K\end{cases}$$

$$\varphi_o(b\theta) = \begin{cases} = & \tau(b) \quad \text{for } b \in B,\, \theta \in K \\ = & 0 \qquad \text{off } B \cdot K \end{cases}$$

Let S be the support of $f_o \varphi_o$.

We estimate the image of

$$(A \cdot K) \cap (B \cdot K)$$

in $(A \cap B)\backslash G$. We have

$$A \cdot K \cap B \cdot K = (A \cap B \cdot K) \cdot K \subset (A \cap B \cdot B_o A_o) \cdot K$$

$$= (A \cap B \cdot A_o) \cdot K = (A \cap B) \cdot A_o K = (A \cap B) \cdot K_o$$

since $K \subset A_o \subset A$, $B_o \subset B$, and A_o and B_o lie inside K_o. Thus, the image is contained in the image of K_o, which is compact.

Therefore, adjusting $f_o \otimes \varphi_o$ by a positive rational constant if necessary (depending upon choice of $\mathbf{Q}$-valued right Haar measure), by elementary integration theory (e.g., surjectivity of the *averaging maps*) we have

$$\int_{(A \cap B)\backslash G} q_x f(g)\, q_x \phi(g)\; dg = \int_{K_o} q_x f(\theta)\, q_x \phi(\theta)\; d\theta = \text{meas}(K_o)$$

since K_o was chosen to be small enough so that both σ and τ are trivial on it. This holds for all x (not only for x for which the original integral was guaranteed to converge). That is, adjusting $f_o \otimes \varphi_o$ by a non-zero rational number if necessary, we have

$$\int_{(A \cap B)\backslash G} q_x f(g)\, q_x \phi(g)\; dg = 1 = \int_{K_o} q_x f(\theta)\, q_x \phi(\theta)\; d\theta$$

Thus, we are guaranteed that *good test vectors exist*.

Thus, we can invoke the Rationality Lemma, concluding that there is an extension of the integral to a meromorphic $\mathcal{O}$-parametrized intertwining Φ.

At this point, we return to the distinction between k and $\tilde{k}$. First, σ and τ were actually $\mathcal{O}$-parametrized, rather than merely $\mathcal{O} \otimes_k \tilde{k}$-parametrized. Also, the construction of the good test vector really yields a test vector in

$$\text{c-Ind}_A^G \sigma \otimes \text{c-Ind}_B^G \tau$$

not merely in the corresponding families of representations with scalars extended from k to $\tilde{k}$. Thus, by the Rationality Lemma the meromorphic family Φ of intertwinings just proven to exist is actually k-rational, rather than merely $\tilde{k}$-rational.

In particular, this entails that for every $f \in \text{c-Ind}_A^G \sigma$ and $\varphi \in \text{c-Ind}_B^G \tau$, the value $\Phi(f \otimes \varphi)$ is in the field $\mathcal{M}$, rather than merely in the larger field $\mathcal{M} \otimes_k \tilde{k}$. *Done.*Now we give a different-looking rationality result for local integrals with parameters. Still, the parameters are potentially the familiar q^{-s}, or several such parameters $q^{-s}, q^{-s_1}, \ldots, q^{-s_n}$, and there is a corresponding rationality result concerning the 'scalars' involved. Again, the obvious follow-up is closer determination of the local integrals without *ad hoc* computation. (This result includes the previous one as a special case).

Let $\tilde{H}$ be a locally compact totally disconnected topological group of which H and G are closed subgroups. Let Y be a closed subset of $\tilde{H}$ stable under left multiplication by H and right multiplication by G. Let Z be a locally compact totally disconnected space on which H acts trivially and upon which G acts continuously on the right. Take

$$\Omega = Y \times Z$$

Let σ be an $\mathcal{O}$-parametrized family of *finite-dimensional* smooth representations of H. Recall the notation that, for a prime ideal x in the parameter space, σ_x is the pointwise representation at x obtained from the family σ.

Let $C_c^\infty(H\backslash\Omega, \sigma)$ be the collection of σ-valued locally constant functions f on Ω the image of whose support is *compact* in the quotient space $H\backslash\Omega$, and which have the left equivariance

$$f(hx) = \sigma(h)\, f(x)$$

for all $h \in H$ and $x \in \Omega$. These functions are **(generalized) test functions** in the sense of Bruhat. The action of G on the right on Ω makes this space of functions a smooth representation space for G.

The **open orbit** hypothesis we impose is the assumption that there is some $\xi \in \Omega$ so that

$$H\,\xi\, G \subset \Omega$$

is an *open subset* of Ω. We want a $\mathbf{Q}$-valued right-G-invariant "integral"

$$f \to \int_{H\backslash\Omega} f(\omega)\, d\omega$$

supported on $H\backslash H\xi G$. By properties of test functions on totally disconnected spaces (as in 3.2), we have a natural G-isomorphism

$$C_c^\infty(H\backslash H\xi G, \sigma) \approx C_c^\infty(\xi^{-1}H\xi \cap G\backslash G, \sigma^\xi)$$

where $\sigma^\xi(x) = \sigma(\xi x \xi^{-1})$. Thus, assuming (as in 1.2) that

$$\sigma^\xi = \frac{\delta_H}{\delta_G}$$

assures the existence of such an integral in extended sense.

For x a *maximal* ideal in the prime spectrum of $\mathcal{O} \otimes_k \tilde{k}$ consider the ($\tilde{k}_x$-valued) **local "integral"** (in the extended sense)

$$I_f(x) = \int_{H\backslash\Omega} f(\omega)\, d\omega$$

where f is in $C_c^\infty(H\backslash\Omega, \sigma)$, and q_x denotes pointwise evaluation as above. Again, this integral is an analytic object rather than an algebraic one, although it may be *p-adic* rather than real or complex. We *assume* that for maximal ideals x in some non-meager subset of $X \otimes_k \tilde{k}$ the integral *converges* absolutely for *all* $f \in C_c^\infty(H\backslash\Omega, \sigma_x)$.

Let us *assume* also the **generic multiplicity-one** condition: that for x in a non-meager subset of $X \otimes_k \tilde{k}$ we have the *multiplicity-one* condition

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G\times\tilde{k}_x}(C_c^\infty(H\backslash\Omega, \sigma_x), \tilde{k}_x) \le 1$$

Then we have the following corollary of the Rationality Lemma, including two rationality assertions: rationality as a function of the parameter $x \in X \otimes_k \tilde{k}$, and rationality of scalars.

Theorem: With the open-orbit hypothesis, with convergence on a non-meager set in $X \otimes_k \tilde{k}$, and with

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G\times\tilde{k}_x}(C_c^\infty(H\backslash\Omega, \sigma_x), \tilde{k}_x) \le 1$$

for x in a non-meager set in $X \otimes_k \tilde{k}$, the map

$$f \to \int_{H\backslash\Omega} f(\omega)\, d\omega$$

extends to an $\mathcal{O} \otimes_k \tilde{k}$-parametrized meromorphic family of intertwining operators

$$\Phi : C_c^\infty(H\backslash\Omega, \sigma) \otimes_k \tilde{k} \to \mathcal{M} \otimes_k \tilde{k}$$

which is *rational over* k. In particular, for each fixed $f \in C_c^\infty(H\backslash\Omega, \sigma)$ the "function" $\Phi(f)$ is in the field $\mathcal{M}$.

Remarks: Thus, as a 'function' of the parameter x, the integral $x \to I_f(x)$ is a *rational function*, and the 'coefficients' of that rational 'function' lie in k, not merely in $\tilde{k}$, for f in the k-rational points $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ of the representation.

Proof: We can assume without loss of generality for the proof that $\tilde{k} = k$.

The explicit hypotheses give us two of the three hypotheses for the Rationality Lemma. All that remains is to find a *good test vector* $f_o \in \mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$, meaning that

$$\int_{H\backslash\Omega} f_o(\omega)\, d\omega = 1$$

(independent of x in the parameter space).

We will choose f_o in the subspace

$$\mathrm{C}_c^\infty(H\backslash H\xi G, \sigma)$$

of the whole test space $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$. Let

$$\Theta_\xi = \{g \in G : \text{ there is } h_g \in H \text{ so that } h_g^{-1}\xi g = \xi\}$$

For $g \in \Theta_\xi$, define

$$\sigma^\xi(g) = \sigma(h_g)$$

From the discussion of orbit filtrations of test functions (section 3.2), we have

$$\mathrm{C}_c^\infty(H\backslash H\xi G, \sigma) \approx \text{c-Ind}_{\Theta_\xi}^G \sigma^\xi$$

Let K be a compact-open subgroup of G small-enough so that σ^ξ is identically 1 on $K \cap \Theta_\xi$. Then put

$$f_o(\theta\ell) = \begin{cases} = & \sigma^\xi(\theta) \quad \text{for } \theta \in \Theta_\xi,\ \ell \in K \\ = & 0 \quad\quad\quad \text{off } \Theta \cdot K \end{cases}$$

The hypothesis on K, necessitating that σ be finite-dimensional, assures that this is well-defined. Then (at first only for x at which the integral is known to converge)

$$I(f_o)(x) = \int_{H\backslash\Omega} f_o(\omega)\, d\omega = \int_{\Theta_\xi\backslash\Theta_\xi K} f_o(\omega)\, d\omega$$

$$= \int_K f_o(\ell)\, d\ell = \text{meas}(K)$$

where $d\ell$ is a suitably-normalized Haar measure on K. Thus, correcting f_o by dividing by meas(K), we have the desired test function. ///

2.3Analytic continuation of intertwinings

Now we return to consideration of the global integrals. So far, we have factored the *inner integral* as

$$\int_{Z'_{\mathcal{A}}\,\Theta_k\backslash\Theta_{\mathcal{A}}} f(\theta\, g)\, d\theta$$

$$= (\text{ bad-primes factor and period }) \times \prod_{v \text{ good}} \Phi_v(e_v)(g_v)$$

where e_v is the normalized spherical vector in the unramified principal series c-$\mathrm{Ind}_{P_v}^{G_v}\chi_v$, and Φ_v is a non-zero intertwining operator from the unramified principal series to $\mathrm{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v}$.

The goal of this section is to express the intertwining operator Φ_v as an *integral*. Ironically, the natural integral expressing the intertwining operator seldom converges for χ_v in the range of characters appearing in the unramified principal series mapping surjectively to the local representations π_v generated by a (square-integrable) cuspform.

To deal with this we enlarge the class of χ_v's to include values for which the natural integral *does* converge, and then prove that the intertwining operator has an *analytic continuation*. In fact, in the cases at hand we present the family of characters χ_v as a parametrized family (in the sense of (2.1)), and invoke the Rationality Lemma and its corollaries (2.2) to prove that the natural integral is identifiable as a meromorphic family of intertwining operators. Thus, the analytic continuation issue will be resolved trivially, by proving that the object in question is in fact *rational*.

As noted in (1.3), in general the hypothesis that the period does not vanish is not trivially fulfilled, but may require *at least* some local condition on the characters χ_v. We will make a parametrized family of characters meeting at least this local condition.

Remark: Unfortunately, still due to the fact that we have limited information on the character χ_v occuring for a cuspform, it seems difficult to give a universal answer to the question of whether or not the intertwining operator Φ_v lies in the family of intertwining operators we construct. In any particular case somewhat more can be said, of course, and in the most fortuitous circumstances (in which there is at most a single intertwining operator for each parameter value) it certainly follows that Φ_v is given by an integral as constructed below.

We inherit several assumptions from 1.1, 1.2, 1.3, and 1.4: we are assuming that $\Theta_v\backslash G_v/P_v$ is finite, that $\xi\Theta_v\xi^{-1}$ is contained in the kernel of $\tilde{\chi}_v$ (so that ε **escapes** from the inner integral as in 1.2), and that (in the notation of 1.4)

$$\sum_{\alpha\in P_v\backslash G_v/\Theta_v} \dim\ \mathrm{Hom}_{H_\alpha}(\chi^\alpha,\ \delta_{\Theta_v}\delta_{H_\alpha}) = 1$$

The latter implies that

$$\dim \mathrm{Hom}_{G_v}(\text{c-}\mathrm{Ind}_{P_v}^{G_v}\,\chi_v\delta_{P_v}^{1/2}, \mathrm{Ind}_{\Theta_v}^{G_v}\,\delta_{\Theta_v}) \le 1$$

To avoid trivial vanishing of the inner integral (1.3), we must assume at least that

$$\dim \mathrm{Hom}_{G_v}(\text{c-}\mathrm{Ind}_{P_v}^{G_v}\,\chi_v\delta_{P_v}^{1/2}, \mathrm{Ind}_{\Theta_v}^{G_v}\,\delta_{\Theta_v}) = 1$$

Let N_v be (the v-adic points of) the unipotent radical of P, and M_v (the v-adic points of) a Levi component of P. Let M_o be the maximal compact subgroup of M_v (unique because P_v is minimal). Let $\varpi_1,\ldots,\varpi_n$ be generators for the abelian group M_v/M_o. Let $z_1,\ldots,z_n$ be transcendental over the groundfield k, and put

$$\mathcal{O}' = k[z_1, z_1^{-1},\ldots,z_n,z_n^{-1}]$$

Define

$$\chi' : M_v \to M_v/M_o \to (\mathcal{O}')^\times$$

by

$$\chi'(\varpi_i) = z_i$$

extending by multiplicativity, and where the first map is the quotient map. Then extend χ' to P_v by taking χ' identically 1 on N_v. This χ' is the (universal) $\mathcal{O}'$-parametrized family of unramified characters on P_v. To avoid trivial vanishing, in general we must use a smaller family, as follows.

As in 3.2, partially order the orbits $P_v\alpha\Theta_v$ (or representatives α) by

$$P_v\alpha\Theta_v \le P_v\beta\Theta_v$$

if $P_v\beta\Theta_v$ lies in the closure of $P_v\alpha\Theta_v$. We have the orbit filtration on $\text{c-Ind}_{P_v}^{G_v}\chi'$ described by

$$(\text{c-Ind}_{P_v}^{G_v}\chi')^{\le\beta} = \{f \in \text{c-Ind}_{P_v}^{G_v}\chi' : \text{ support } f \text{ is in } \bigcup_{\alpha\le\beta} P_v\alpha\Theta_v\}$$

Also put

$$(\text{c-Ind}_{P_v}^{G_v}\chi')^{<\beta} = \{f \in \text{c-Ind}_{P_v}^{G_v}\chi' : \text{ support } f \text{ is in } \bigcup_{\alpha<\beta} P_v\alpha\Theta_v\}$$

From 3.2, the β^{th} graded piece of this filtration is naturally isomorphic as a Θ_v-representation to

$$\text{c-Ind}_{H_\beta}^{\Theta_v}(\chi')^\beta$$

where

$$H_\beta = \beta^{-1}P_v\beta \cap \Theta_v$$

and for $\theta \in H_\beta$

$$(\chi')^\beta(\theta) = \chi'(\beta\,\theta\,\beta^{-1})$$

(These are none other than the sort of graded pieces considered in using such a filtration to estimate

$$\dim_{k_x} \text{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v}\chi_v,\ \text{Ind}_{\Theta_v}^{G_v}\delta_{\Theta_v})$$

as we did in 1.4.)

As in 1.4, it is an elementary computation that

$$\dim_{k_x} \text{Hom}_{\Theta_v}(\text{c-Ind}_{H_\beta}^{\Theta_v}(\chi')^\beta, \delta_{\Theta_v}) \le 1$$

and that we get equality if and only if

$$(\chi')^\beta|_{H_\beta} = \delta_{H_\beta}$$

For representative β, for each $h \in H_\beta$ the condition

$$(\chi')^\beta(h) = \delta_{H_\beta}(h)$$

is something of the form

$$z_1^{a_1(h)} \ldots z_n^{a_n(h)} = c(h)$$

where the $a_i(h)$ are integers and $c(h) \in \mathbf{Q}^\times$ (since the group is totally disconnected). Let I_β be the radical of the ideal in $\mathcal{O}'$ generated by all such relations for $h \in H_\beta$.

Let $P_v\beta\Theta_v$ be *minimal* (in the partial order above) so that $I_\beta \ne \mathcal{O}'$. (That the condition *can* ever be met is a consequence of the existence of a non-trivial intertwining operator for *some* pointwise representation obtained from χ'). Then let

$$\mathcal{O} = \mathcal{O}'/I_\beta$$

for such β. Thus, $\mathcal{O}$ parametrizes a family of characters so that the β^{th} graded piece of the orbit filtration admits an intertwining operator to δ_{H_β}.

Remark: In practice, it seems that usually if $P_v\beta\Theta_v < P_v\alpha\Theta_v$ and $I_\beta \ne \mathcal{O}'$ then

$$I_\beta \subset\subset I_\alpha \subset \mathcal{O}'$$

with the first containment *strict*. Further, it often happens that there is a *unique* minimal β whose associated graded piece admits an intertwining operator. In such circumstances there is of course no ambiguity in choice of $\mathcal{O}$.

Then the natural integral associated to this choice of β and $\mathcal{O}$ is

$$\Phi(f)(g) = \int_{\beta^{-1} P_v \beta \cap \Theta_v \backslash \Theta_v} f(\beta\, \theta\, g)\, \delta_{\Theta_v}(\theta)\; d\theta$$

It is immediate that, *if* this converges, then it is an intertwining operator from $\text{c-Ind}_{P_v}^{G_v} q_x \chi'$ to $\text{Ind}_{\Theta_v}^{G_v} \delta_{\Theta_v}$. Yet, while this integral is actually a finite sum for f in $(\text{c-Ind}_{P_v}^{G_v} \chi')^{\le \beta}$, convergence for general f in $\text{c-Ind}_{P_v}^{G_v} \chi'$ is an issue. Let $X = \operatorname{spec} \mathcal{O}$, and let q_x be the evaluation map(s) at x.

Theorem: Assume that for a non-meager collection of $x \in X \otimes_k \tilde{k}$ and for all f in $\text{c-Ind}_{P_v}^{G_v} \chi'$ the integral for $\Phi(q_x\, f)$ is absolutely convergent, and that

$$\dim_{\tilde{k}_x} \operatorname{Hom}_{G_v}(\text{c-Ind}_{P_v}^{G_v} q_x \chi', \text{Ind}_{\Theta_v}^{G_v} \delta_{\Theta_v}) \le 1$$

Then this integral extends to give a meromorphic family Φ of intertwining operators from $\text{c-Ind}_{P_v}^{G_v} \chi'$ to $\text{Ind}_{\Theta_v}^{G_v} \delta_{\Theta_v}$, in the sense of 2.1.

Proof: Fix $g \in G_v$ and denote the map given by the integral whenever it converges absolutely by φ_g. This φ_g is an intertwining operator

$$\text{Res}_{\Theta_v}^{G_v}\, \text{c-Ind}_{P_v}^{G_v} q_x \chi' \to \delta_{\Theta_v}$$

(observing that the representation δ_{Θ_v} is a *constant* family.) The assumption on the dimension of the space of intertwining operators immediately translates by Frobenius Reciprocity to

$$\dim_{k_x} \operatorname{Hom}_{\Theta_v}(\text{Res}_{\Theta_v}^{G_v}\, \text{c-Ind}_{P_v}^{G_v} q_x \chi', \delta_{\Theta_v}) \le 1$$

This and the assumptions of convergence allow us to invoke the theorem(s) of 2.2, concluding that φ_g is a *meromorphic family* of intertwining operators from $\text{Res}_{\Theta_v}^{G_v}\, \text{c-Ind}_{P_v}^{G_v} q_x \chi'$ to δ_{Θ_v}.

Since

$$\Phi(f)(g) = \varphi_g(f)$$

by the definition of meromorphy of a family of intertwining operators (from 2.1) we have the conclusion of the theorem. *Done.*

Remark: The meromorphy asserted here might be construed as a *weak* meromorphy, since there is not necessarily any uniform estimate on poles as $g \in G_v$ varies. This flaw will be corrected in 2.5, when *strong* meromorphy is discussed.

2.4Rationality of the Euler factors

By this point, we have managed to rewrite the global integral as a product of the *period* and an *Euler product* with v^{th} factor

$$\int_{\tilde{Z}_v \Theta_v \backslash G_v} \varepsilon_v(\xi g)\; \Phi_v(e_v)(g)\; dg$$

(using the notation of 1.2). In the last section we found an integral expression giving (when analytically continued) a meromorphic family of intertwining operators which we *assume* contains Φ_v. Namely, we put

$$\Phi(f) = \int_{\beta^{-1} P_v \beta \cap \Theta_v \backslash \Theta_v} f(\beta\, \theta\, g)\, \delta_{\Theta_v}(\theta)\; d\theta$$

We inherit several assumptions: we are assuming that $\Theta_v \backslash G_v / P_v$ is finite, and that $\xi \Theta_v \xi^{-1}$ is contained in the kernel of $\tilde{\chi}_v$ (so that ε **escapes** from the inner integral). To use the integral representation of the last section, we must assume that the character χ_v lies among the pointwise representations from the family χ' described in the last section (as well as that the intertwining operator Φ_v lies in the family Φ).

Now we replace Φ_v in the outer integral by its integral expression from the last expression: the v^{th} Euler factor is the outer integral, which is of the form

$$\int_{\tilde{Z}_v \Theta_v \backslash G_v} \varepsilon_v(\xi g) \int_{\beta^{-1} P_v \beta \cap \Theta_v} f(\beta\,\theta\,g)\,\delta_{\Theta_v}(\theta)\,d\theta\,dg$$

where instead of simply the spherical vector e_v we now use an arbitrary f in some pointwise representation arising from the family $\text{c-Ind}_{P_v}^{G_v}\chi'$ constructed in the last section. By integration theory (see 1.2), we can immediately rewrite this as a single integral

$$\int_{\tilde{Z}_v H_\beta \backslash G_v} \varepsilon_v(\xi g)\ f(\beta\,\theta\,g)\ dg$$

where

$$H_\beta = \beta^{-1} P_v \beta \cap \Theta_v$$

Now we assume *further* that

$$\tilde{P}_v \backslash \tilde{G}_v / P_v \qquad \text{is finite}$$

for almost all primes v. We may as well note the following:
Proposition: If $\tilde{P}_v \backslash \tilde{G}_v / P_v$ is finite then $\Theta_v \backslash G_v / P_v$ is finite.

Proof: $\Theta = \xi^{-1}\tilde{P}\xi \cap G$. *Done.*

In addition to postulating that the local character χ_v arising from the cuspform is also a pointwise representation from the $\mathcal{O}$-parametrized family χ' constructed in the last section, we must also present the character $\tilde{\chi}_v$ (from the Eisenstein series) in the same fashion. Just as in general we must restrict the parameter space for χ' to avoid trivial vanishing of the inner integral (see 1.3), we must restrict the family in which $\tilde{\chi}_v$ lies in order to assure that the integrand is $\tilde{Z}_v$-invariant.

The corresponding construction is even simpler than that done in the last section, so without repeating it we assume now that we have the commutative Noetherian ring $\tilde{\mathcal{O}} = k[z, z^{-1}]$ whose spectrum $\tilde{X}$ parametrizes a family $\tilde{\chi}'$ of characters on $\tilde{P}_v$, so that the integrand above is $\tilde{Z}_v$-invariant.

Recall also from 2.3 that the ring $\mathcal{O}$ parametrizing the family of characters χ' is a quotient of $k[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$. We abusively use the symbols z_i to refer to their images in $\mathcal{O}$.
Theorem: Let v be a prime so that π_v is spherical. Suppose that the double-coset space

$$\tilde{P}_v \backslash \tilde{G}_v / P_v$$

is *finite*. Suppose that for x in a non-meager subset of $\operatorname{spec} \tilde{\mathcal{O}} \otimes \mathcal{O} \otimes_k \tilde{k}$ for all $f \in \text{c-Ind}_{P_v}^{G_v} q_x \chi'$ the integral

$$\int_{\tilde{Z}_v H_\beta \backslash G_v} \varepsilon_v(\xi g)\ f(\beta\,\theta\,g)\ dg$$

is absolutely convergent, and also

$$\dim_{k_x} \operatorname{Hom}_{G_v}(\operatorname{Res}_{G_v}^{\tilde{G}_v} \left(\text{c-Ind}_{\tilde{P}_v}^{\tilde{G}_v} \tilde{\chi}_v\right) \otimes \text{c-Ind}_{P_v}^{G_v} \chi_v, \tilde{k}_x) \leq 1$$

Then this integral extends to give a meromorphic family of intertwining operators

$$\operatorname{Res}_{G_v}^{\tilde{G}_v} \text{c-Ind}_{\tilde{P}_v}^{\tilde{G}_v} \tilde{\chi}' \otimes \text{c-Ind}_{P_v}^{G_v} \chi' \to (\tilde{\mathcal{O}} \otimes \mathcal{O} \otimes_k \tilde{k})^\times$$

Corollary Under the hypotheses of the previous theorem, the v^{th} Euler factor is a *rational function* of $z, z_1, z_2, \ldots, z_n$.

Remark: We think of z as q^{-s} where this s is the "live" variable occurring in the Eisenstein series, while $z_i = q^{-s_i}$ map homomorphically to give the Satake parameters of the cuspform.

Proofs: The hypotheses are those necessary to apply the rationality theorems of 2.2, so we obtain this theorem at once. By integration theory, the present convergence hypothesis implices convergence of the integral of 2.3 used to rewrite the inner integral. The Corollary is obtained by replacing the general f in the parametrized family of unramified principal series by the spherical vector e' in that parametrized family, uniformly normalized (as is done pointwise) by the condition

$$e'(1) = 1$$

Done.

Remark: Pursuing the quantitative issue of the computation or estimation of the rational expressions which occur in this result, we will prove later that if the double-coset space $\tilde{P}_v\backslash\tilde{G}_v/P_v$ is finite then the poles of the rational expressions so occurring have leading terms in their Laurent expansions which are interpretable as *anomalous intertwining operators* (in a sense made precise later (section 2.6)). *This will put very strong limitations on the possible denominators of the rational expressions possibly occurring in the local integrals.*

Further, much as was done in 1.4 for the inner integral, we can obtain an estimate on the dimension of the indicated space of intertwining operators using the orbit filtration method from 3.2, as follows. By dualizing and applying Frobenius Reciprocity, the space of intertwining operators becomes (pointwise, for x in the parameter space)

$$\dim_{k_x} \mathrm{Hom}_{P_v}(\mathrm{Res}^{\tilde{G}_v}_{P_v} \text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} q_x\tilde{\chi}', (q_x\chi')^{-1}\delta_{P_v}) = 1$$

Using the $\tilde{P}_v \times P_v$-orbit filtration on the test functions

$$\mathrm{Res}^{\tilde{G}_v}_{P_v}\left(\text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} q_x\tilde{\chi}'\right)$$

we obtain graded pieces

$$\text{c-Ind}^{G_v}_{H_\beta} q_x\tilde{\chi}'^{\beta}$$

where $\beta \in \tilde{P}_v\backslash\tilde{G}_v/P_v$,

$$H_\beta = \beta^{-1}\tilde{P}_v\beta \cap P_v$$

and

$$q_x\tilde{\chi}'^{\beta}(p) = q_x\tilde{\chi}'(\beta p \beta^{-1})$$

for $p \in H_\beta \subset P_v$.

Thus, we obtain an inequality

$$\dim_{k_x} \mathrm{Hom}_{G_v}(\mathrm{Res}^{\tilde{G}_v}_{G_v}\left(\text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} q_x\tilde{\chi}'\right) \otimes \text{c-Ind}^{G_v}_{P_v} q_x\chi',\, k_x)$$

$$\leq \sum_\beta \dim_{k_x} \mathrm{Hom}_{P_v}(\text{c-Ind}^{P_v}_{H_\beta} q_x\tilde{\chi}'^{\beta}, (q_x\chi')^{-1}\delta_{P_v})$$

Dualizing and applying Frobenius Reciprocity once more, the latter sum is

$$\sum_\beta \dim_{k_x} \mathrm{Hom}_{H_\beta}(q_x\chi'\delta_{P_v}^{-1}, q_x\tilde{\chi}'^{\beta}\frac{\delta_{H_\beta}}{\delta_{P_v}}) = 1$$

Simplifying and dualizing again, we get

$$\dim_{k_x} \mathrm{Hom}_{G_v}(\mathrm{Res}^{\tilde{G}_v}_{G_v}\left(\text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} q_x\tilde{\chi}'\right) \otimes \text{c-Ind}^{G_v}_{P_v} q_x\chi',\, k_x)$$

$$\leq \sum_\beta \dim_{k_x} \operatorname{Hom}_{H_\beta}(q_x\chi', q_x\tilde{\chi}'^\beta \delta_{H_\beta})$$

The latter sum is easy to compute, since all the representations are one-dimensional.

Further, one may notice that the the orbit-filtration estimate on multiplicity for the inner integral in 1.4 is simply a *subsum* of the present sum

$$\sum_{\beta \in \tilde{P}_v \xi \tilde{G} P_v} \dim_{k_x} \operatorname{Hom}_{H_\beta}(q_x\chi', q_x\tilde{\chi}'^\beta \delta_{H_\beta})$$

because $\xi^{-1}\Theta_v\xi$ was assumed to lie in the kernel of the character $q_x\tilde{\chi}'$. Thus, we have

Observation: If the graded-piece estimate *generically* fails to show multiplicity one for the inner integral, it will also fail for the outer integral.

Remarks: Since the multiplicity-one condition need not be satisfied for *every* parameter value, but only for a sufficiently large set of such, fewer additional devices are needed to get to the desired *generic* multiplicity-one condition than a *strict* multiplicity-one condition.

If the program were successful up to this point, the discussion would begin another phase, in which the outer local integrals should be (literally) *computed*. By this point, one would already know that the integrals were rational functions of parameters such as q^{-s}, and also of less-obvious parameters such as the Satake parameters q^{-s_i}, which become visible when spherical representations are treated as quotients of unramified principal series.

Yet, rather than engage in *ad hoc* computations, in the sequel we will interpret the 'poles' of the integrals as being special kinds of intertwining operators, referred to below as *anomalous intertwining operators*.

2.5Orbit criterion for strong meromorphy

To study local integrals further, we must attach meaning to the *zeros* and *poles* in the rational expressions proven above to occur. Not surprisingly, the study of poles is the easier, since (as in other scenarios as well) the *residue* (or more generally *leading term*) in a *Laurent expansion* of an intertwining operator at a *pole* has demonstrably special properties. Thus, to estimate the possible poles, one instead classifies the possible **anomalous intertwining operators**, in a sense described precisely below in 2.6.

One minor complication is that in general, and in many situations of interest (e.g., intertwinings involving unramified principal series) the parameter space $X = \operatorname{Spec} \mathcal{O}$ is of dimension greater than 1, so *poles*, being codimension 1, are of positive dimension.

Generally, for a $G \times \mathcal{O}$-module V and for a height-one prime x in $\mathcal{O}$, say that a (meromorphic $\mathcal{O}$-parametrized family of) intertwining(s)

$$\Phi \in \operatorname{Hom}_{G\times\mathcal{O}}(V, \mathcal{M})$$

is **locally strongly meromorphic at** x if there is some integer power ϖ^N of a local parameter $\varpi = \varpi_x$ at x so that

$$\varpi^N \Phi \in \operatorname{Hom}_{G\times\mathcal{O}}(V, \mathcal{O}_x)$$

where $\mathcal{O}_x$ is the local ring at x. If Φ is locally strongly meromorphic at every height-one prime x in $\mathcal{O}$, then Φ is **locally strongly meromorphic**.

Remarks: The locally strong meromorphy condition at x precludes there being a sequence of vectors $v_i \in V$ so that the *rational functions* $\Phi(v_i) \in \mathcal{M}$ have deeper and deeper poles along the hypersurface given by x. On the other hand, even though our parameter space $\operatorname{Spec} \mathcal{O}$ is an affine variety, the present definition of (local) strong meromorphy makes no assertion about finiteness of the collection of hypersurfaces along which there might be a pole. Such claims will be made later, depending upon interpretation of Laurent expansions at poles as giving *anomalous intertwinings*.

Now we can describe a fairly general scenario in which locally strong meromorphy is provably assured. (The general construction of the set Ω is the same as earlier).

Let $\tilde{H}$ be a locally compact totally disconnected topological group of which H and G are closed subgroups. Let Y be a closed subset of $\tilde{H}$ stable under left multiplication by H and right multiplication by G. Let Z be a locally compact totally disconnected space on which H acts trivially and upon which G acts continuously on the right. Take

$$\Omega = Y \times Z$$

Let σ be an $\mathcal{O}$-parametrized family of *finite-dimensional* smooth representations of H.

Let $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ be the collection of σ-valued locally constant functions f on Ω the image of whose support is *compact* in the quotient space $H\backslash\Omega$, and which have the left equivariance

$$f(hx) = \sigma(h)\, f(x)$$

for all $h \in H$ and $x \in \Omega$. These functions we view as **(generalized) test functions**.

Let P be a closed subgroup of G, with closed normal subgroup N, and closed subgroup M so that P is a semi-direct product of M and N. We suppose that N is an ascending union of compact open subgroups. And suppose that there is a compact open subgroup M_o of M so that with idempotent

$$e = \frac{\text{characteristic function of } M_o}{\text{measure of } M_o}$$

in the Hecke algebra $\mathcal{G}_M$ of M the subalgebra $e\mathcal{G}_M e$ is *Noetherian*.

Let χ be an $\mathcal{O}$-parametrized family of *finite-dimensional* smooth representations of $M \approx P/N$, so as to give an $\mathcal{O}$-parametrized family of representations of P, trivial on N. We denote this by the same letter χ. We suppose that the image $e \cdot \chi$ of the $\mathcal{O}$-module χ is a *Noetherian* $e\mathcal{G}_M e$-module, where the idempotent e is as just above.

The previous hypotheses are met very often in practice. In particular, the hypotheses on P, M, N and e are met for *unramified principal series*, where P is a parabolic subgroup of a p-adic reductive group G, N is the unipotent radical of P, and so on.

The critical hypothesis here is that the double coset space

$$H\backslash\Omega/P$$

be **finite**. In that case, we have
Theorem: Assuming finiteness of the double coset space $H\backslash\Omega/P$, any meromorphic $\mathcal{O}$-parametrized family of intertwining operators

$$\Phi \in \mathrm{Hom}_{G\times\mathcal{O}}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_{\mathcal{O}} \text{c-Ind}_P^G\, \chi, \mathcal{M})$$

is *locally strongly meromorphic. (Proof given in 3.5).*

2.6 Denominators and anomalous intertwinings

Granting that a meromorphic $\mathcal{O}$-parametrized family of intertwining operators is local strongly meromorphic allows meaning to be attached to the poles. In particular, the *residues* (or more general *leading terms*), in a sense described below, are *anomalous intertwining operators*. Thus, the representation theory of the situation gives *a priori* restrictions on the possible locations of poles of (parametrized) intertwining operators.

Thus, viewing a local integral as a certain kind of (parametrized) intertwining operator, we can give *a priori* restrictions on poles of the integral.

Now we add the hypothesis that $\mathcal{O}$ is *integrally closed*, in addition to being Noetherian (and commutative), and let x be a *height one* prime in $\mathcal{O}$. Then the local ring $\mathcal{O}_x$ is an integrally closed Noetherian integral domain in which every non-zero prime (here only $x\mathcal{O}_x$ in fact) is maximal. Thus, $\mathcal{O}_x$ is a *Dedekind domain* with one non-zero prime, so is a *valuation ring.* Thus, there exists a *local parameter* $\varpi = \varpi_x$, a generator for the unique maximal ideal in $\mathcal{O}_x$.

Suppose that a meromorphic family $\Phi : \sigma \to \tau$ of intertwinings between $\mathcal{O}$-parametrized families of representations σ and τ is (locally) **strongly meromorphic**. Let $\varpi = \varpi_x$ be a local parameter for a height-one prime x. Then for some power ϖ^N of the local parameter at x the family $\varpi^N\Phi$ of intertwinings is *holomorphic* at x. The least integer N so that $\varpi^N\Phi$ is *holomorphic* at x is the **order of the pole** of Φ at x.

Assuming that a meromorphic family Φ of intertwining operators $\sigma \to \tau$ is locally strongly meromorphic and has a pole of order at least 1 at a height-one prime x, we can consider a new intertwining operator, the **leading term** of Φ at x. Let N be the least integer so that $\varpi^N\Phi$ is holomorphic at x. Then the *pointwise* intertwining operator

$$q_x(\varpi^N\Phi) : \sigma_x \to \tau_x$$

is the **leading term** of Φ at x. In the extreme case that the order of the pole is 1, this leading term is the **residue** of Φ at x.

Remarks: Unless maximal primes x in $\mathcal{O}$ are height one, the corresponding residue fields k_x are of positive transcendence degree over the base field k. Thus, except in the special case that maximal primes are height one, residues or other leading term intertwining operators are intertwining operators between vectorspace representations over fields of positive transcendence degree over k.

We will look at local integrals of the sort considered in the *second* version of the *rationality of local integrals* results in 2.2 above. Fix a field extension $\tilde{k}$ of k, and suppose that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain, with $\mathcal{M} \otimes_k \tilde{k}$ the field of fractions of $\mathcal{O} \otimes_k \tilde{k}$. Assume that $\tilde{k}$ and finite extensions $\tilde{k}_x$ of it are *complete.* (As earlier, taking $\tilde{k}$ to be $\mathbf{R}$ or $\mathbf{C}$ or finite extensions of $\mathbf{Q}_p$ suffices).

As before, let $\tilde{H}$ be a locally compact totally disconnected topological group of which H and G are closed subgroups. Let Y be a closed subset of $\tilde{H}$ stable under left multiplication by H and right multiplication by G. Let Z be a locally compact totally disconnected space on which H acts trivially and upon which G acts continuously on the right. Take

$$\Omega = Y \times Z$$

Let σ be an $\mathcal{O}$-parametrized family of *finite-dimensional* smooth representations of H.

Again, let $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ be the collection of σ-valued locally constant functions f on Ω the image of whose support is *compact* in the quotient space $H\backslash\Omega$, and which have the left equivariance

$$f(hx) = \sigma(h)\, f(x)$$

for all $h \in H$ and $x \in \Omega$.

The **open orbit** hypothesis we impose is the assumption that there is some $\xi \in \Omega$ so that

$$H\,\xi\,G \subset \Omega$$

is an *open subset* of Ω.

And we require that there is a right G-invariant $\mathbf{Q}$-valued measure $d\omega$ on Ω, supported (for simplicity) on the open orbit $H\xi G$. For x a *maximal* ideal in the prime spectrum of $\mathcal{O} \otimes_k \tilde{k}$ consider the ($\tilde{k}_x$-valued) **local integral**

$$I_f(x) = \int_{H\backslash\Omega} f(\omega)\, d\omega$$

where f is in $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$, and q_x denotes pointwise evaluation as above.

As earlier, we *assume* that for maximal ideals x in some non-meager subset of $X \otimes_k \tilde{k}$ the integral *converges* absolutely for *all* $f \in \mathrm{C}_c^\infty(H\backslash\Omega, \sigma_x)$, and that we have the *generic multiplicity one* condition

$$\dim_{\tilde{k}_x} \mathrm{Hom}_{G\times\tilde{k}_x}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma_x), \tilde{k}_x) \le 1$$

We have shown that

$$f \to \int_{H\backslash\Omega} f(\omega)\, d\omega$$

extends to an $\mathcal{O} \otimes_k \tilde{k}$-parametrized meromorphic family of intertwining operators

$$\Phi : \mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_k \tilde{k} \to \mathcal{M} \otimes_k \tilde{k}$$

rational over k.

Theorem *(on Anomalous Intertwinings)***:** Now we assume that the ring $\mathcal{O}$ is integrally closed, Noetherian, commutative. Let Φ be the extension just described of the local integral

$$\int_{H\backslash\Omega} f(\omega)\, d\omega$$

Let x be a height-one prime at which the meromorphic $\mathcal{O}$-parametrized family of intertwining operators Φ given in $\mathrm{Hom}_{G\times\mathcal{O}}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma), \mathcal{O})$ has a pole of order $N > 0$. Let

$$\rho_x = q_x(\varpi^N \Phi)$$

be the *leading term* of Φ at x, where ϖ is a local parameter at x. Then ρ_x is a non-zero intertwining operator

$$\rho_x \in \mathrm{Hom}_{G\times k_x}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_{\mathcal{O}} k_x, k_x)$$

and

$$\rho_x(\mathrm{C}_c^\infty(H\backslash H\xi G, \sigma) \otimes_{\mathcal{O}} k_x) = \{0\}$$

In particular, *there is no pole* at a hypersurface x *unless*

$$\dim_{k_x} \mathrm{Hom}_{G\times k_x}(\mathrm{C}_c^\infty(H\backslash(\Omega - H\xi G), \sigma), k_x) > 0$$

Remarks: By definition, a k_x-linear intertwining operator from the space of test functions

$$\mathrm{C}_c^\infty(H\backslash(\Omega - H\xi G), \sigma_x)$$

on the complement of the open orbit $H\xi G$ to k_x is an **anomalous intertwining operator**. When composed with the surjective (see 3.2) map

$$\mathrm{C}_c^\infty(H\backslash\Omega, \sigma_x) \to \mathrm{C}_c^\infty(H\backslash(\Omega - H\xi G), \sigma_x)$$

we obtain (equivalently) intertwining operators which are zero on the subspace $\mathrm{C}_c^o(H\backslash H\xi G, \sigma_x)$.

Remarks: From a slightly different viewpoint: while the associated *Mackey-Bruhat* distribution of the intertwining operators at holomorphic points of Φ have *support* with non-trivial interior, the leading-term intertwining operators such as ρ_x have support contained in much smaller sets (e.g., with empty interior).

Proof of Theorem: The fact that the leading term ρ_x is an intertwining operator is straightforward. And the idea of the rest of the proof is clear: the family Φ when restricted to the smaller space of test functions does *not* have a pole at x, so when Φ is multiplied by the local parameter ϖ it *vanishes* at x.

Functions f_o in the subspace $\mathrm{C}_c^o(H\backslash H\xi G, \sigma)$ of $\mathrm{C}_c^o(H\backslash\Omega, \sigma)$ have compact support left-modulo H, so the integral (defining Φ pointwise)

$$I(f_o) = \int_{H\backslash\Omega} f_o(\omega)\, d\omega$$

has a compactly-supported (locally constant) integrand. Thus, for such f_o, we can integrate over $\mathcal{O}$ itself, since such integrals are really just algebraic objects.

In particular, for any $N > 0$, for such f_o,

$$\varpi^N \int_{H\backslash\Omega} f_o(\omega)\, d\omega \in \varpi\mathcal{O}_x$$

Thus, the evaluation map q_x at q sends this to 0, as claimed. *Done*.

Corollary: Suppose that we are in the situation of 2.4, with the v^{th} Euler factor of our global integral expressed as a value of a parametrized family of intertwining operators. Let 0 denote the zero ideal in the parameter space, and k_o the associated residue field. Suppose that

$$\dim_{k_o} \mathrm{Hom}_{G_v}(\mathrm{Res}^{\tilde{G}_v}_{G_v} \text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} \tilde{\chi}' \otimes \text{c-Ind}^{G_v}_{P_v} \chi',\ k_o) = 1$$

Then (up to units in the parametrizing ring $\tilde{\mathcal{O}} \otimes \mathcal{O}$) the denominator of the v^{th} Euler factor is a product of factors of the form

$$1 - c\, z_1^{m_1} \ldots z_n^{m_n} z^m$$

for integers $m, m_1, \ldots, m_n$, and c a non-zero element of the basefield k.

Remark: The fact that c lies in the basefield means that it is "independent" of the parameters $z, z_1, \ldots, z_n$. And we imagine that $z = q^{-s}$ and $z_i = q^{-s_i}$, so this is to assert that the denominator has factors only of the form

$$1 - c\, q^{-m_1 s_1} \ldots q^{-m_n s_n} q^{-ms}$$

with c independent of the $s, s_1, \ldots, s_n$.

Proof: Let Ψ denote the meromorphic family of intertwining operators given by the integral

$$\int_{\tilde{Z}_v H_\beta \backslash G_v} \varepsilon_v(\xi g)\, f(\beta\, \theta\, g)\, dg$$

of 2.4. Any denominator gives rise to a pole (at least over an algebraic closure of the base field), which can only occur at a hypersurface along which there is an anomalous intertwining operator. As before, using the $\tilde{P}_v \times P_v$-orbit filtration on the test functions

$$\mathrm{Res}^{\tilde{G}_v}_{P_v} \left(\text{c-Ind}^{\tilde{G}_v}_{\tilde{P}_v} q_x \tilde{\chi}'\right)$$

we obtain graded pieces

$$\text{c-Ind}^{G_v}_{H_\alpha} q_x \tilde{\chi}'^{\alpha}$$

where $\alpha \in \tilde{P}_v \backslash \tilde{G}_v / G_v$,

$$H_\alpha = \alpha^{-1} \tilde{P}_v \alpha \cap P_v$$

and

$$q_x \tilde{\chi}'^{\alpha}(p) = q_x \tilde{\chi}'(\alpha p \alpha^{-1})$$

for $p \in H_\alpha \subset P_v$. The intertwining operator Ψ is an extension from an intertwining operator on one of these graded pieces.

The anomalous intertwining operators can only occur at points x at which a *second* graded piece in the orbit filtration has a non-zero intertwining operator to k_x. The condition for this to occur is of the general form

$$(\tilde{\chi}' \otimes \chi')^{\alpha}|_{H_\alpha} = \delta_{H_\alpha}$$

as computed in both 1.4 and 2.4. As noted in 2.4, in terms of the coordinates $z, z_1, \ldots, z_n$, this condition is of the indicated form. *Done.*

Remarks: Further, the isomorphisms among irreducible unramified principal series whose characters differ by an element of the Weyl group implies a corresponding symmetry in the poles of the intertwining operator, thus implying corresponding symmetry of the denominator. Further, the degenerate principal series (locally generated by the Eisenstein series on $\tilde{G}$) also have natural intertwining operators which give additional symmetry in the "variable" s. Thus, these intertwining operators cause symmetries which tend to assure that the denominators of such local integrals look "automorphic".

2.7Illustrative examples

Here we apply the previous machinery to some simple examples, where one already knows that there is an Euler product, and also knows what the local integrals should be. First we look at the local zeta integrals of Tate's thesis [Tate 1950] and [Godement-Jacquet 1972]. Although such integrals certainly can be understood in a direct computational manner, it is interesting that we can reach conclusions without computation.
Tate's thesis revisited:

Let q be the cardinality of the residue field of a non-archimedean local field k. Let $\mathcal{S} = C_c^\infty(k)$ be the space of $\mathbf{C}$-valued test functions on k. Following Tate's thesis, we consider the *local zeta integrals*

$$\zeta(\phi, s) = \int_{k^\times} \phi(x)\chi(x)|x|^s \, d^\times x$$

where $d^\times x$ refers to a multiplicative Haar measure, $\phi \in \mathcal{S}$, χ is a fixed character on $k^\times$, and $||$ is a normalization of the norm so that $|\varpi| = q^{-1}$ for a local parameter ϖ. Write

$$\chi_s(x) = \chi(x)|x|^s$$

We observe that this zeta integral is an intertwining operator in

$$\mathrm{Hom}_{k^\times}(\mathcal{S} \otimes \chi_s, \mathbf{C})$$

The family of representation spaces χ_s is parametrized in the following manner: Let

$$\mathcal{O} = \mathbf{C}[z, z^{-1}]$$

Let V be a one-dimensional complex vectorspace, and for $g \in k^\times$ define

$$\pi(g)(v) = \chi(g) z^{\mathrm{ord}\, g} \cdot v$$

where the ord function is as usual defined by

$$g\mathbf{o} = \varpi^{\mathrm{ord}\, g}\mathbf{o}$$

where $\mathbf{o}$ is the valuation ring in k. Then the pointwise representations are recovered by mapping

$$z \to q^{-s}$$

From the orbit filtration on test functions, we have an exact sequence

$$0 \to C_c^\infty(k^\times) \to C_c^\infty(k) \to C_c^\infty(\{0\}) \to 0$$

And

$$\mathrm{Hom}_{k^\times}(C_c^\infty(k^\times) \otimes \chi_s, \mathbf{C}) \approx \mathrm{Hom}_{k^\times}(C_c^\infty(k^\times), \chi_{-s}) =$$

$$\mathrm{Hom}_{k^\times}(\text{c-Ind}_{\{1\}}^{k^\times}\mathbf{C}, \chi_s^{-1}) \approx \mathrm{Hom}_{k^\times}(\chi_s, \mathrm{Ind}_{\{1\}}^{k^\times}\mathbf{C})$$

By Frobenius Reciprocity this is

$$\mathrm{Hom}_{\{1\}}(\chi_s|_{\{1\}}, \mathbf{C}) \approx \mathbf{C}$$

Thus, this space is *one-dimensional.* The Tate local zeta integral is convergent for all $s \in \mathbf{C}$ for $\phi \in C_c^\infty(k^\times)$, so gives such a $k^\times$-homomorphism.

The smaller orbit $\{0\}$ gives

$$C_c^\infty(\{0\}) \approx \text{ trivial } k^\times\text{-representation}$$

so unless $\chi_s = 1$ this smaller orbit *cannot* 'support' any intertwining operator, and so when $\chi_s = 1$ we have an *anomalous* intertwining operator.

For $\mathrm{Re}\,(s)$ sufficiently large the integral defining the zeta integral on the whole space $C_c^\infty(k)$ is absolutely convergent, yielding non-zero intertwining operators on the *non-meager* set of complex s with large real part. Thus, so far, we conclude that off at most countably many points the dimension of the space of intertwining operators is at most one-dimensional. That is, we have the *generic multiplicity-one* condition.

And taking $\phi_o \in C_c^\infty(k)$ to be a suitable scalar multiple of the characteristic function of a sufficiently small compact open subgroup, we have

$$\int_{k^\times} \phi_o(x)\chi_s(x)\, d^\times x = 1$$

for all s. This fulfills the *good test vector* condition. Note that matters are simplified here in that the representation $\mathcal{S} = C_c^\infty(k)$ is 'constant' (i.e., did not depend upon the parameters).

Thus, without any further work, we have the *qualitative* conclusion, from the results on rationality of local integrals (section 2.2), that *for any $\phi \in \mathcal{S}$ the local zeta integral is a rational function of $z = q^{-s}$.*

Quantitatively: Let $\mathcal{H}$ be the Hecke algebra of the totally disconnected group $k^\times$. Let Θ be a sufficiently small compact open subgroup of $k^\times$ so that χ_s is identically 1 on Θ, and let $e \in \mathcal{H}$ be the corresponding idempotent

$$e = \text{ characteristic function of } \Theta / \text{ measure } \Theta$$

Then $e\mathcal{H}e$ is a commutative Noetherian ring. We suppose that χ is unramified, so that Θ is the whole unit group $\mathbf{o}^\times$, and then

$$e\mathcal{H}e \approx \mathbf{C}[x, x^{-1}]$$

In that case, we may as well take $\chi = 1$, so that the character is completely specified by the parameter s. With these observations, by the *orbit criterion for strong meromorphy* (section 2.5), as intertwining operator on parametrized families of representations *the local zeta integral is strongly meromorphic.*

Therefore, the only possible pole of the local zeta integral can be at the point where there is an *anomalous intertwining operator*, that is, at parameter values where there is an intertwining operator which is 0 on the subrepresentation $\mathrm{C}_c^\infty(k^\times)$ of $\mathrm{C}_c^\infty(k)$. Thus, with $z = q^{-s}$, the only possible pole is at $z = 1$.

In other words, the only possible linear factor of the denominator of the local zeta integral is

$$1 - z = 1 - q^{-s}$$

(Of course, here z, z^{-1} are units).

And we can easily verify that the collection of all possible values of the local zeta integral is a *fractional ideal* of $\mathcal{O} = \mathbf{C}[z, z^{-1}]$ containing 1.

Thus, in summary, without computing anything directly we can see that in the unramified case there is some test function so that the local zeta integral applied to it yields

$$\frac{1}{(1 - q^{-s})^m}$$

for a non-negative integer m. To prove that the 'pole' actually occurs, and with order 1, would require a little further investigation.

In the case of *ramified* character we directly conclude that there is no 'denominator', and that the greatest common divisor of all 'numerators' is just 1. Further, if we choose *any* test function, we are assured that the result will be a *polynomial* in q^{-s}, in any case.

Godement-Jacquet revisited:

Now we recall some of the local results of [Godement-Jacquet 1972] in the case of $GL(2)$, and then recover several of the (non-archimedean) local results at good primes in a substantially non-computational manner. We restrict our attention to $GL(2)$ for simplicity, although the case of $GL(n)$ (and other simple algebras) is entirely analogous. (The general Godement-Jacquet result subsumes Tate's thesis as a special case.)

Let k be a non-archimedean local field of characteristic zero, and let $G = GL(2, k)$. Given an irreducible smooth representation π of G, given $v \in \pi$ and λ in the smooth dual $\check{\pi}$ of π, given a test function ϕ on the space of 2×2 matrices over k, and given $s \in \mathbf{C}$ with $\mathrm{Re}\,(s)$ sufficiently positive, define a local integral

$$Z(v, \lambda, \phi, s) = \int_G |det(g)|^s \, c_{v,\lambda}(g) \, \phi(g) \, dg$$

Here $c_{v,\lambda}$ is the ('matrix'-) coefficient-function

$$c_{v,\lambda}(g) = \langle \pi(g)v, \lambda \rangle$$

and $\langle , \rangle$ is the canonical bilinear map

$$\langle , \rangle : \pi \times \check{\pi} \to \mathbf{C}$$

This integral converges for real part of s sufficiently positive.

We assume for simplicity that the central character of π is unramified, since otherwise this particular form of zeta integral vanishes identically. The more general version with arbitrary central character is treated quite analogously.

For any list of data π, v, λ, ϕ, s, it is true that the local integral is a *rational expression in* q^s with complex coefficients (the latter depending upon π, v, λ, ϕ), where q is the cardinality of the residue field. This is not obvious.

Fixing π, the collection of all local integrals with $v \in \pi$ and $\lambda \in \check{\pi}$ is a *fractional ideal* of $\mathbf{C}[q^s, q^{-s}]$ in its fraction field $\mathbf{C}(q^s)$. This is not obvious. Further, this fractional ideal contains 1, so contains $\mathbf{C}[q^s, q^{-s}]$.

Then the L-factor attached to π is defined to be the *greatest common divisor* in $\mathbf{C}[q^s, q^{-s}]$ of all the local integrals attached to π. Since $\mathbf{C}[q^s, q^{-s}]$ is a principal ideal domain, this fractional ideal does have a single generator. Further, since this fractional ideal contains 1, the ambiguity (by units) of choice of generator is eliminated by taking the unique greatest common divisor of the form

$$\frac{1}{(\text{monic polynomial in } q^{-s})}$$

Now we reconsider these results within the present framework. Let $G \times G$ act on the space $\mathrm{C}_c^\infty(E)$ of test functions on the space E of 2×2 matrices over k, by

$$(g_1, g_2)f(x) = L_{g_1} R_{g_2} f(x) = f(g_1^{-1} x g_2)$$

Then

$$Z(\pi(g_2)v, \check{\pi}(g_1)\lambda, L_{g_1} R_{g_2}\phi, s) = |det\, g_1|^s \, |det\, g_2|^{-s} \, Z(v, \lambda, \phi, s)$$

That is, the map

$$v \otimes \lambda \otimes \phi \to Z(v, \lambda, \phi, s)$$

is a $G \times G$-homomorphism

$$(\pi \otimes \check{\pi}) \otimes \mathrm{C}_c^\infty(E) \to |det\, *\,|^s \, |det\, *\,|^{-s}$$

Or, writing

$$\pi_s = \pi \otimes |det *|^s$$

we can rearrange this all to view the map as

$$(\pi_{-s} \otimes \check{\pi}_s) \otimes C_c^\infty(E) \to \mathbf{C}$$

Or, we can rearrange this to

$$C_c^\infty(E) \to \check{\pi}_s \otimes \pi_s$$

where the admissibility of π entails that π is *reflexive*, that is, that the natural inclusion $\pi \to \check{\check{\pi}}$ is an *isomorphism.*

We filter $C_c^\infty(E)$ by $G \times G$-orbits: there are 3 orbits E_r of $G \times G$ on E, indexed by *rank* r of the matrices. Then $C_c^\infty(E_2)$ naturally imbeds in $C_c^\infty(E)$. Let Δ be the diagonal copy of G in $G \times G$. This is the isotropy group of the point $1 \in E$. Then

$$C_c^\infty(E_2) \overset{G\times G}{\approx} \text{c-Ind}_\Delta^{G\times G}\mathbf{C}$$

Thus, we compute

$$\text{Hom}_{G\times G}(C_c^\infty(E_2), \check{\pi}_s \otimes \pi_s) \approx \text{Hom}_{G\times G}(\pi_{-s} \otimes \check{\pi}_s, \text{Ind}_\Delta^{G\times G}\mathbf{C}) \approx$$

$$\approx \text{Hom}_G(\pi_{-s} \otimes \check{\pi}_s, \mathbf{C}) \qquad \text{(by Frobenius Reciprocity)}$$

$$\approx \text{Hom}_G(\pi_{-s}, \check{\check{\pi}}_s) \approx \text{Hom}_G(\pi_{-s}, \pi_{-s})$$

which is *one-dimensional* for all irreducibles π_{-s}, by Schur's Lemma for smooth representations of countable dimension (for separable groups).

We will see that, for given π, for all but finitely-many $s \in \mathbf{C}$ each intertwining operator just above has at most one extension to $C_c^\infty(E)$. This will provide the requisite *generic multiplicity-one* hypothesis toward application of our results.

Let

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

$$N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$$

and let χ be a one-dimensional smooth representation of P factoring through P/N. Let δ be the modular function of P. Let

$$I_\chi = \text{Ind}_P^G \chi\delta^{1/2}$$

denote that associated *principal series* representation of G. Its smooth dual $\check{\pi}$ is

$$\check{I}_\chi \approx \text{Ind}_P^G \chi^{-1}\delta^{1/2} = I_{\chi^{-1}}$$

Again, all spherical representations occur as quotients and subrepresentations of unramified principal series. Indeed, 'generically' I_χ is irreducible, so I_χ is spherical (e.g., see [Casselman 1980]). However, as usual, the objects which best lend themselves to examination here are *not* the spherical representations but the *unramified principal series.*

Take $\pi = I_\chi$. Let

$$\omega_s(g) = |\det(g)|^s$$

and (as before) write

$$\pi_s = \pi \otimes \omega_s$$

We wish to show that for fixed χ and for all but finitely-many s

$$\mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E_1 \cup E_0), \check{\pi}_s \otimes \pi_{-s}) = 0$$

This would imply that each intertwining operator in

$$\mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E_2), \check{\pi}_s \otimes \pi_{-s})$$

has at most one extension to an element of

$$\mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E), \check{\pi}_s \otimes \pi_{-s})$$

Since we showed that for any π and for any s

$$\dim_{\mathbf{C}} \mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E_2), \check{\pi}_s \otimes \pi_{-s}) = 1$$

we will obtain the desired *generic multiplicity-one* result.

For $p \in P$ write

$$\chi_s(p) = \chi(p)\omega_s(p)$$

By Frobenius Reciprocity,

$$\mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E_1 \cup E_0), \check{\pi}_s \otimes \pi_{-s}) \approx$$

$$\mathrm{Hom}_{P\times P}(\mathrm{C}_c^\infty(E_1 \cup E_0), \chi_s^{-1}\delta^{1/2} \otimes \chi_{-s})$$

Now we filter the 2×2-matrices of less than full rank by $P \times P$-orbits. Fix $G \times G$-representatives

$$\varepsilon_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varepsilon_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Clearly $P\varepsilon_0 P = \{\varepsilon_0\}$ is one $P \times P$-orbit. Let

$$Q = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ 0 & a \end{pmatrix}\}$$

denote the isotropy group Q in $(P \times P)\backslash(G \times G)$ of $(P \times P) \cdot \varepsilon_1$. This group Q is just slightly smaller than the full product $P \times P$. To first break the $G \times G$-orbit E_1 into $P \times P$-orbits amounts to computing

$$(P \times P)\backslash(G \times G)/Q$$

Via a Bruhat decomposition, we find four irredundant representatives

$$1 \times 1, \;\; 1 \times w, \;\; w \times 1, \;\; w \times w$$

Indeed, $Q \supset N \times N$, and already

$$(P \times P)\backslash(G \times G)/N \times N \approx \{1 \times 1, 1 \times w, w \times 1, w \times w\}$$

On the other hand, $Q \subset P \times P$, and we have the same representatives for

$$(P \times P)\backslash(G \times G)/(P \times P)$$

Projected to Q, the isotropy groups of these representatives are, respectively,

$$\begin{array}{rcll} Q &=& \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ 0 & a \end{pmatrix}\} & (\text{for } 1 \times 1) \\ Q_1 &=& \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & 0 \\ 0 & a \end{pmatrix}\} & (\text{for } 1 \times w) \\ Q_2 &=& \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ 0 & a \end{pmatrix}\} & (\text{for } w \times 1) \\ Q_* &=& \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & 0 \\ 0 & a \end{pmatrix}\} & (\text{for } w \times w) \end{array}$$

Let Θ be any one of these isotropy groups. The *graded pieces* associated to this filtering are

$$\mathrm{Hom}_{P\times P}(\text{c-Ind}_\Theta^{P\times P}\mathbf{C}, \chi_s^{-1}\delta^{1/2} \otimes \chi_{-s}\delta^{1/2})$$

By dualizing and by Frobenius Reciprocity, we have

$$\mathrm{Hom}_{P\times P}(\text{c-Ind}_\Theta^{P\times P}\mathbf{C}, \chi_s^{-1}\delta^{1/2} \otimes \chi_{-s}\delta^{1/2}) \approx$$

$$\mathrm{Hom}_{P\times P}(\chi_{-s}\delta^{-1/2} \otimes \chi_s^{-1}\delta^{-11/2}, \mathrm{Ind}_\Theta^{P\times P}\delta_\Theta\delta_{P\times P}^{-1}) \approx$$

$$\mathrm{Hom}_\Theta(\chi_{-s}\delta^{-1/2} \otimes \chi_s^{-1}\delta^{-1/2}, \delta_\Theta\delta_{P\times P}^{-1})$$

This is $\{0\}$ unless

$$\Big(\chi_{-s}\delta^{-1/2} \otimes \chi_s^{-1}\delta^{-1/2}\Big)\,|_\Theta = \big(\delta_\Theta\delta_{P\times P}^{-1}\big)\,|_\Theta$$

in which case it is one-dimensional. The latter condition simplifies a little, to

$$\Big(\chi_{-s}\delta^{1/2} \otimes \chi_s^{-1}\delta^{1/2}\Big)\,|_\Theta = \delta_\Theta$$

We will examine this condition for each of the four orbits inside the less-than-full-rank matrices.

Write

$$\chi(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}) = \chi_1(a)\chi_2(d)$$

Then the respective conditions for the four orbits are as follows. For the first, we have

$$\chi_1(a)\chi_2(d)|ad|^{-s}|a/d|^{1/2}\chi_1^{-1}(a')\chi_2^{-1}(a)|a'a|^s|a'/a|^{1/2} = |a/d||a'/a|$$

This is equivalent to the system of equations

$$\begin{array}{rcll} \chi_1 &=& \chi_2 & (\text{from the } a\text{-component}) \\ \chi_2 &=& |*|^{s-1/2} & (\text{from the } d\text{-component}) \\ \chi_1 &=& |*|^{s-1/2} & (\text{from the } a'\text{-component}) \end{array}$$

Unless $\chi_1 = \chi_2$ this system of equations is not satisfied for *any* value of s. And, even when $\chi_1 = \chi_2$, there is only one s which satisfies it. So *generically s does not satisfy this system.*

For the orbit with isotropy group Q_1:

$$\chi_1(a)\chi_2(d)|ad|^{-s}|ad|^{1/2}\chi_1^{-1}(a)\chi_2^{-1}(d')|ad'^{-1}|^s|a/d'|^{1/2} = |a/d|$$

which is equivalent to

$$\chi_2 = |*|^{s-1/2} \quad \text{from } d \text{ and/or from } d'$$

since the 'scalar equation' arising from a is $0 = 0$, and since the scalar equation from d is identical to that from d'. Unless χ_2 is of the form $|*|^{s'}$, i.e., is *unramified*, this system of equations cannot be satisfied by *any* s. Even when the conditions is met, there is only one s which satisfies the system. So *generically s does not satisfy this system.*

Similar (symmetrical) remarks apply to the orbit with isotropy group Q_2: the conditions simplify to

$$\chi_1 = |*|^{s-1/2}$$

Unless χ_1 is of the form $|*|^{s'}$, i.e., is *unramified*, this system of equations cannot be satisfied by any s. Even when the conditions is met, there is only one s which satisfies the system. So *generically s does not satisfy this system.*

For the orbit with isotropy group Q_*:

$$\chi_1(a)\chi_2(d)|ad|^{-s}|a/d|^{1/2}\chi_1^{-1}(d)\chi_2^{-1}(d')|dd'|^s|d/d'|^{1/2} = 1$$

which is equivalent to

$$\begin{array}{ccll} \chi_1 & = & |*|^{s-1/2} & \text{from } a \\ \chi_1 & = & \chi_2 & \text{from } d \\ \chi_2 & = & |*|^{s-1/2} & \text{from } d' \end{array}$$

Unless $\chi_1 = \chi_2$ this system has no solutions s. Even when this condition is satisfied, there is only one s which satisfies it. So *generically s does not satisfy this system.*

(The rank-zero orbit is even simpler to treat than these four orbits, since there the representation of $G \times G$ which arises is just $\mathbf{C} \otimes \mathbf{C}$.)

We conclude that for $\chi_1 \neq \chi_2$ *generically there is at most a one-dimensional space of intertwining operators*

$$\mathrm{Hom}_{G\times G}(\mathrm{C}_c^\infty(E) \otimes \left((I_\chi)_{-s} \otimes (I_{\chi^{-1}})_s\right), \mathbf{C})$$

(And all such intertwining operators are uniquely determined by their restrictions to the space $\mathrm{C}_c^\infty(E_2)$ of test functions supported on the open orbit E_2.)

The family of representations

$$\mathrm{C}_c^\infty(E) \otimes \left((I_\chi)_{-s} \otimes (I_{\chi^{-1}})_s\right)$$

considered above is a parametrized family of representations, over the ring

$$\mathcal{O} = \mathbf{C}[z, z^{-1}]$$

where the pointwise representations are recovered by $z \to q^{-s}$. Taking the test function ϕ_o to be a suitable scalar multiple of the characteristic function of a sufficiently small neighborhood of $1 \in E$, the local zeta integral is just

$$Z(v, \lambda, \phi_o, s) = 1$$

Thus, we have the *generic multiplicity-one and good-test-vector properties* required to conclude that *all zeta integrals are rational functions of q^{-s}.*

Now we will estimate the possible denominators of these local zeta integrals (at good primes) in terms of *anomalous intertwining operators.* To this end we will re-use the computations just above. Specifically, we saw that for $\chi_1 \neq \chi_2$ the only possible anomalous intertwining operators occur for

$$\chi_1 = |*|^{s-1/2} \qquad \text{or} \qquad \chi_2 = |*|^{s-1/2}$$

Indeed, at any other points the intertwining operators from $\mathrm{C}_c^\infty(E_2)$ have unique extensions to $\mathrm{C}_c^\infty(E)$. Let

$$\chi_1 = |*|^\alpha \qquad \chi_2 = |*|^\beta$$

Then the general result on denominators and anomalous intertwining operators asserts that the only possible linear factors of the denominator in the unramified principal series case are

$$(1 - q^{\alpha+1/2}q^{-s}) \qquad (1 - q^{\beta+1/2}q^{-s})$$

This matches the direct computation in [Godement-Jacquet 1972] of these local integrals.

Note that the Weyl group symmetry in this case would simply interchange α and β. Indeed, the expression we obtain is symmetrical in this regard.

3. Completions of Proofs

3.1Proof of the Rationality Lemma

Proof of Theorem: This argument requires some preparation. Several parts of this discussion have their origins in the work of J. Bernstein mentioned without proof in [Gelbart, Piatetski-Shapiro, Rallis 1987]. This is related to, but also a bit different from, the more algebraic results from [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1977], and [Bernstein 1984]

Let k be a field of characteristic zero, and V a k-vectorspace. A **linear system** (over k, with coefficients in V) is a set Ξ of ordered pairs (v_i, c_i) where $v_i \in V$ and $c_i \in k$ and $i \in I$ for some index set I. A **solution** to the linear system is $\lambda \in V^*$ so that, for all indices i,

$$\lambda(v_i) = c_i$$

where $V^* = \mathrm{Hom}_k(V, k)$ is the k-linear dual space of V. Obviously the set of all solutions is an affine subspace of V^*, so has a sense of **dimension**. A system Ξ is **homogeneous** if all the constants c_i are 0. In that case, the collection of solutions is a vector subspace of V^*. Two systems are **equivalent** if they have the same set of solutions. It is immediate that any system is equivalent to a system with at most one constant c_i non-zero.

Let $\tilde{k}$ be an extension field of k. Let W be a k-vectorspace. There is a natural inclusion $W \to W \otimes_k \tilde{k}$. For any k-subspace W' of W there is naturally associated the $\tilde{k}$-subspace $W' \otimes_k \tilde{k}$ of $W \otimes_k \tilde{k}$. Any $\tilde{k}$-subspace of $W \otimes_k \tilde{k}$ occurring as $W' \otimes_k \tilde{k}$ for a k-subspace W' of W is a k-**rational subspace** of $W \otimes_k \tilde{k}$.

Let $\Xi = \{(v_i, c_i)\}$ be a linear system over k with coefficients in a $k$vectorspace V. For an extension field $\tilde{k}$ of k we can **extend scalars** by looking for solutions λ not only in $\mathrm{Hom}_k(V, k)$ but also in the larger space

$$\mathrm{Hom}_k(V, \tilde{k}) \approx \mathrm{Hom}_{\tilde{k}}(V \otimes_k \tilde{k}, \tilde{k})$$

A solution $\lambda \in \mathrm{Hom}_{\tilde{k}}(V \otimes_k \tilde{k}, \tilde{k})$ is k-**rational** if it is in the image of the natural inclusion

$$\mathrm{Hom}_k(V, k) \subset \mathrm{Hom}_k(V, \tilde{k}) \approx \mathrm{Hom}_{\tilde{k}}(V \otimes_k \tilde{k}, \tilde{k})$$

We may write $\Xi \otimes_k \tilde{k}$ for the linear system obtained by extending scalars.

Proposition: *(Existence, Uniqueness, and Rationality)* Let $\Xi = \{(v_i, c_i) : i \in I\}$ be a k-linear system with coefficients v_i in a k-vectorspace V. Suppose that there is at most one index $i_o \in I$ so that $c_{i_o} \neq 0$.
• If v_{i_o} does not lie in the k-span of $\{v_i : i \neq i_o\}$ then there is at least one solution to the linear system.
• If the coefficient vectors $\{v_i : i \in I\}$ span V, then there is at most one solution.
• If Ξ is *homogeneous* then and if the solution space N in $\mathrm{Hom}_k(V, k)$ is *finite-dimensional* then, for any extension field $\tilde{k}$ of k, the solution space of $\Xi \otimes_k \tilde{k}$ is the k-rational subspace $N \otimes_k \tilde{k}$ of $\mathrm{Hom}_k(V, \tilde{k}) \approx \mathrm{Hom}_k(V, k) \otimes_k \tilde{k}$.
• If for some field extension $\tilde{k}$ of k $\Xi \otimes_k \tilde{k}$ has a *unique* solution λ, then that solution is in fact k-rational, and the original system Ξ therefore has a unique solution.

Proof: If v_{i_o} does not lie in the span of the other coefficient vectors, then (via the Axiom of Choice) there is a linear functional λ in the dual space V^* so that $\lambda(v_i) = 0$ for $i \neq i_o$, but $\lambda(v_o) = 1$. Then $c_{i_o}\lambda$ is a solution of the system Ξ.

Next, for two solutions λ, λ' of Ξ the difference $\mu = \lambda - \lambda'$ is a solution of the homogeneous system $\Xi_o = \{(v_i, 0)\}$. If the v_i span V, then the condition

$$\mu(v_i) = 0 \quad \text{for all } i \in I$$

implies that $\mu = 0$. This is the uniqueness.

Now suppose that Ξ is homogeneous with finite-dimensional solution space N, and let $\tilde{k}$ be an extension field of k. Given a solution Λ of $\Xi \otimes_k \tilde{k}$, we must find finitely-many solutions λ_j of Ξ and scalars $a_j \in \tilde{k}$ so that

$$\Lambda = \sum_k a_j \lambda_j$$

Whatever the $\tilde{k}$-span of the coefficient vectors $v_i \otimes 1$ in $V \otimes_k \tilde{k}$ may be, it is $W \otimes_k \tilde{k}$ where W is the k-span of the v_i in k. Let $e_1, \ldots, e_n$ be a k-basis for a necessarily finite-dimensional complementary subspace to W inside V. Then the $e_j \otimes 1$ form a $\tilde{k}$-basis for a $\tilde{k}$-subspace complementary to $W \otimes_k \tilde{k}$ in $V \otimes_k \tilde{k}$. Let $\lambda_1, \ldots, \lambda_n$ be in the solution space N so that $\lambda_i(e_j)$ is 1 or 0 depending on whether $i = j$ or not. Then it is easy to check that

$$\tilde{\lambda} = \sum_j \lambda(e_j)\lambda_j$$

Finally, let $\tilde{\lambda}$ be the unique solution to $\Xi \otimes_k \tilde{k}$. By the first part of this lemma, the uniqueness of the solution implies that the $v_i \otimes 1$ span $V \otimes_k \tilde{k}$ over $\tilde{k}$, but that $v_{i_o} \otimes 1$ does not lie in the $\tilde{k}$-span of the other vectors. This immediately implies that the vectors v_i span V over k, and that v_{i_o} is not in the k-span of the other vectors. Thus, again by the first part of this lemma, there is a unique solution λ in $\mathrm{Hom}_k(V, k)$. The natural inclusion and isomorphism

$$\mathrm{Hom}_k(V, k) \subset \mathrm{Hom}_k(V, \tilde{k}) \approx \mathrm{Hom}_{\tilde{k}}(V \otimes_k \tilde{k}, \tilde{k})$$

necessarily send λ to $\tilde{\lambda}$, so $\tilde{\lambda}$ is k-rational, as asserted. *Done.*For an extension field $\tilde{k}$ of k so that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain, with $\mathcal{M} \otimes_k \tilde{k}$ the field of fractions of $\mathcal{O} \otimes_k \tilde{k}$, as a notational device let

$$X \otimes_k \tilde{k} = \text{ prime ideal spectrum of } \mathcal{O} \otimes_k \tilde{k}$$

Let V be an $\mathcal{O}$-module. A **parametrized linear system** over $\mathcal{O}$ (or over X) with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$ is a collection Ξ of ordered pairs (μ_i, f_i) with $\mu_i \in M \otimes_{\mathcal{O}} \mathcal{M}$, $f_i \in \mathcal{M}$.

A **generic solution** to such a parametrized system is

$$\lambda \in (V \otimes_{\mathcal{O}} \mathcal{M})^* = \mathrm{Hom}_{\mathcal{M}}(V \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{M})$$

so that for all indices i

$$\lambda(\mu_i) = f_i$$

That is, a generic solution is simply a *solution* (in the previous sense) to the $\mathcal{M}$-linear system on the $\mathcal{M}$-vectorspace $V \otimes_{\mathcal{O}} \mathcal{M}$.

Note that we do not require the coefficient vectors μ_i to be in the module V, but only in $V \otimes_{\mathcal{O}} \mathcal{M}$, and likewise the f_i need not be in $\mathcal{O}$, but only in $\mathcal{M}$. Of course, the same collection of generic solutions would be obtained if each (μ_i, f_i) were replaced by $(g_i\mu_i, g_i f_i)$ for non-zero $g_i \in \mathcal{O}$. Thus, one *could* assume without loss of generality that all the μ_i are in V and the f_i are in $\mathcal{O}$, but it is not necessary to do so.

For $x \in X$, $f \in \mathcal{M}$ is **holomorphic at** x if $f \in \mathcal{O}_x$. An element μ of $V \otimes_{\mathcal{O}} \mathcal{M}$ is **holomorphic at** x if $\mu \in M \otimes_{\mathcal{O}} \mathcal{O}_x$. A parametrized system $\Xi = \{(\mu_i, f_i)\}$ is **holomorphic at** x if for all indices i both f_i and μ_i are holomorphic at x. (From the definitions, *every* parametrized system is holomorphic at the generic point $0 \in X$).

For any $x \in X$ at which the parametrized system Ξ is holomorphic, we have the **associated pointwise system** Ξ_x, obtained by replacing all the elements $\mu_i \in M \otimes_{\mathcal{O}} \mathcal{O}_x$ by their images in $V \otimes_{\mathcal{O}} k_x$, and likewise by replacing the $f_i \in \mathcal{O}_x$ by their images in k_x. Thus, Ξ_x is a k_x-linear system with coefficients in $V \otimes_{\mathcal{O}} k_x$.

A **pointwise solution** λ_x **at** x to the parametrized linear system Ξ is just a solution to the linear system Ξ_x. Thus, it is

$$\lambda \in \mathrm{Hom}_{\mathcal{O}}(V, k_x) \approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

so that for all indices i

$$\lambda(\mu_i) = f_i \mod x\mathcal{O}_x$$

Let $y \subset x$ be two prime ideals in X. Then $S_y \supset S_x$, $\mathcal{O}_y \supset \mathcal{O}_x$, and there is a natural inclusion

$$\mathcal{O}_x/y\mathcal{O}_x \subset \mathcal{O}_y/y\mathcal{O}_y = k_y$$

A solution λ to the pointwise system Ξ_y at y is **holomorphic** at x if

$$\lambda(V \otimes_{\mathcal{O}} k_y) \subset \mathcal{O}_x/y\mathcal{O}_x \subset k_y$$

In that case, the solution λ to Ξ_y gives a solution to the pointwise system Ξ_x by taking the image of λ under the natural map

$$\mathrm{Hom}_{k_y}(V \otimes_{\mathcal{O}} k_y, \mathcal{O}_x/y\mathcal{O}_x) \approx \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}_x/y\mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{O}}(V, k_x)$$
$$\approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

where the last map arises from the quotient map

$$\mathcal{O}_x/y\mathcal{O}_x \to \mathcal{O}_x/x\mathcal{O}_x = k_x$$

Lemma: Suppose that $\mathcal{O}$ is Noetherian. Given $f \neq 0$ in $\mathcal{M}$, there is a finite collection $\eta_1, \ldots, \eta_n$ of irreducible hyperplanes so that $1/f \in \mathcal{O}_x$ for x not in the union $\eta_1 \cup \ldots \cup \eta_n$. In particular, the hyperplanes are those attached to the isolated primes in a primary decomposition of the ideal $f\mathcal{O}$.

Proof: For a prime ideal $x \in X$, if $f \notin x$ then $1/f \in \mathcal{O}_x$. Thus, we must show that if $f \in x$ then there is a height-one prime y so that $f \in y \subset x$.

Let $J = \bigcap_i Q_i$ be a primary decomposition of an ideal $J \subset x$, where Q_i is primary with associated prime x_i. Taking radicals,

$$x = \mathrm{rad}\, x \supset \bigcap_i \mathrm{rad}\, Q_i = \bigcap_i x_i$$

so, for some index i, $x \supset x_i$. Thus, x must contain some one of the minimal (i.e., isolated) primes among the primes associated to J.

Krull's Principal Ideal Theorem asserts that every prime *minimal* among those occurring in a primary decomposition of $f\mathcal{O}_x$ is height-one. This gives the result. *Done.*

Lemma: Let Y be a non-meager subset of X, the prime spectrum of $\mathcal{O}$, where $\mathcal{O}$ is a Noetherian integral domain. Then

$$\bigcap_{x \in Y} x\mathcal{O}_x = \{0\}$$

Proof: Let $r \neq 0$ be in the indicated intersection. By the previous lemma, there would be finitely-many hypersurfaces $\eta_1, \ldots, \eta_n$ so that for $x \in X$ *off* the union of these hypersurfaces we would have both $r \in \mathcal{O}_x$ and $1/r \in \mathcal{O}_x$. Removing from Y the intersections of Y with these hypersurfaces would still leave a non-meager set Y_r. In particular, Y_r would be non-empty, and, by construction, for all $x \in Y_r$ both r and $1/r$ would lie in $\mathcal{O}_x$. But this would contradict the hypothesis $r \in x\mathcal{O}_x$. Thus, it must be that the indicated intersection is $\{0\}$, as claimed. *Done.*

Lemma: Let $\Xi = \{(\mu_i, f_i) : i \in I\}$ be a $\mathcal{O}$-parametrized system with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$. Then

- There is a union $\bigcup_{i \in I} \eta_i$ of irreducible hyperplanes η_i off which the system Ξ is holomorphic.
- Let $\{m_j : j \in J\}$ be a generating set for V over $\mathcal{O}$. For a generic solution λ of Ξ, there is a union $\bigcup_{j \in J} \eta_i$ of irreducible hyperplanes η_i off which λ is holomorphic.

Proof: Let $\{g_i : i \in I\}$ be a collection of non-zero elements of $\mathcal{O}$ so that, for for all i, $g_i\mu_i \in M$ and $g_i f_i \in \mathcal{O}$. Then Ξ is holomorphic off the union of the hyperplanes attached to the isolated primes occurring in the primary decompositions of the ideals $g_i\mathcal{O}$.

Similarly, let $\{g_j : i \in J\}$ be a collection of non-zero elements of $\mathcal{O}$ so that, for for all $j \in J$, $g_i\lambda(m_j) \in \mathcal{O}$. Then λ is holomorphic off the union of the hyperplanes attached to the isolated primes occurring in the primary decompositions of the ideals $g_j\mathcal{O}$. *Done.*

Theorem: Let k be a field, and $\mathcal{O}$ a commutative Noetheria k-algebra. Let $\tilde{k}$ be an extension field of k so that $\mathcal{O} \otimes_k \tilde{k}$ is still an integral domain, with $\mathcal{M} \otimes_k \tilde{k}$ the field of fractions of $\mathcal{O} \otimes_k \tilde{k}$. Let $\Xi = \{(\mu_i, f_i) : i \in I\}$ be an $\mathcal{O}$-parametrized linear system with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$ where V is a countably-generated $\mathcal{O}$-module, where also the index set I is countable. Let $\Xi \otimes_k \tilde{k}$ be the system obtained by extending scalars from k to $\tilde{k}$, and let $X \otimes_k \tilde{k}$ be the prime spectrum of $\mathcal{O} \otimes_k \tilde{k}$.

• If there is a unique generic solution λ to $\Xi \otimes_k \tilde{k}$, then for $x \in X \otimes_k \tilde{k}$ off a meager set there is a unique solution λ_x to the pointwise linear system $(\Xi \otimes_k \tilde{k})_x$ at x, and this pointwise solution is obtained from λ via the natural map

$$\mathrm{Hom}_{\mathcal{O}\otimes_k \tilde{k}}(V \otimes_k \tilde{k}, (\mathcal{O} \otimes_k \tilde{k})_x) \to \mathrm{Hom}_{\mathcal{O}\otimes_k \tilde{k}}(V \otimes_k \tilde{k}, \tilde{k}_x)$$

$$\approx \mathrm{Hom}_{\tilde{k}_x}(V \otimes_{(\mathcal{O}\otimes_k \tilde{k})} \tilde{k}_x, \tilde{k}_x)$$

• Suppose that Y is a *non-meager* subset of $X \otimes_k \tilde{k}$ so that for $x \in Y$ the system $\Xi \otimes_k \tilde{k}$ is holomorphic at x, and so that for $x \in Y$ the pointwise system $(\Xi \otimes_k \tilde{k})_x$ at x has a unique solution λ_x. Then there is a unique *generic solution* to $\Xi \otimes_k \tilde{k}$, and this solution is *rational* over k.

Proof: First, we reduce to the case that at most one of the f_i is not $0 \in \mathcal{M}$. If all f_i are already 0, we are done. So suppose that some f_{i_o} is non-zero. For another index i, replace the condition

$$\lambda(v_i) = f_i$$

by the condition

$$\lambda(v_i - \frac{f_i}{f_{i_o}} v_{i_o}) = 0$$

The collection of generic solutions is unchanged by such an adjustment, and off a meager subset of X this change gives pointwise systems equivalent to the original pointwise system Ξ_x. Thus, overlooking a meager subset of X, we can assume without loss of generality that for at most one index i_o is f_{i_o} not 0.

Also, we ignore the countably-many hypersurfaces on which the system Ξ fails to be holomorphic.

If there is a unique generic solution λ, then by the proposition the vectors μ_i span $V \otimes_{\mathcal{O}} \mathcal{M}$, and if f_{i_o} is not 0 then μ_{i_o} does not lie in the $\mathcal{M}$-span of the other vectors.

Let $v_1, v_2, \ldots$ be a countable collection of generators for the $\mathcal{O}$-module V. The images of these v_j span all the vectorspaces $V \otimes_{\mathcal{O}} k_x$ for $x \in X$. Then $v_1 \otimes 1, v_2 \otimes 1, \ldots$ is an $\mathcal{M}$-basis for $V \otimes_{\mathcal{O}} \mathcal{M}$. Express each $m_j \otimes 1$ as an $\mathcal{M}$-linear combination of the μ_i as

$$v_j \otimes 1 = \sum_i g_{ji}\, \mu_i$$

with $g_{ji} \in \mathcal{M}$. Off the meager set of points $x \in X$ where some one of the g_{ji} fails to be holomorphic, tensoring with $k_x = \mathcal{O}_x / x\mathcal{O}_x$ expresses the image of v_j as a k_x-linear combination of the images of μ_i in $V \otimes_{\mathcal{O}} k_x$. (We have already excluded the meager set on which any one of the μ_i fails to be holomorphic). That is, we conclude that off a meager set the pointwise system Ξ_x has at most one solution.

Then, given a generic solution λ, off the meager set where λ fails to be holomorphic, the image of λ under the natural map

$$\mathrm{Hom}_{\mathcal{M}}(V \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{O}_x) \approx \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}_x)$$

$$\to \mathrm{Hom}_{\mathcal{O}}(V, k_x) \approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

is a solution of the pointwise system Ξ_x. Thus, *uniqueness of the generic solution implies uniqueness of solution to pointwise systems Ξ_x for x off a meager set.*

On the other hand, suppose that for x in a non-meager subset Y (on which Ξ is holomorphic) the pointwise system Ξ_x has a unique solution. To prove that a generic solution exists, we must show that μ_{i_o} does not lie in the $\mathcal{M}$-span of the other μ_i. Indeed, if

$$\mu_{i_o} = \sum_{i \neq i_o} g_i \mu_i$$

with $g_i \in \mathcal{M}$, then, off the meager set where some one of the g_i or μ_i fails to be holomorphic, the image of μ_{i_o} in $V \otimes \mathcal{O}k_x$ would be a k_x-linear combination of the images of the other elements μ_i with $i \neq i_o$. The proposition above assures that this cannot happen. Thus, there is at least one generic solution.

Suppose that there were two distinct generic solutions, and call their difference δ. The pointwise uniqueness hypothesis assures that, for x in a non-meager set Y (on which we may assume δ holomorphic), the image of δ in $\mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$ is 0. Thus, for all $x \in Y$, for all of the countably-many $\mathcal{O}$-generators m_j for V,

$$\delta(m_j \otimes 1) \subset x\mathcal{O}_x$$

But we saw above that the intersection of sets $x\mathcal{O}_x$ for x ranging over any non-meager set is $\{0\}$. Thus, $\delta(m_j \otimes 1) = 0$, proving uniqueness. *Done.*Now we can return to discussion of parametrized families of smooth representations, in the setting of linear systems and parametrized linear systems.

Let $\{t_i\}$ be a countable set of $\mathcal{O}$-generators for V. Let $\{g_j\}$ be a countable dense subset of G. The condition that an $\mathcal{O}$-linear map

$$\lambda : M \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{M}$$

be an $\mathcal{O}$-parametrized family of intertwining operators is that

$$\lambda(\pi(g_j)t_i) = \lambda(t_i)$$

for all indices i, j, since the isotropy group of each t_i is *open* (by smoothness). This is

$$\lambda(\pi(g_j)t_i - t_i) = 0$$

The countable collection of such conditions is a homogeneous parametrized linear system Ξ with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$.

First we prove the Rationality Lemma without concern for extending scalars. That is, we suppose that $\tilde{k} = k$. Then the hypothesis of the theorem is that the associated pointwise system Ξ_x has a solution space for dimension less than or equal 1 for x in the non-meager subset Y of X.

Consider the single further condition

$$\lambda(\mu_o) = 1$$

and let Ξ^+ be the system obtained from the homogeneous system Ξ by adjoining this condition. On a non-meager set Ξ^+ has the *unique* solution μ_o. Thus, by the previous theorem, there is a unique generic solution φ to Ξ^+. (By earlier results, necessarily $\varphi_x = \psi_{(x)}$ on a non-meager subset). And, by the previous theorem, the uniqueness of the generic solution to Ξ^+ implies that off a meager subset of X the pointwise solution is unique.

Finally, consider the issue of extension of scalars. The previous discussion applies as well to the representation $\pi \otimes_k \tilde{k}$ of G on $V \otimes_k \tilde{k}$, yielding an intertwining

$$\tilde{\varphi} : \pi \otimes_k \tilde{k} \to \mathcal{O} \otimes_k \tilde{k}$$

Further, by the theorem above, this intertwining is k-rational. Thus, in particular, rather than merely being able to assert that $\varphi(\mu_o) \in \mathcal{M} \otimes_k \tilde{k}$, we know that actually $\varphi(\mu_o) \in \mathcal{M}$. *Done.*

3.2Orbit filtrations on test functions

The material of this section has its origin in [Bruhat 1961], although some technical adjustments are necessary for present use. See also treatment of related matters in [Bernstein-Zelevinsky 1976], [Bernstein-Zelevinsky 1976], [Bernstein 1984].

Let G be a locally compact, Hausdorff topological group. For a *closed* subgroup H of G, the quotient G/H is Hausdorff because H is closed. Let Ω be a Hausdorff topological space with a continuous transitive action of G upon it. Suppose that G has a *countable basis.* Let x_0 be any fixed element of Ω, and let

$$G_{x_0} := \{g \in G : gx_0 = x_0\}$$

be the *isotropy group* of x_0 in G. Recall the basic
Lemma: The natural map

$$G/G_{x_0} \to \Omega$$

by $gG_{x_0} \to gx_0$ is a homeomorphism.

Proof: The map $gG_{x_0} \to gx_0$ is a continuous bijection, by assumption. We need to show that it is *open.* Let U be an open subset of G, and take a compact neighborhood V of $1 \in G$ so that $V^{-1} = V$ and $gV^2 \subset U$ for fixed $g \in U$.

Since G has a countable basis, there is a countable list $g_1, g_2, \ldots$ of elements of G so that $G = \bigcup_i g_iV$. Let $W_n = g_nVx_o$. By the transitivity, $\Omega = \bigcup_i W_i$. Now W_n is compact, being a continuous image of a compact set, so is closed since it is in the Hausdorff space Ω.

Since Ω is locally compact and Hausdorff, by Urysohn's Lemma it is *regular.* In particular, if no W_n contained an open set, then there would be a sequence of non-empty open sets U_n with compact closure so that

$$U_{n-1} - W_{n-1} \supset \bar{U}_n$$

and

$$\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \ldots$$

Then $\bigcap \bar{U}_i \neq \emptyset$, yet this intersection fails to meet any W_n, contradiction.

Therefore, some $W_m = g_mVx_0$ contains an open set S of Ω. For $h \in V$ so that $hx_0 \in S$,

$$gx_0 = gh^{-1}hx_0 \in gh^{-1}S \subset gh^{-1}Vx_0 \subset gV^{-1} \cdot Vx_0 \subset Ux_0$$

Therefore, gx_0 is an interior point of Ux_0, for all $g \in U$. *Done.*Let Ω be any locally compact Hausdorff space or a complete metric space. Recall that a locally compact Hausdorff space with a countable basis *is* metrizable (and complete). Let G be a locally compact Hausdorff topological group, with a countable basis, so that G has a countable dense subset. Suppose that G acts continuously on Ω. From the Baire category theorem we can deduce several properties of the orbits of G on Ω, as follows.
Lemma: Suppose that the collection $G\backslash\Omega$ of G-orbits on Ω is *countable.* Then there is at least one open G-orbit in Ω, and every G-orbit is open in its own closure in Ω.

Proof: By the Baire category theorem, at least one orbit G_ox (with $x_o \in \Omega$) has non-empty interior. Let U be an open subset of Gx_o, and $g_ox_o \in U$ with $g_o \in G$. Then for arbitrary $g \in G$

$$gx_o = (gg_o^{-1})(g_ox_o) \in (gg_o^{-1})U \subset Gx_o$$

Thus, every point gx_o of Gx_o is an interior point, so Gx_o is open.

Now fix $x_1 \in \Omega$. The closure Ω_1 of Gx_1 in Ω is still locally compact, Hausdorff, and countably-based, so is complete metrizable. Also, it is still acted upon continuously by G. By the previous argument, there is at least one open orbit Gx_2 inside Ω_1. Every point of Gx_2, including interior points, is in the closure of Gx_1, so necessarily $Gx_2 = Gx_1$, as claimed. *Done.*Let $\mathcal{O}$ be a commutative ring with identity. As earlier, for a totally disconnected space Ω let $C_c^\infty(\Omega)$ denote the collection of compactly-supported locally constant $\mathcal{O}$-valued **test functions** on Ω. The following two lemmas are elementary:

Lemma: Let $\Omega_1, \ldots, \Omega_n, \ldots$ be disjoint open sets in a totally disconnected space Ω. Then the direct sum of restriction maps

$$f \to \oplus (f|_{\Omega_i})$$

$$C_c^\infty(\bigcup_i \Omega_i) = \bigoplus_i C_c^\infty(\Omega_i)$$

is an isomorphism. *Done.*

Lemma: For an open subset Y of a totally disconnected space Ω we have an exact sequence

$$0 \to C_c^\infty(Y) \to C_c^\infty(\Omega) \to C_c^\infty(\Omega - Y) \to 0$$

where the map $C_c^\infty(\Omega) \to C_c^\infty(\Omega - Y)$ is by restriction. *Done.*

Analogues of the above lemmas for somewhat more general spaces of test functions are necessary. As earlier, the space $C_c^\infty(H\backslash\Omega, \sigma)$ is defined to be those σ-valued functions f on Ω which are compactly-supported left modulo H and so that

$$f(hx) = \sigma(h)(f(x))$$

Here σ is a smooth representation of H on an $\mathcal{O}$-module (the latter also denoted σ).

The obvious analogues are not true in much greater generality than when Ω is constructed as follows, as in the theorem giving the orbit criterion for locally strong meromorphy, which will suffice for our purposes. Let $\tilde{H}$ be a locally compact totally disconnected topological group of which H is a closed subgroup. Let Y be a closed subset of $\tilde{H}$ stable under left multiplication by H. Let Z be a locally compact totally disconnected space on which H acts trivially. Then take

$$\Omega = Y \times Z$$

Proposition: Let U be an open subset of Ω, with Ω as just above. Let σ be an $\mathcal{O}$-parametrized smooth representation of H on an $\mathcal{O}$-module V. Then we have an exact sequence

$$0 \to C_c^\infty(H\backslash U, \sigma) \to C_c^\infty(H\backslash\Omega, \sigma) \to C_c^\infty(H\backslash(\Omega - U), \sigma) \to 0$$

where the first map is 'extend-by-zero', and the second is restriction of functions from Ω to $\Omega - U$.

Proof: If extending elements of $C_c^\infty(H\backslash U, \sigma)$ by zero off Y really were to yield elements of $C_c^\infty(H\backslash\Omega, \sigma)$, then the injectivity would be clear. To justify extending by zero, we must show that a subset C of U which is compact left modulo H *in* U is also compact left modulo H *in* Ω. Here that we use the fact that all this is happening inside a larger topological group $\tilde{H}$. In particular, since H is a *closed* subgroup of $\tilde{H}$, the quotient $H\backslash\tilde{H}$ is *Hausdorff.* Then $H\backslash C$ is assumed compact in $H\backslash U$, and we are to show that it is compact in $H\backslash\Omega$. But these *quotient* topologies are the same as the *subspace* topologies from $H\backslash\tilde{H}$, from the first lemma above. Thus, compactness of $H\backslash HC$ in $H\backslash U$ implies compactness of $H\backslash HC$ in $H\backslash\tilde{H}$. Then since $H\backslash\Omega$ is closed and $H\backslash\tilde{H}$ is Hausdorff, $H\backslash HZ$ is surely compact in $H\backslash\Omega$. Thus, we have proven that the extension-by-zero map makes sense, and it is certainly an injection.

Restriction to $\Omega - U$ of a function with compact support left mod H inside U certainly gives 0; and, if the restriction to $\Omega - U$ of a function is 0, then its support must be contained in U. Again, compactness of support is assessed in the quotient $H\backslash\tilde{H}$. This proves exactness at the middle joint.

What remains to be checked is the surjectivity of the restriction map from $C_c^\infty(H\backslash\Omega, \sigma)$ to $C_c^\infty(H\backslash(\Omega-U), \sigma)$ for an open H-stable subset U of Ω. It is elementary that the restriction map

$$C_c^\infty(\Omega, \mathcal{O}) \to C_c^\infty(\Omega - U, \mathcal{O})$$

is *surjective.* Thus, by the right exactness of tensor products, tensoring over $\mathcal{O}$ with the representation space V of σ still gives a surjective restriction map

$$C_c^\infty(\Omega, \mathcal{O}) \otimes_{\mathcal{O}} V \to C_c^\infty(\Omega - U, \mathcal{O}) \otimes_{\mathcal{O}} V$$

The averaging maps

$$\alpha_\Omega : \mathrm{C}_c^\infty(\Omega, \mathcal{O}) \otimes V \to \mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$$
$$\alpha_{\Omega-U} : \mathrm{C}_c^\infty(\Omega - U, \mathcal{O}) \otimes V \to \mathrm{C}_c^\infty(H\backslash(\Omega - U), \sigma)$$

given (in both cases) by the formula

$$f \otimes v \ \to \ (x \to \int_H f(hx)\, \sigma(h^{-1}) v \, dh$$

are *surjective.*

Thus, all arrows except possibly the lower horizontal one in the commuting square

$$\begin{array}{ccc} \mathrm{C}_c^\infty(\Omega, \mathcal{O}) \otimes V & \to & \mathrm{C}_c^\infty(\Omega - U, \mathcal{O}) \otimes V \\ \alpha_\Omega \downarrow & & \downarrow \alpha_{\Omega-U} \\ \mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes V & \to & \mathrm{C}_c^\infty(H\backslash(\Omega - U), \sigma) \otimes V \end{array}$$

are *surjections.* But then it is clear that the lower horizontal one is a surjection, as well. *Done.*Let H and Ω be as just above, and let G be another totally disconnected locally compact topological group acting (continuously) on the right on Ω. Suppose that there are *finitely*-many $G \times H$-orbits on Ω. From above, each orbit is open in its closure.

Write $z \geq w$ for $G \times H$-orbits z, w on Ω if w is in the closure $\bar{z}$ of z. Since z, w are both $G \times H$-orbits, $\bar{z}$ and $\bar{z} \cap w$ are unions of orbits, and if w meets $\bar{z}$ (non-trivially) we can conclude that

$$w \cap \bar{z} = w$$

Thus, if w merely *meets* the closure $\bar{z}$ of z, then $z \geq w$. Since the closure of the closure of a set is the closure, the relation $\leq$ is a partial ordering.

For an orbit z, let

$$[z] = \bigcup_{w \leq z} w$$

$$(z) = \bigcup_{w < z} w = [z] - z$$

Let M be a module. Let T be a poset with order $<$, and suppose that for each $t \in T$ we have a submodule $M_t \subset M$, and that $s < t$ implies that $M_s \subset M_t$. Then the collection

$$\{M_t : t \in T\}$$

is a **filtration** of M indexed by T. For $t \in T$ the **associated graded piece** of M is the quotient object

$$\sum_{s \leq t} M_s / \sum_{s < t} M_s$$

Thus, in the situation under consideration at present, we would consider $M = \mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$, take T to be the collection of $G \times H$ orbits on Ω, and

$$M_z = \mathrm{C}_c^\infty(H\backslash[z], \sigma)$$

With the hypothesis that there are *finitely-many orbits*, the results above concerning test functions yield:
Corollary: Suppose that there are finitely-many $G \times H$-orbits on Ω. Then the filtration of $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ indexed by the partially ordered set of $G \times H$-orbits has associated graded pieces

$$\frac{\mathrm{C}_c^\infty(H\backslash[z], \sigma)}{\mathrm{C}_c^\infty(H\backslash(z), \sigma)} \approx \mathrm{C}_c^\infty(H\backslash z, \sigma)$$

That is, each orbit's own space of equivariant test functions appears among the graded pieces of the $G \times H$-orbit filtration on Ω, and these are all of the graded pieces. *Done.*

3.3Exactness of some functors

Now we begin preparations for proof of the orbit criterion for locally strong meromorphy. Here we prove the exactness of some relatively elementary functors on smooth representation spaces. Some of this occurs in the unpublished notes [Casselman 1976] in a different form.

Let $\mathcal{O}$ be a commutative ring with identity, and $\mathcal{A}$ an associative $\mathcal{O}$-algebra. Say that $\mathcal{A}$ is an *idempotented algebra* if for any finite collection $\eta_1, \ldots, \eta_n$ of elements of $\mathcal{A}$ there is an idempotent $e \in \mathcal{A}$ so that $e\eta_i = \eta_i = \eta_i e$ for all i.

A module M over an idempotented algebra $\mathcal{A}$ is *smooth* if, for every finite list $m_1, \ldots, m_n$ of elements of M there is an idempotent e in $\mathcal{A}$ so that $em_i = m_i$ for all i. For example, $\mathcal{A}$ is smooth as a (left) module over itself.

Given *any* module M over an idempotented algebra $\mathcal{A}$, we form the submodule M^∞ of *smooth vectors* in M by taking the set of $m \in M$ so that there exists some idempotent $e \in \mathcal{A}$ so that $em = m$. Then M^∞ is the unique maximal smooth submodule of M.

An $\mathcal{O}$-algebra $\mathcal{A}$ is *augmented* if it is equipped with an $\mathcal{O}$-algebra homomorphism $\varepsilon : \mathcal{A} \to \mathcal{O}$ which maps every *sufficiently small* idempotent to 1. For fixed augmentation ε, the *trivial* $\mathcal{A}$-module denoted by ε is the $\mathcal{O}$-module $\mathcal{O}$ itself on which $\eta \in \mathcal{A}$ acts by

$$\eta r = \varepsilon(\eta) \cdot r$$

for all $r \in \mathcal{O}$.

In the case of a Hecke algebra $\mathcal{G}$ of a totally disconnected group G, the *standard augmentation* is simply the *integral* of a locally-constant, compactly-supported $\mathcal{O}$-valued function: thus,

$$\varepsilon(\mathrm{ch}_{KgK}) = \mathrm{meas}(KgK)$$

for every $g \in G$, where ch_{KgK} is the characteristic function of KgK and *meas* is a $\mathbf{Q}$-valued right Haar measure. (These 'integrals' are in fact finite sums, so there is no analysis, and this makes sense over an arbitrary field of characteristic zero, at least).

Proposition: For any fixed idempotent e in an idempotented $\mathcal{O}$-algebra $\mathcal{A}$ the functor $V \to eV$ is an *exact functor* from smooth $\mathcal{A}$-modules to $e\mathcal{A}e$-modules.

Proof: Suppose that

$$A \overset{\phi}{\to} B \overset{\psi}{\to} C$$

is an exact sequence of $\mathcal{A}$-modules. Then for $x \in A$

$$\psi(\phi(ex)) = e(\psi(\phi(x)) = e \cdot 0 = 0$$

so the kernel of ψ on eB is contained in the image $\phi(eA) = e\phi(A)$. On the other hand, if $\psi(ey) = 0$ for $y \in B$, then invoking the exactness of the original sequence take $x \in A$ so that $\phi(x) = ey$. But then also

$$\phi(ex) = e\phi(x) = e(ey) = ey$$

since e is idempotent. This verifies the asserted exactness. *Done.*We can abstract the notion of *Jacquet module*. Say that an augmented idempotented algebra $\mathcal{N}$ has **large idempotents** if, given a finite collection $f_1, \ldots, f_n \in \mathcal{N}$, there is an idempotent e so that

$$ef_i = f_i e = \varepsilon(f) \cdot e$$

where ε is the augmentation. (This abstracts the situation in which $\mathcal{N}$ is the Hecke algebra of a unipotent p-adic group).

The ε**-co-isotype** V_ε is the largest quotient of V so that any $\mathcal{N}$-homomorphism

$$\varphi : V \to \varepsilon$$

factors through the quotient map $V \to V_\varepsilon$. This is the natural dual notion to the more common usage of ε*isotype*, meaning the smallest submodule V^ε so that every homomorphism $\varepsilon \to V$ factors through the inclusion $V^\varepsilon \subset V$.

We may define the **Jacquet functor** $\mathcal{J}$ on $\mathcal{N}$-modules to be that which associates to a smooth $\mathcal{N}$-module V the ε-co-isotype V_ε. Again we have an exactness result which does not depend upon the nature of the 'scalars' $\mathcal{O}$:

Proposition: For an idempotented $\mathcal{O}$-algebra $\mathcal{N}$ with 'large idempotents', and with augmentation ε, the functor $\mathcal{J}$ which takes the ε-co-isotype V_ε of a smooth $\mathcal{N}$-module V is *exact*.

Proof: First, it is elementary to check that V_ε is the quotient of V by the $\mathcal{O}$-submodule generated by all expressions of the form

$$\eta \cdot v - \varepsilon(\eta)v$$

with $\eta \in \mathcal{N}$ and $v \in V$.

Second, we claim that an element $v \in V$ is in the kernel of the quotient map $q : V \to V_\varepsilon$ of the ε co-isotype if and only if there is an idempotent e in $\mathcal{N}$ so that $ev = 0$.

If $v = \sum_i (\eta_i v - \varepsilon(\eta_i)v)$ is in the kernel of q, then take an idempotent e so that $e\eta_i = \varepsilon(\eta_i)e$ for all i. Then

$$e(\eta_i v - \varepsilon(\eta_i)v) = 0$$

On the other hand if $ev = 0$, then

$$v = 1 \cdot v - 0 \cdot v = \varepsilon(e)v - ev$$

which lies in the kernel of q. This proves the claim.

Suppose that

$$A \overset{f}{\to} B \overset{g}{\to} C$$

is an exact sequence of $\mathcal{N}$-modules.

Suppose that $(\mathcal{J}g)(q_B b) = 0$ where $q_B b$ is the image in $\mathcal{J}B$ of $b \in B$ via the quotient map $q_B : B \to \mathcal{J}B$. Then by definition of $\mathcal{J}$ we find that $q_C(gb) = 0$, where $q_C : C \to \mathcal{J}C$ is the quotient map. Thus, by the first observation of this proof, for a sufficiently large idempotent e of $\mathcal{N}$ we have $e(gb) = 0$. Since g is $\mathcal{N}$-linear, this implies that $g(eb) = 0$. By the exactness, there is $a \in A$ so that $f(a) = eb$. Then

$$f(ea) = ef(a) = e(eb) = eb$$

That is, the kernel of $\mathcal{J}g$ is contained in the image of $\mathcal{J}f$.

And

$$(\mathcal{J}g)(\mathcal{J}f)(q_A a) = q_C((g \circ f)(a)) = q_C(0) = 0$$

simply follows from our (correct) presumption that $\mathcal{J}$ is a functor. *Done.*

3.4Frobenius Reciprocity for parametrized families

For a locally compact Hausdorff totally disconnected group G, let $\mathcal{G} = \mathcal{G}_{\mathbf{Q}}$ be the Hecke algebra of locally-constant compactly-supported $\mathbf{Q}$-valued functions on the totally disconnected group G. For any $\mathbf{Q}$-algebra $\mathcal{A}$, let

$$\mathcal{G}_{\mathcal{A}} = \mathcal{G} \otimes_{\mathbf{Q}} \mathcal{A}$$

which can be naturally identified with the collection of $\mathcal{A}$-valued compactly-supported locally-constant functions on G. The product is the convolution

$$(\eta * \zeta)(g) = \int_G \eta(gh^{-1})\,\zeta(h)\,dh$$

where dh denotes a $\mathbf{Q}$-valued right Haar measure on G. The latter exists since G is totally disconnected. Note that such an integral with compact support and locally-constant integrand uses only *finite* additivity, and thus we can make sense of such integrals with integrands having values in any k-vectorspace whatsoever.

For an $\mathcal{O}$-parametrized family π of smooth representations of G on an $\mathcal{O}$-module V, there is a $\mathcal{G}_\mathcal{O}$-module structure on V by

$$\eta \cdot m = \int_G \eta(g)\,\pi(g)m\,dg$$

Thus, $\mathcal{O}$-linear representations of G become $\mathcal{G}\otimes\mathcal{O}$-modules.

For a compact-open subgroup K of G, let e_K be the element of $\mathcal{G}_\mathbf{Q} \subset \mathcal{G}_\mathcal{O}$ given by

$$e_K = \frac{\text{characteristic function of } K}{\text{measure of } K}$$

A $\mathcal{G}_\mathcal{O}$-module V is **smooth** if for every finite collection $v_1, \ldots, v_n$ of elements of V there is a compact-open subgroup K of G so that $e_K m_i = m_i$ for all indices i. (The functor of the previous paragraph attaches *smooth* $\mathcal{G}_\mathcal{O}$-modules to $\mathcal{O}$-parametrized families of smooth representations of G). Then from any smooth $\mathcal{G}_\mathcal{O}$-module V one can recover a smooth representation π of G by

$$\pi(g)m = \chi_{gK} \cdot m$$

where χ_{gK} is the characteristic function of the subset gK of G, and where K is a small-enough compact-open subgroup of G so that $e_K m = m$. This functor is the inverse of the functor of the previous paragraph.

The $\mathcal{G}$-module $\mathcal{G}$ itself has at least two natural (left) $\mathcal{G}$-module structures on it. The first is $\mathcal{G}$ itself: the multiplication is

$$\eta \cdot f = \eta * f \quad \text{(convolution)}$$

which directly reflects the ring structure. On the other hand, let $M_{\rm rt}$ be the left $\mathcal{G}$-module consisting of $\mathcal{G}$ itself, but with the module structure

$$\eta \cdot m = m * \check{\eta}$$

for $\eta \in \mathcal{G}$ and $m \in M_{\rm rt}$, where

$$\check{\eta}(g) = \frac{\eta(g^{-1})}{\delta_G(g)}$$

with δ_G being the modular function of G, normalized by

$$\delta_G(g_o) = \frac{d(g_o g)}{dg}$$

Direct computation verifies the isomorphism of $\mathcal{G}$ modules

$$\Psi : M_{\rm rt} \to \mathcal{G}$$

by

$$\Psi(m) = \check{m}$$

Let $\mathcal{H}$ be the Hecke algebra of the closed subgroup H of G. Give $\mathcal{G}$ the natural *right* $\mathcal{H}$-module structure as follows:

$$(f \cdot \zeta)(g_o) = \int_H f(g_o h^{-1})\,\zeta(h)\,dh$$

where $\zeta \in \mathcal{H}$ and dh refers to a **Q**-valued right Haar measure on H. The associativity follows from the associativity of convolution on H.

Proposition: Let σ be a smooth $\mathcal{O}$-parametrized family of representations of a closed subgroup H of a totally disconnected group G. Let $\mathcal{G}$ be the Hecke algebra of G, and $\mathcal{H}$ the Hecke algebra of H. Then the map

$$\Phi(f \otimes v) = \alpha_\sigma(\check{f} \otimes v)$$

defines a $\mathcal{G}_\mathcal{O}$-isomorphism

$$\Phi : \mathcal{G} \otimes_\mathcal{H} (\sigma \otimes \frac{\delta_G}{\delta_H}) \to \text{c-Ind}_H^G \sigma$$

where $\mathcal{G}$ has the $\mathcal{G} \times \mathcal{H}$-bimodule structure as above.

Proof: The only serious issue is that this is well-defined: we must check that

$$\Phi(f\zeta \otimes v) = \Phi(f \otimes \zeta v)$$

Computing:

$$\Phi(f\zeta \otimes v)(g) = \alpha_\sigma((f\zeta)\check{} \otimes v)(g)$$

$$= \int_H \sigma(h^{-1}) \left[\int_H \frac{f(hg)^{-1}h_2^{-1})}{\delta_G(hg)}\, \zeta(h_2)\, dh_2\; v\right] dh$$

$$= \int_H \int_H \sigma(h^{-1})\, \frac{f(g^{-1}h^{-1})}{\delta_G(hg)}\, \frac{\sigma(h_2)\delta_H(h_2^{-1})}{\delta_G(h_2^{-1})} \zeta(h_2)$$

by replacing h by $h_2^{-1}h$. Then this is

$$\int_H \sigma(h^{-1})\, \check{f}(hg) \left(\sigma \frac{\delta_G}{\delta_H}\right) (\zeta) v\; dh\, dh_2 = \alpha_\sigma(\check{f} \otimes \zeta v)(g) = \Phi(f \otimes \zeta v)(g)$$

which verifies that the map is well-defined. Then the fact that the indicated map is an isomorphism follows easily. *Done.*The group-theoretic **Frobenius Reciprocity** isomorphism

$$\text{Hom}_{G\times k}(\pi, \text{Ind}_H^G \sigma) \approx \text{Hom}_{H\times k}(\text{Res}_H^G\, \pi, \sigma)$$

does not hold in general for $\mathcal{O}$-linear representations, but only for linear representations over *fields* k, where σ is a k-linear smooth representation of H and π is a k-linear representation of G. The isomorphism is given by

$$\Phi \to \varphi_\Phi$$

described by

$$\varphi_\Phi(v) = \Phi(v)(1_G)$$

for $v \in \pi$, with inverse

$$\varphi \to \Phi_\varphi$$

given by

$$\Phi_\varphi(v)(g) = \varphi(\pi(g)v)$$

On the other hand, in the language of modules over Hecke algebras, we have:

Proposition:

$$\text{Hom}_{\mathcal{G}_\mathcal{O}}(\pi \otimes \text{c-Ind}_H^G \sigma, \mathcal{O}) \approx \text{Hom}_{\mathcal{G}_\mathcal{O}}(\pi \otimes (\mathcal{G}_\mathcal{O} \otimes_{\mathcal{H}_\mathcal{O}} \sigma \frac{\delta_G}{\delta_H}), \mathcal{O})$$

$$\approx \text{Hom}_{\mathcal{H}_\mathcal{O}}(\pi \otimes \sigma \frac{\delta_G}{\delta_H}, \mathcal{O})$$

and the map is $\Phi \to \varphi_\Phi$ described by

$$\varphi_\Phi(v \otimes x) = \Phi(v \otimes (e \otimes x))$$

where $v \in \pi$, $x \in \sigma$, and $e \in \mathcal{G}_\mathcal{O}$ is any idempotent 'small enough' so that $ev = v$. The inverse map is $\varphi \to \Phi_\varphi$ given by

$$\Phi_\varphi(v \otimes (\eta \otimes x)) = \varphi(\check{\eta} v \otimes x)$$

for $\eta \in \mathcal{G}_\mathcal{O}$. *Done.*

3.5Proof of the orbit criterion for strong meromorphy

Keep notation as in the statement of the orbit criterion for locally strong meromorphy. With the preparation of the previous sections, we can prove the orbit-criterion theorem.

Let Φ be a meromorphic $\mathcal{O}$-parametrized family of intertwining operators

$$\Phi \in \mathrm{Hom}_{G\times\mathcal{O}}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_\mathcal{O} \text{c-Ind}_P^G \chi, \mathcal{M})$$

Let

$$\varphi_\Phi \in \mathrm{Hom}_{P\times\mathcal{O}}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_\mathcal{O} \chi \frac{\delta_G}{\delta_H}, \mathcal{M})$$

be the element corresponding to it by Frobenius Reciprocity. That is, with $f \in \mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ and $v \in \chi\frac{\delta_G}{\delta_H}$,

$$\varphi_\varphi(f \otimes v) = \Phi(f \otimes (e' \otimes v))$$

for any idempotent $e' \in \mathcal{G}$ sufficiently small so that $e' f = f$. Thus, for convencience, let

$$\chi' = \chi \frac{\delta_G}{\delta_H}$$

Lemma: Let x be a height one prime in $\mathcal{O}$. The meromorphic family Φ is locally strongly meromorphic at x if and only if the meromorphic family φ_Φ associated to it by Frobenius Reciprocity (as just above) is locally strongly meromorphic.

Proof: It is important to use the tensor product and Hecke-algebra version of Frobenius Reciprocity, rather than the more group-theoretic, in order to verify this result.

Certainly the values of φ_Φ are among the values of Φ, by the definition of φ_Φ in terms of Φ. Conversely, for $f \in \mathrm{C}_c^\infty(H\backslash\mathbf{C}, \sigma)$, $\eta \in \mathcal{G}$, $v \in \frac{\delta_G}{\delta_H}$ we have

$$\Phi(f \otimes (\eta \otimes v)) = \phi(\check{\eta} f \otimes v)$$

Thus, the collection of values of $\psi\Phi$ is identical to the collection of values assumed by $\psi\varphi_\Phi$. Therefore, if $\varpi_x^N \Phi$ takes values in $\mathcal{O}_x$ (rather than merely $\mathcal{M}$), then $\varpi_x^N \varphi_\Phi$ has the same property, and conversely. *Done.*Thus, we wish to prove that any meromorphic $\mathcal{O}$-parametrized family of intertwining operators

$$\varphi \in \mathrm{Hom}_{P\times\mathcal{O}}(\mathrm{C}_c^\infty(H\backslash\Omega, \sigma) \otimes_\mathcal{O} \chi', \mathcal{M})$$

is ineluctably *locally strongly meromorphic everywhere.*

Since we will only consider one height-one prime x at a time, we may as well suppose that $\mathcal{O}$ is *already* a discrete valuation ring with unique non-zero prime x. Thus, we may as well suppose that $\mathcal{O} = \mathcal{O}_x$ already. This simplification will allow us to write simply 'π' rather than '$\pi \otimes_\mathcal{O} \mathcal{O}_x$' when convenient.

Lemma: Let π be an $\mathcal{O}$-parametrized family of smooth representations of a totally disconnected group G. Let Φ be a meromorphic $\mathcal{O}$-parametrized family of intertwining operators Φ in Hom)$G \times \mathcal{O}(\pi, \mathcal{M})$. Define

$$V_i = \{v \in \pi \otimes_\mathcal{O} \mathcal{O}_x : \varpi^i \Phi(v) \in \mathcal{O}_x\}$$

(with local parameter ϖ for x), obtaining a filtration

$$V_o \subset V_1 \subset V_2 \subset \ldots \subset \pi \otimes_{\mathcal{O}} \mathcal{O}_x$$

If Φ fails to be locally strongly meromorphic at x then for every index i

$$\mathrm{Hom}_{G\times\mathcal{O}_x}(V_{i+1}/V_i, k_x) \approx \mathrm{Hom}_{G\times k_x}(V_{i+1}/V_i, k_x) \neq \{0\}$$

Proof: (Suppress the '$\otimes_{\mathcal{O}}\mathcal{O}_x$'). Certainly $\varpi V_i \subset V_{i-1}$, so the quotients V_i/V_{i-1} are indeed $k_x$vectorspaces.

If Φ is *not* locally strongly meromorphic at x, then no one of the V_i contains π entirely. Further, since $\mathrm{ord}_x\Phi(v)$ takes arbitrarily large negative values, by simply multiplying through by powers of ϖ we can obtain *every* value of $\mathrm{ord}_x\Phi(v)$. If $\mathrm{Hom}_{G\times k_x}(V_i/V_{i-1}, k_x)$ is non-trivial then actually

$$\mathrm{Hom}_{G\times\mathcal{O}_x}(V_{i-\ell}/V_{i-1-\ell}, k_x)$$

is also non-trivial, simply by multiplying everything by ϖ. Thus, *every* quotient V_i/V_{i-1} is a non-zero k_x-vectorspace and

$$\mathrm{Hom}_{G\times\mathcal{O}_x}(V_i/V_{i-1}, k_x)$$

is non-zero for *all* indices $i \geq 1$. *Done*.Generally, if a $G\times\mathcal{O}_x$-module V has the property that for any ascending chain

$$V_0 \subset V_1 \subset \ldots$$

of $G\times\mathcal{O}_x$-submodules only *finitely-many* of the spaces

$$\mathrm{Hom}_{G\times k_x}(V_i/V_{i-1}, k_x)$$

are non-zero, then say that V is **co-iso-Noetherian**. References to the prime x and to the group G are suppressed.

Thus, the last lemma can be paraphrased as asserting that if π is co-iso-Noetherian, then Φ is locally strongly meromorphic.

So suppose that φ is *not* locally strongly meromorphic at x, and let V_i be a filtration of

$$C_c^\infty(H\backslash\Omega, \sigma) \otimes_{\mathcal{O}} \chi'$$

as in the lemma.

Next, we use the finiteness of the double-coset space $H\backslash\Omega/P$ to obtain a *finite* $H\times P$-orbit filtration of $C_c^\infty(H\backslash\Omega, \sigma)$ as $P\times\mathcal{O}_x$-module:

Lemma: Let

$$\{0\} = U_o \subset U_1 \subset \ldots \subset U_n = V$$

be a *finite* filtration by $G\times\mathcal{O}_x$-modules of a $G\times\mathcal{O}_x$-module V. Let Φ be a meromorphic $\mathcal{O}$-parametrized family in

$$\mathrm{Hom}_{G\times\mathcal{O}_x}(V, \mathcal{O}_x)$$

and let

$$V_i = \{v \in V : \varpi^i\Phi(v) \in \mathcal{O}_x\}$$

where ϖ is a local parameter for x. Suppose that

$$V_o = \{0\} \subset V_1 \subset \ldots \subset V$$

is an *infinite* filtration of V. Then for some index i the filtration

$$\frac{(U_i\cap V_o)+U_{i-1}}{U_{i-1}} \subset \frac{(U_i\cap V_1)+U_{i-1}}{U_{i-1}} \subset \ldots \subset \frac{(U_i\cap V_n)+U_{i-1}}{U_{i-1}} \subset \ldots$$

of U_i/U_{i-1} is *infinite*. Let i be the smallest such index. Then, further, letting

$$\Gamma_j = \frac{(U_i \cap V_j) + U_{i-1}}{U_{i-1}}$$

there are infinitely-many indices j so that

$$\mathrm{Hom}_{G\times k_x}(\Gamma_j/\Gamma_{j-1}, k_x) \neq \{0\}$$

Proof: This is an extension of a Jordan-Holder argument.

Suppose the assertion of the lemma were false, i.e., that the induced filtrations of all of the quotients U_i/U_{i-1} were finite. For each of the finitely-many indices i let $j(i)$ be the smallest index so that $j(i) \geq j(i-1)$ and so that

$$(U_i \cap V_{j(i)}) + U_{i-1} = U_i$$

Since $U_o = \{0\}$, $U_1 \subset V_{j(1)}$. We prove $U_i \subset V_{j(i)}$ by induction:

$$V = U_i \subset (U_i \cap V_{j(i)}) + U_{i-1} \subset (U_i \cap V_{j(i)}) + (U_{i-1} \cap V_{j(i-1)})$$

$$\subset (U_i \cap V_{j(i)}) + (U_{i-1} \cap V_{j(i)}) \subset U_i \cap V_{j(i)}$$

since

$$U_{i-1} \cap V_{j(i)} \subset U_i \cap V_{j(i)}$$

Then

$$V = U_n \subset V_{j(n)}$$

contradicting the fact that the filtration of the V_j is infinite.

Let i be the smallest index such that this filtration is infinite. The argument just given *does* prove that for $i' < i$

$$U_{i'} \subset V_{j(i')}$$

for some index j. In particular,

$$U_{i-1} \subset V_{j(i-1)}$$

Thus, for $j \geq j(i-1)$,

$$U_i \cap V_j \supset U_i$$

and

$$(U_i \cap V_j) + U_{i-1} = U_i \cap V_j$$

Thus, restricting $j \geq j(i-1)$, the chain

$$U_i \cap V_j \subset U_i \cap V_{j+1} \subset U_i \cap V_{j+2} \subset \ldots$$

must be infinite. Thus, for given j_o no matter how large, there is $j > j_o$ so that

$$\frac{U_i \cap V_j}{U_i \cap V_{j-1}} \neq 0$$

In particular, some element v of V_j but not in V_{j-1} lies in U_i. Thus, $\varpi^j \Phi$ composed with reduction modulo ϖ is non-trivial on such $v \in U_i$, proving that Γ_j/Γ_{j-1} has non-trivial homomorphisms to k_x. *Done.* We have already shown that the graded pieces of the $H \times P$-orbit filtration on $\mathrm{C}_c^\infty(H\backslash\Omega, \sigma)$ are of the form

$$\text{c-Ind}_{\Theta^\xi}^P \sigma_\xi$$

where

$$\sigma_\xi(p) = \sigma(\xi p \xi^{-1})$$

for $p \in P \cap \xi^{-1} H \xi$. Thus, if the original Φ were *not* locally strongly meromorphic at x, then for some $\xi \in H\backslash\Omega/P$ the there would be an infinite chain of subspaces V_i of

$$\text{c-Ind}_{\Theta^\xi}^P \sigma_\xi \otimes_{k_x} \chi' \otimes k_x$$

so that

$$\text{Hom}_{P\times k_x}(V_i/V_{i-1}, k_x) \neq \{0\}$$

Now $\text{c-Ind}_{\Theta^\xi}^P \sigma_\xi$ is an image of $\mathrm{C}_c^\infty(P, \mathcal{O}_x) \otimes \sigma_\xi$ under an averaging map, where in the latter space P acts *trivially* on σ_ξ. Thus, if

$$\text{c-Ind}_{\Theta^\xi}^P \sigma_\xi \otimes \chi'$$

fails to have the co-iso-Noetherian propery so will

$$\mathrm{C}_c^\infty(P, \mathcal{O}_x) \otimes \sigma_\xi \otimes \chi'$$

Since σ_ξ is finite-dimensional, this can happen only if already

$$\mathrm{C}_c^\infty(P, \mathcal{O}_x) \otimes \chi'$$

fails to have the co-iso-Noetherian property.

Next, we use the fact that χ is a finite sum of trivial representations of N (since χ is finite-dimensional and trivial on N), and the assumption that N is an ascending union of compact open subgroups. The latter hypothesis assures that the *Jacquet functor* (i.e., trivial co-isotype functor) for N is *exact*, as shown earlier. And since N is normal in P, such co-isotypes are still P-representations. The exactness assures that

$$\mathcal{J}V_i/\mathcal{J}V_{i-1} \approx \mathcal{J}(V_i/V_{i-1})$$

for submodules V_i of $\mathrm{C}_c^\infty(P, \mathcal{O}_x) \otimes \chi'$. Since generally (by the defining property of this co-isotype functor $\mathcal{J}$)

$$\text{Hom}_{P\times\mathcal{O}_x}(V, k_x) \approx \text{Hom} P \times \mathcal{O}_x(\mathcal{J}V, k_x)$$

we conclude that

$$\mathcal{J}(\mathrm{C}_c^\infty(P, \mathcal{O}_x) \otimes \chi') \approx (\mathcal{J}(\mathrm{C}_c^\infty(P, \mathcal{O}_x)) \otimes \chi'$$

fails to have the co-iso-Noetherian property.

But now the Jacquet module (co-isotype) $\mathcal{J}\mathrm{C}_c^\infty(P, \mathcal{O}_x)$ is readily computed to be simply $\mathrm{C}_c^\infty(N\backslash P, \mathcal{O}_x)$. Thus, from this point we can restrict our attention to M-representations. Thus, to this point, we have concluded that the original supposed failure of locally strong meromorphy implies that

$$\mathrm{C}_c^\infty(M, \mathcal{O}_x) \otimes \chi'$$

fails to have the co-iso-Noetherian condition as $M \times \mathcal{O}_x$-module.

By hypothesis, for a certain idempotent e in the Hecke algebra $\mathcal{G}_M$ of M, $e\chi = \chi$ (referring to the representation space of χ). Certainly δ_H and δ_G are trivial on compact subgroups, so also $e\chi' = \chi'$. Therefore,

$$\text{Hom}_{M\times\mathcal{O}_x}(\mathrm{C}_c^\infty(M, \mathcal{O}_x) \otimes \chi', k_x) \approx \text{Hom}_{e\mathcal{G}_M e\times\mathcal{O}_x}(e\mathrm{C}_c^\infty(M, \mathcal{O}_x) \otimes \chi', k_x)$$

We saw earlier that the map $E \to eE$ from $\mathcal{G}_M$-modules to $e\mathcal{G}_M e$-modules is *exact* quite generally. Thus, it must be that $e\mathrm{C}_c^\infty(M, \mathcal{O}_x) \otimes \chi'$ fails to be co-iso-Noetherian as a $e\mathcal{G}_M e \times \mathcal{O}_x$-module.

But by hypothesis $e\mathcal{G}_M e$ and $\mathcal{O}_x$ are Noetherian. By the finite-generation hypothesis on $e\mathrm{C}_c^\infty(M,\mathcal{O}_x) = e\mathcal{G}_M$ and on $\chi' = e\chi'$, the $e\mathcal{G}_M e \times \mathcal{O}_x$-module

$$e\mathrm{C}_c^\infty(M,\mathcal{O}_x) \otimes \chi'$$

is Noetherian. This certainly contradicts the conclusion that it was not co-iso-Noetherian, since Noetherian-ness is a stronger condition than co-iso-Noetherian. So we conclude that, after all, Φ *was* locally strongly meromorphic.

This completes the proof of the orbit criterion for strong meromorphy. *Done*.

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