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Some facts about discrete series (holomorphic, quaternionic)

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This is a very brief overview (for novices) of certain facts concerning discrete series representations of real Lie groups, especially **holomorphic** and **quaternionic**. In particular, the table at the end tells which classical real Lie groups have discrete series, holomorphic discrete series, and quaternionic discrete series. See also the extensive historical notes in [Knapp 1986] and [Knapp, Vogan 1995].

Recall that a **discrete series** representation of a unimodular [1] topological group [2] G is an irreducible unitary representation of G on a Hilbert space [3] V such that at least one **(matrix) coefficient function**

$$g \to \langle g \cdot v, w\rangle \quad \text{(for fixed } v, w \in V)$$

is square-integrable [4] on G. The basic fact [5] is

Theorem: If one coefficient function of an irreducible unitary Hilbert space representation V of a unimodular topological group G is square integrable, then all are. Equivalently, $V \subset L^2(G)$ with the right regular representation [6] of G on $L^2(G)$. ///

For G a **linear connected semi-simple real Lie group** [7] [Harish-Chandra 1966] proved what he had conjectured at the 1954 International Congress of Mathematicians [8] , namely

Theorem: G has discrete series representations if and only if for a maximal compact subgroup K of G

$$\text{rank} K = \text{rank} G$$

Equivalently, there is a *compact* Cartan subgroup in G. ///

Corollary: A real Lie group G obtained from a *complex* semi-simple Lie group by forgetting the complex structure [9] *never* has discrete series representations. [10] In particular, for maximal compact K, always $\text{rank} G = 2 \cdot \text{rank} K$ so that Harish-Chandra's rank condition cannot be met. ///

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[1] *Unimodular* in the sense that a left Haar measure is a right Haar measure.

[2] As usual, these means that G is locally compact, Hausdorff, and probably has a countable basis, in order to ensure regularity of measures called into existence by application of the Riesz-Markov-Kakutani theorem.

[3] First, we do insist on Hilbert space representations rather than any more general class of topological vector spaces. In this context, the irreducibility requires that there be no proper *topologically closed* G-stable subspace. The unitariness is, as usual, that $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$ and u, v in the Hilbert space.

[4] If G has a non-compact center Z, Schur's lemma assures that an irreducible unitary has a *(unitary) central character*, and then it makes sense to require that the coefficient functions be square-integrable on G/Z rather than G. Indeed, for non-compact center Z *no* coefficient function can be integrable or square-integrable on G.

[5] Such a result was proven concretely in [Bargmann 1947], and abstractly in [Godement 1947]. See, for example, [Garrett 2004] for the proof due to Godement, using the closed graph theorem.

[6] As usual, the right regular representation of G on $L^2(G)$ is by $R_h f(g) = f(gh)$. The continuity of this representation is proven by using the density of $C_c^o(G)$ in $L^2(G)$.

[7] For example, classical groups such as $SL(n, \mathbf{R})$, $Sp(n, \mathbf{R})$, $SO(p, q)$. The conditions can be relaxed, but are sufficiently generous for discussion of our examples.

[8] [Knapp 1986] page 729.

[9] This includes $SL(n, \mathbf{C})$, $Sp(n, \mathbf{C})$, and $SO(n, \mathbf{C})$, for example.

[10] To see this, note that a complex semi-simple group has a *compact* real form $K = G_c$ which is a maximal compact subgroup of G. To compute the rank, complexify both: G_c is complexified to G, and the complexification of an already complex Lie group with the complex structure forgotten is two copies $G \times G$ of the original. That is, the rank of G in this situation is always twice that of its maximal compact.

In particular, this is why [Gelfand, Graev 1950] did not *need* discrete series in the Plancherel formula [11] for classical complex groups such as $SL(n, \mathbf{C})$. That is, there are *no* irreducibles occuring *discretely* as summands in $L^2(SL(2, \mathbf{C}))$. The Plancherel formula in such cases is purely an integral.

The only explicit discrete series known prior to the 1960's were the **holomorphic discrete series**, in [Harish-Chandra 1955], [Harish-Chandra 1956a], and [Harish-Chandra 1956b]. For classical groups, the existence of these representations is easily motivated from the theory of holomorphic automorphic forms on classical domains. Their models (due to Harish-Chandra) do literally involve holomorphic functions, similar to the Borel-Weil theorem [12] for *compact* groups.

Theorem: A simple Lie group [13] has holomorphic discrete series representations if and only if a maximal compact subgroup has a non-finite center [14] (which will therefore contain a circle group). ///

The first example of discrete series other than holomorphic was in [Dixmier 1961] for $SO(4, 1)$. [15]

[Harish-Chandra 1966] did not *construct* discrete series representations, but, rather, discussed them via their *characters*, and proved that this viewpoint was sufficient for several purposes. Construction of discrete series in Dolbeault cohomology was approached systematically in [Schmid 1967] and [Schmid 1968]. Construction of discrete series as kernels of Dirac operators was first done in [Parthasarathy 1972]. Further results on the pattern of K*-types* occurring in discrete series (*Blattner's conjecture*) appeared in [Hecht, Schmid 1975] and [Wallach 1976]. A different sort of construction was initiated in [Zuckerman 1977], and given an extensive treatment in [Knapp, Vogan 1995]. [Wallach 1988] gives another treatment of the various constructions. See also [Enright, Wallach 1997] for constructions. And see the overview [Huang, Pandžić 2004] for comparison of Dirac cohomology to other constructions.

For groups $Spin(2n, 1)$, $SU(n, 1)$, $Sp^*(n, 1)$, and a rank-one real form of the exceptional group F_4, not-holomorphic discrete series representations were systematically treated [16] in [Baldoni-Silva 1980], [Baldoni-Silva 1981], [Baldoni-Silva, Kraljevi¸1980].

Among all the real forms of the *exceptional* groups G_2, F_4, E_6, E_7, and E_8, only certain real forms of E_6 and E_7 have holomorphic discrete series. A more special family of discrete series representations which nevertheless appears for [17] *all* exceptional groups is the **quaternion discrete series**. Let $\mathbf{H}^1$ be the group of Hamiltonian quaternions of norm 1. One has $\mathbf{H}^1 \approx SU(2)$. For the following theorem, let $\mathbf{g}$ be the Lie algebra of G and $\mathbf{k}$ the Lie algebra of K.

Theorem: [Gross, Wallach 1996] If a maximal compact subgroup K of a simple Lie group G has a normal subgroup isomorphic to $SU(2) = \mathbf{H}^1$, and if the quotient $\mathbf{g}/\mathbf{k}$ as an H^1 space under Ad [18] is a sum of copies of the left multiplication of $\mathbf{H}^1$ on $\mathbf{H}$, then G has (quaternion) discrete series. [19] ///

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[11] Roughly, the Plancherel formula for a group G expresses (*decomposes*) $L^2(G)$ as a direct sum and/or integral of irreducible unitary representations. It is possible to prove the *existence* of such a decomposition in the abstract, without explicit exhibition of irreducibles.

[12] For example, see [Knapp 1986].

[13] For present purposes, this means that the Lie *algebra* is simple, and we may want the group to be connected, and possibly linear.

[14] That the latter hypothesis of non-discrete center implies that there is a compact Cartan subgroup is an exercise in structure theory.

[15] [Knapp 1986] page 729.

[16] In a discussion of imbedding irreducibles into *principal series* representations.

[17] suitable real forms of

[18] The adjoint action of a Lie group on its Lie algebra, for matrices typically given by the obvious $(\mathrm{Ad}g)(x) = gxg^{-1}$.

[19] As with the holomorphic discrete series, that this condition on the maximal compact subgroup implies Harish-Chandra's rank condition is not immediate.

Relatively recently, there has been progress of various sorts in demonstrating that automorphic forms whose archimedean components are quaternion discrete series have features parallel to holomorphic automorphic forms, for example B. Gross's treatment of G_2. An essential positivity feature of Fourier coefficients was proven in [Wallach 2003].

The following lists the classical (non-exceptional) simple real Lie groups, their maximal compact subgroups, the ranks of both, whether they do or do not have discrete series, whether or not they have holomorphic discrete series, and whether or not they have quaternion discrete series. The notation $\lfloor x \rfloor$ is the *floor* function, that is, the greatest integer less than or equal x.

	max cpt	rank	rank max cpt	discrete series?	holo?	quaternion?
$SL(n,\mathbf{C})$	$SU(n)$	$2n-2$	$n-1$	–	–	–
$SL(n,\mathbf{R})$	$SO(n)$	$n-1$	$\lfloor \frac{n}{2} \rfloor$	$n=2$	$n=2$	–
$SL(n,\mathbf{H})$	$Sp^*(n)$	$2n-1$	n	–	–	–
$SU(p,q)$	$S(U(p)\times U(q))$	$p+q-1$	$p+q-1$	yes	yes	$SU(n,2)$
$SO(n,\mathbf{C})$	$SO(n)$	$2\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	–	–	–
$SO(p,q)$	$S(O(p)\times O(q))$	$\lfloor \frac{p+q}{2} \rfloor$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$	pq even	$SO(n,2)$	$SO(n,4)$
$O^*(2n)$	$U(n)$	n	n	yes	yes	$n=2$
$Sp(n,\mathbf{C})$	$Sp^*(n)$	$2n$	n	–	–	–
$Sp(n,\mathbf{R})$	$U(n)$	n	n	yes	yes	–
$Sp^*(p,q)$	$Sp^*(p)\times Sp^*(q)$	$p+q$	$p+q$	yes	–	$Sp^*(n,1)$

From the table, note that among classical groups, the groups $SO(n,4)$ and $Sp^*(n,1)$ are the only ones that have quaternion discrete series but *not* holomorphic discrete series.

To understand that $SO(n,4)$ meets the Gross-Wallach criterion one must realize that $SO(4)$ does have a normal subgroup isomorphic to $SU(2)$.

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