(*July 29, 2020*)

The Hilbert transform

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[This document is http://www.math.umn.edu/~garrett/m/v/hilbert_transform.pdf]

Formulaically, the Cauchy principal-value functional η attached to $1/x$ is

$$\eta f \;=\; \text{principal-value functional of } f \;=\; P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

This is a fragile presentation, since the apparent integral is *not* a literal integral.

The uniqueness proven below helps prove plausible properties like the *Sokhotski-Plemelj theorem* from [Sokhotski 1871], [Plemelji 1908], with a possibly unexpected leading term:

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\,dx \;=\; -i\pi\, f(0) + P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx$$

In the physics literature, similar properties are often called *Kramers-Kronig* relations after [Kramers 1926], [Kronig 1927]. See [wiki 2020], and [Gelfand-Silov 1964]. Uniqueness also best certifies many heuristically plausible identities, such as

$$\eta f \;=\; \tfrac{1}{2}\int_{\mathbb{R}} \frac{f(x)-f(-x)}{x}\,dx \qquad\qquad (\text{for } f\in C^1(\mathbb{R})\cap L^1(\mathbb{R}))$$

using the canonical continuous extension of $\frac{f(x)-f(-x)}{x}$ at 0. Also,

$$\eta f \;=\; \int_{\mathbb{R}} \frac{f(x)-f(0)\cdot e^{-x^2}}{x}\,dx \qquad\qquad (\text{for } f\in C^o(\mathbb{R})\cap L^1(\mathbb{R}))$$

The *Hilbert transform* of a function f on $\mathbb{R}$ is awkwardly described as a principal-value integral

$$(\mathcal{H}f)(x) \;=\; \frac{1}{\pi}P.V.\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt \;=\; \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|t-x|>\varepsilon} \frac{f(t)}{x-t}\,dt$$

with the leading constant $1/\pi$ understandable with sufficient hindsight: we will see that this adjustment makes $\mathcal{H}$ extend to a *unitary* operator on $L^2(\mathbb{R})$. The formulaic presentation of $\mathcal{H}$ makes it appear to be a *convolution* with (a constant multiple of) the principal-value functional. The fragility of these presentations make existence, continuity properties, and range of applicability of $\mathcal{H}$ less clear than one would like.

The technicalities here are prototypes for similar technicalities regarding pseudo-differential operators, for example.

1. *The principal-value functional*

The principal-value functional η is better characterized as the unique (up to a constant multiple) odd *distribution* on $\mathbb{R}$, positive-homogeneous of degree 0 as a distribution (see below). This characterization allows unambiguous comparison of various limiting expressions closely related to the principal-value functional. Further, the characterization of the principal-value functional makes discussion of the Hilbert transform less fragile and more convincing.

Again, let

$$\eta f \;=\; \text{principal-value functional of } f \;=\; P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

We prove that η is a tempered distribution:

[1.1] Claim: For $f \in \mathscr{S}$, for every $h > 0$,

$$|\eta(f)| \;\le\; \int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx + 2h \sup_{|x|\le h} |f'(x)|$$

Thus, u is a tempered distribution:

$$|\eta(f)| \;\ll\; \sup_x (1+x^2)|f(x)| + \sup_x |f'(x)| \qquad\qquad \text{(implied constant independent of } f)$$

Proof: Certainly

$$|\eta(f)| \;=\; \int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx + \lim_{\varepsilon\to 0^+} \Big|\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx\Big|$$

By the Mean Value Theorem, $f(x) = f(0) + x\, f'(\xi_x)$ for some ξ_x between 0 and x. Thus,

$$\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx \;=\; \int_{\varepsilon<|x|\le h} \frac{f(0)}{x}\,dx + \int_{\varepsilon<|x|\le h} f(\xi_x)\,dx \;=\; 0 + \int_{\varepsilon<|x|\le h} f(\xi_x)\,dx$$

because $1/x$ is *odd* and $x \to f(0)$ is *even*. Thus,

$$\Big|\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx\Big| \;=\; \Big|\int_{\varepsilon<|x|\le h} f(\xi_x)\,dx\Big| \;\le\; \int_{\varepsilon<|x|\le h} |f(\xi_x)|\,dx$$

$$\le\; 2h\cdot \sup_{\varepsilon<|x|\le h} |f'(x)| \;\le\; 2h\cdot \sup_{|x|\le h} |f'(x)|$$

The latter is independent of ε. With $0 < h \le 1$,

$$\int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx \;=\; \int_{|x|\ge 1} \Big|\frac{f(x)}{x}\Big|\,dx + \int_{h\le|x|\le 1} \Big|\frac{f(x)}{x}\Big|\,dx$$

$$\le\; \int_{|x|\ge 1} |f(x)|\,dx + \frac{1}{h}\int_{h\le|x|\le 1} |f(x)|\,dx \;\le\; \int_{|x|\ge 1} (1+x^2)|f(x)|\cdot\frac{1}{1+x^2}\,dx + 2h\sup_x |f(x)|$$

$$\le\; \sup_x (1+x^2)|f(x)| \cdot \int_{|x|\ge 1} \frac{1}{1+x^2}\,dx + 2h\sup_x |f(x)| \;\ll\; \sup_x (1+x^2)|f(x)|$$

Of course, $\sup_{|x|\le h} |f'(x)| \le \sup_x |f'(x)|$. ///

To make the *degree* of a positive-homogeneous function φ agree with the degree of the integration-against-φ distribution u_φ, due to change-of-measure, we find that

$$(u_\varphi \circ t)(f) \;=\; u_\varphi(f\circ t^{-1}) \;=\; \int_{\mathbb{R}} u_\varphi(x)\, f(t^{-1}x)\, dx \;=\; \int_{\mathbb{R}} u_\varphi(tx)\, f(x)\, d(tx) \;=\; t\cdot u_{\varphi\circ t}(f)$$

Thus, for agreement of the notion of homogeneity for distributions and (integrate-against) functions, the dilation action $u\to u\circ t$ on *distributions* should be

$$(u\circ t)(f) \;=\; \frac{1}{t} u(f\circ t^{-1}) \qquad\qquad \text{(for } t>0\text{, test function } f\text{, and distribution } u\text{)}$$

Parity is as expected: u is *odd* when $u(x\to f(-x)) = -u(x\to f(x))$ for all test functions f, and is *even* when $u(x\to f(-x)) = +u(x\to f(x))$ for all test functions f.

[1.2] Claim: The principal-value distribution η is positive-homogeneous of degree -1, and is *odd*.

Proof: The ε^{th} integral in the limit definition of η is itself *odd*, by changing variables:

$$\eta(f\circ(-1)) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(-x)}{x}\, dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(x)}{-x}\, d(-x)$$

$$=\; \lim_{\varepsilon\to 0^+} -\int_{|x|\ge\varepsilon} \frac{f(x)}{x}\, dx \;=\; -\lim_{\varepsilon\to 0^+} \int_{|x|\ge\varepsilon} \frac{f(x)}{x}\, dx \;=\; -\eta(f)$$

as claimed. For the degree of homogeneity, for $t>0$ and test function f,

$$(\eta\circ t)(f) \;=\; \frac{1}{t}\eta(f\circ\frac{1}{t}) \;=\; \frac{1}{t}\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(\frac{x}{t})}{x}\, dx \;=\; \frac{1}{t}\lim_{\varepsilon\to 0^+}\int_{|tx|\ge\varepsilon} \frac{f(x)}{tx}\, d(tx) \;=\; \frac{1}{t}\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon/t} \frac{f(x)}{x}\, d(x)$$

$$=\; \frac{1}{t}\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(x)}{x}\, d(x) \;=\; \frac{1}{t}\eta(f)$$

That is, η is homogeneous of degree -1. ///

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2. *Other characterizations of the principal-value functional*

[2.1] Claim: Let φ be any *even* function in $L^1(\mathbb{R})\cap C^1(\mathbb{R})$, with $\varphi(0)=1$. Then

$$P.V.\int_{\mathbb{R}} \frac{f(x)}{x}\, dx \;=\; \int_{\mathbb{R}} \frac{f(x)-f(0)\cdot\varphi(x)}{x}\, dx \qquad\qquad \text{(for } f\in C^o(\mathbb{R})\cap L^1(\mathbb{R})\text{, for example)}$$

where $\big(f(x)-f(0)\cdot\varphi(x)\big)/x$ is extended by continuity at $x=0$.

Proof: By the demonstrated odd-ness of the principal-value integral, it is 0 on the even function φ. Extending $\frac{f(x)-f(0)\cdot\varphi(x)}{x}$ by continuity at 0,

$$\int_{\mathbb{R}} \frac{f(x)-f(0)\cdot\varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(x)-f(0)\cdot\varphi(x)}{x}\, dx$$

$$=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{f(x)}{x}\, dx + f(0)\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon} \frac{\varphi(x)}{x}\, dx \;=\; P.V.\int_{\mathbb{R}} \frac{f(x)}{x}\, dx + 0$$

as claimed. ///

The leading term in the following might be unexpected:

[2.2] Claim: *(Sokhotski-Plemelj)*

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{f(x)}{x+i\varepsilon}\,dx \;=\; -i\pi\,f(0)+P.V.\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx$$

Proof: Aiming to apply the previous claim,

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{f(x)}{x+i\varepsilon}\,dx \;=\; \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{f(x)-\frac{f(0)}{1+x^2}}{x+i\varepsilon}\,dx + f(0)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{1/(1+x^2)}{x+i\varepsilon}\,dx$$

For $0<\varepsilon<1$, the right-most integral is evaluated by residues, moving the contour through the upper half-plane:

$$\int_{\mathbb{R}}\frac{1/(1+x^2)}{x+i\varepsilon}\,dx \;=\; 2\pi i\,\mathrm{Res}_{x=i}\frac{1}{(1+x^2)(x+i\varepsilon)} \;=\; 2\pi i\frac{1}{(i+i)(i+i\varepsilon)} \;\longrightarrow\; \frac{\pi}{i} \;=\; -i\pi$$

as claimed. ///

[2.3] Claim: With f continuously differentiable at 0, thereby extending $\frac{f(x)-f(-x)}{x}$ by continuity at 0,

$$\tfrac{1}{2}\int_{\mathbb{R}}\frac{f(x)-f(-x)}{x}\,dx \;=\; P.V.\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx \qquad (\text{for } f\in C^1(\mathbb{R})\cap L^1(\mathbb{R}))$$

Proof: For test function f, since $\frac{f(x)-f(-x)}{x}$ is continuous as 0, using the odd-ness of the principal-value functional,

$$\tfrac{1}{2}\int_{\mathbb{R}}\frac{f(x)-f(-x)}{x}\,dx \;=\; \tfrac{1}{2}\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon}\frac{f(x)-f(-x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon}\frac{f(x)}{x}\,dx$$

as claimed. ///

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3. *Hilbert transform*

The most immediate description of the Hilbert transform $\mathcal{H}f$ is

$$(\mathcal{H}f)(y) \;=\; \frac{1}{\pi}P.V.\int_{\mathbb{R}}\frac{f(y)}{x-y}\,dy \;=\; \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|x|\ge\varepsilon}\frac{f(y)}{x-y}\,dy$$

From the previous discussion of the principal-value functional η, $\mathcal{H}f$ exists at least as a point-wise function. In fact, with $(L_y)f(x)=f(x-y)$, $y\to L_yf$ is a smooth $\mathscr{S}$-valued function, so $f\to\eta(L_yf)$ is in $C^\infty(\mathbb{R})$, but growth properties are unclear.

The usual heuristic is

$$P.V.\int_{\mathbb{R}}\frac{f(y)}{x-y}\,dy \;=\; (\eta * f)(x)$$

so

$$\mathcal{H}f \;=\; \frac{1}{\pi}\eta * f \qquad (\text{for } f\in\mathscr{S}(\mathbb{R}))$$

Granting this for a moment, taking Fourier transform would seem to give

$$(\mathcal{H}f)\widehat{\ } \;=\; \frac{1}{\pi}\widehat{\eta}\cdot\widehat{f}$$

We will *prove* that this heuristic is correct. The principal-value functional η is a tempered distribution, so its Fourier transform makes sense at least as a tempered distribution.

Recall the unsurprising

[3.1] Claim: The Fourier transform of an odd/even distribution on $\mathbb{R}$ of positive-homogeneity degree s is odd/even of positive-homogeneity degree $-(s+1)$.

Proof: Since Schwartz functions are dense, the interaction of dilation and Fourier transform can be examined via functions with point-wise values and Fourier transforms defined by literal integrals, although none of these can be homogeneous. It is immediate by changing variables that parity is preserved. With $(f\circ t)(x) = f(tx)$, for *functions* f,

$$(f\circ t)\widehat{\ }(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x} f(tx)\,dx \;=\; \frac{1}{t}\int_{\mathbb{R}} e^{-2\pi i\xi x/t} f(tx)\,dx \;=\; \frac{1}{t}(\widehat{f}\circ\frac{1}{t})(\xi)$$

That is,

$$\widehat{f}\circ t \;=\; \frac{1}{t}\cdot f\circ\frac{1}{t}$$

Thus, without trying to use pointwise sense of a tempered distribution,

$$(\widehat{u}\circ t)(f) \;=\; \widehat{u}(\frac{1}{t}\,f\circ\frac{1}{t}) \;=\; \frac{1}{t}u\Big((f\circ\frac{1}{t})\widehat{\ }\Big) \;=\; \frac{1}{t}u\Big(t\cdot\widehat{f}\circ t\Big) \;=\; \frac{1}{t}(u\circ\frac{1}{t})(\widehat{f})$$

$$=\; \frac{1}{t}\Big((\frac{1}{t})^{s}\cdot u\Big)(\widehat{f}) \;=\; t^{-(s+1)}\cdot u(\widehat{f}) \;=\; t^{-(s+1)}\cdot\widehat{u}(f)$$

as claimed. ///

[3.2] Corollary: The principal-value distribution η, has Fourier transform has degree 0 and of odd parity. In particular, $\widehat{\eta} = -i\pi\,\mathrm{sgn}\,x$.

Proof: By uniqueness of distributions of a given degree and parity, it suffices to evaluate $\widehat{\eta}$ and $\mathrm{sgn}x$ on a given odd Schwartz function, for example, $x\to xe^{-\pi x^2}$. On one hand,

$$\widehat{\eta}(xe^{-\pi x^2}) \;=\; \eta\Big((xe^{-\pi x^2})\widehat{\ }\Big) \;=\; \eta\Big(i^{-1}xe^{-\pi x^2}\Big) \;=\; i^{-1}\int_{\mathbb{R}}\frac{xe^{-\pi x^2}}{x}\,dx \;=\; i^{-1}\int_{\mathbb{R}} e^{-\pi x^2}\,dx \;=\; i^{-1}$$

On the other,

$$\int_{\mathbb{R}}\mathrm{sgn}\,x\cdot xe^{-\pi x^2}\,dx \;=\; \int_{\mathbb{R}}|x|\,e^{-\pi x^2}\,dx \;=\; 2\int_0^\infty x\,e^{-\pi x^2}\,dx \;=\; 2\int_0^\infty x^2\,e^{-\pi x^2}\,\frac{dx}{x} \;=\; \int_0^\infty x\,e^{-\pi x}\,\frac{dx}{x}$$

$$=\; \pi^{-1}\int_0^\infty x\,e^{-x}\,\frac{dx}{x} \;=\; \pi^{-1}\Gamma(1) \;=\; \pi^{-1}$$

Thus, $\widehat{\eta} = -i\pi\,\mathrm{sgn}\,x$. ///

With the factors of π cancelling, this suggests defining the Hilbert transform at least on Schwartz functions f by

[3.3] Theorem: $\mathcal{H}f \;=\; \Big(-i\,\mathrm{sgn}\,x\cdot\widehat{f}\Big)^{\vee}$

[3.4] Corollary: The Hilbert transform continuously extends to an isometry $L^2 \to L^2$. ///

(Proof below.)

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4. *Some multiplier operators on* H^∞

In describing the Hilbert transform in terms of Fourier transform and pointwise multiplication, there is an implicit issue, namely, that what appears to be a convolution action of η on $\mathscr{S}$ should have usual structural properties, such as *associativity.* That is, we should identify a class of distributions containing η, a class of nice functions containing $\mathscr{S}$, and an action (temporarily denoted by $\#$) of such distributions φ, ψ on the nice functions f such that $\varphi\#f$ is again in the space of nice functions, and

$$\varphi\#(\psi\#f) \;=\; (\varphi * \psi)\#f \qquad\qquad \text{(associativity)}$$

The iconic cautionary example about failure of associativity

$$1 * (\delta' * H) \;=\; 1 * \delta \;=\; 1 \;\neq\; 0 \;=\; 0 * H \;=\; (1 * \delta') * H \qquad\qquad \text{(with Heaviside function } H\text{)}$$

shows that *some* restrictions are necessary in order to preserve associativity. Further, tempered distributions need not have useful pointwise values, so trying to define $f \to \varphi\#f$ as a *multiplier operator* by $(\varphi\#f) = \widehat{\varphi} \cdot \widehat{f}$ cannot be quite right. Even when $\widehat{\varphi}$ *does* have pointwise values almost everywhere, it need not be smooth, so $\widehat{\varphi} \cdot \widehat{f}$ need not be smooth, so could not be Schwartz.

One reasonable resolution of these difficulties, still using Fourier transforms, is to consider the space M (for *multiplier*) of tempered distributions with Fourier transforms that are locally L^1, and of *polynomial growth.* That is, for $\varphi \in M$, $\widehat{\varphi}$ has pointwise values almost everywhere, and there is an exponent N such that $|\widehat{\varphi}(x)| \ll (1 + x^2)^N$ for almost all x. Distributions $\varphi \in M$ do act on the Sobolev space H^∞ by multiplier operators

$$(\varphi\#f)\widehat{\ } \;=\; \widehat{\varphi} \cdot \widehat{f} \qquad\qquad (\text{for } \varphi \in M \text{ and } f \in H^\infty)$$

with $\widehat{\varphi} \cdot \widehat{f}$ being pointwise (almost everywhere) multiplication. The spectral characterization of Sobolev spaces in terms of weighted L^2 spaces shows that this does map $H^\infty \to H^\infty$.

We *anticipate* that the appropriate *convolution* on M is given by pointwise multiplication on the Fourier transform side, namely,

$$(\varphi * \psi)\widehat{\ } \;=\; \widehat{\varphi} \cdot \widehat{\psi}$$

The pointwise product is well-defined almost-everywhere on the spectral side. However, the possibility of writing the obvious formula does not quite prove that it is the incarnation of convolution appropriate for this context. Still, unsurprisingly, the heuristic that discovers the *formula* for appropriate convolution on $H^{-\infty}$, by requiring associativity, does recover the formulaically suggested convolution:

$$\Big(\varphi\#(\psi\#f)\Big)\widehat{\ } \;=\; \widehat{\varphi} \cdot (\psi\#f)\widehat{\ } \;=\; \widehat{\varphi} \cdot (\widehat{\psi} \cdot \widehat{f}) \;=\; (\widehat{\varphi} \cdot \widehat{\psi}) \cdot \widehat{f}$$

Thus, indeed, $(\varphi * \psi)\widehat{\ } = \widehat{\varphi} \cdot \widehat{\psi}$. ///

Since the principal value functional η has Fourier transform a constant multiple of the sign function, $\eta \in M$.

A different sort of action, of compactly-supported distributions on smooth functions, is treated in an appendix. That variant does not apply to the action of η on H^∞.

5. *Hilbert transform on L^2*

[5.1] Corollary: The Hilbert transform extends by continuity from a map $\mathscr{S} \to H^\infty \subset L^2$ to an isometric isomorphism of L^2 to itself.

Proof: Fourier transform is an isometry of L^2 to itself, and multiplication by *sgn* is an isometry, so the Hilbert transform is an L^2-isometry of $\mathscr{S} \subset L^2$ to a subspace of $H^\infty \subset L^2$, and extends by continuity to $L^2 \to L^2$. ///

[5.2] Corollary: $(\mathcal{H} \circ \mathcal{H})f = -f$ for $f \in \mathscr{S}$ or $f \in L^2$.

Proof: This should be a direct computation.

$$(\mathcal{H} \circ \mathcal{H})f = \mathcal{H}(\mathcal{H}f) = \mathcal{H}((-i \cdot \mathrm{sgn} \cdot \widehat{f})^\vee) = \Big(-i \cdot \mathrm{sgn} \cdot \big((-i \cdot \mathrm{sgn} \cdot \widehat{f})^\vee\big)^\wedge \Big)^\vee$$

$$= \Big(-i \cdot \mathrm{sgn} \cdot (-i \cdot \mathrm{sgn} \cdot \widehat{f}) \Big)^\vee = -f$$

as claimed. ///

6. *Extending Hilbert transform to certain tempered distributions*

The collection M of multiplier operators also stabilizes a considerably larger space, namely, a space of tempered distributions on whose Fourier transforms multiplication by *sign* is sensible: let $X \subset \mathscr{S}^*$ be the set of tempered distributions u such that $\widehat{u}$ is in L^1_{loc} on some neighborhood of $0 \in \mathbb{R}$.

[6.1] Claim: : For $u \in X$, the Hilbert transform $\mathcal{H}u$ is again in X.

Proof: Use the description of $\mathcal{H}$ as intertwined by Fourier transform with (essentially) multiplication by sgn. Let φ be a real-valued test function supported on a neighborhood of 0 on which u is L^1, and such that φ is identically 1 on a small neighborhood of 0. Multiplication by $1 - \varphi$ stabilizes Schwartz functions, so stabilizes tempered distributions. Also, $(1 - \varphi) \cdot \mathrm{sgn}$ is smooth. Thus,

$$\mathrm{sgn} \cdot \widehat{u} = \mathrm{sgn} \cdot (\varphi + (1 - \varphi)) \cdot \widehat{u} = \mathrm{sgn} \cdot (\varphi \cdot \widehat{u}) + (\mathrm{sgn} \cdot (1 - \varphi)) \cdot \widehat{u}$$

In the latter expression, the first summand is still L^1_{loc} and compactly supported. The second summand is a tempered distribution, with support not including 0. Thus, the sum is again in the space of Fourier transforms of tempered distributions from X. ///

[6.2] Corollary: For $f \in X$ such that $\mathcal{H}f$ is also in X,

$$(\mathcal{H} \circ \mathcal{H})f = -f$$

Proof: The same proof as earlier, with the sense of the symbols suitably extended. ///

7. *Example computations*

The (suitably-interpreted) description of the Hilbert transform as

$$\mathcal{H}f = \big(-i \cdot \mathrm{sgn} \cdot \widehat{f} \big)^\vee$$

facilitates some illustrative examples.

[7.1] Example: $\mathcal{H}\sin \;=\; \Big(-i\cdot\mathrm{sgn}\cdot\widehat{\sin}\Big)^\vee \;=\; \Big(-i\cdot\mathrm{sgn}\cdot\frac{\delta_1-\delta_{-1}}{2i}\Big)^\vee \;=\; \Big(-\frac{\delta_1+\delta_{-1}}{2}\Big)^\vee \;=\; -\cos$

[7.2] Example: Since $\widehat{\mathrm{ch}_{[-1,1]}} = \frac{\sin 2\pi\xi}{\pi\xi}$,

$$\mathcal{H}\frac{\sin 2\pi x}{\pi x} \;=\; \Big(-i\,\mathrm{sgn}\,x\cdot\mathrm{ch}_{[-1,1]}\Big)^\vee \;=\; i\Big(\mathrm{ch}_{[-1,0]}-\mathrm{ch}_{[0,1]})\Big)^\vee \;=\; i\frac{e^{-2\pi i\xi}-1-1+e^{2\pi i\xi}}{2\pi i\xi} \;=\; \frac{\cos 2\pi\xi-1}{\pi\xi}$$

[7.3] Example: Recall that on $\mathbb{R}$ (the meromorphically continued tempered distribution) $\frac{1}{|x|^s}$ has Fourier transform

$$\frac{1}{|x|^s}^{\widehat{}} \;=\; \frac{1}{|x|^{1-s}}\cdot c_s^+ \qquad\text{with}\qquad c_s^+ \;=\; \frac{\pi^{\frac{s-1}{2}}\,\Gamma(\frac{1-s}{2})}{\pi^{\frac{-s}{2}}\,\Gamma(\frac{s}{2})}$$

Similarly, $\mathrm{sgn}(x)/|x|^s$ has Fourier transform

$$\Big(\frac{\mathrm{sgn}(x)}{|x|^s}\Big)^{\widehat{}} \;=\; \frac{\mathrm{sgn}(x)}{|x|^{1-s}}\cdot c_s^- \qquad\text{with}\qquad c_s^- \;=\; \frac{\pi^{\frac{s-2}{2}}\,\Gamma(\frac{2-s}{2})}{-i\,\pi^{\frac{s+1}{2}}\,\Gamma(\frac{s}{2})}$$

Thus, without concern for local integrability of the Fourier transform at 0, up to constants, the Hilbert transform would interchange $1/|x|^s$ and $\mathrm{sgn}(x)/|x|^s$:

$$\mathcal{H}\frac{1}{|x|^s} \;=\; \Big(-i\,\mathrm{sgn}(x)\cdot\big(\frac{1}{|x|^s}\big)^{\widehat{}}\Big)^\vee \;=\; \Big(-i\,\mathrm{sgn}(x)\cdot\frac{1}{|x|^{1-s}}\cdot c_s^+\Big)^\vee \;=\; -i\cdot c_s^+\cdot c_{1-s}^-\cdot\frac{\mathrm{sgn}(x)}{|x|^s}$$

For this computation to be legitimate (by at least one criterion), $\mathrm{sgn}(x)/|x|^{1-s}$ must be locally L^1 near 0. That is, we need $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) > 1$, the distribution $1/|x|^s$ only makes sense via meromorphic continuation, since it is not locally L^1 at 0.

[7.4] Example: Suitable tempered distributions with Fourier transforms supported in $[0,\infty)$ are $-i$ eigenfunctions for the Hilbert transform. An iconic example is the following. With

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x<0) \\ x^{s-1}e^{-x} & (\text{for } x>0)\end{cases}$$

using the identity principle from complex analysis to extend the obvious identity beyond the range where it is given by a literal change-of-variables, the *inverse* Fourier transform is

$$f^\vee(\xi) \;=\; \int_0^\infty e^{2\pi i\xi x}\,x^{s-1}\,e^{-x}\,dx \;=\; \int_0^\infty e^{-x(1+2\pi i\xi)}\,x^s\,\frac{dx}{x} \;=\; (1+2\pi i\xi)^{-s}\cdot\int_0^\infty e^{-x}\,x^s\,\frac{dx}{x} \;=\; \frac{\Gamma(s)}{(1+2\pi i\xi)^s}$$

For f to be locally L^1 near 0 is $\mathrm{Re}(s) > 0$. Thus, for any positive constant c, $1/(1+icx)^s$ has Fourier transform supported in $[0,+\infty)$. Thus, for $\mathrm{Re}(s) > 0$,

$$\mathcal{H}\frac{1}{(1+ix)^s} \;=\; \Big(-i\cdot\mathrm{sgn}x\cdot\big(\frac{1}{(1+ix)^s}\big)^{\widehat{}}\Big)^\vee \;=\; -i\cdot\Big(\big(\frac{1}{(1+ix)^s}\big)^{\widehat{}}\Big)^\vee \;=\; -i\cdot\frac{1}{(1+ix)^s}$$

[7.5] Example: In contrast to the previous example, where the Fourier transform of a function naturally vanished off a half-line, we can also consider artificially-vanishing modifications of f in $L^2(\mathbb{R})$ and other reasonable (generalized) function spaces: let

$$\widetilde{f}(\xi) \;=\; \int_0^\infty e^{2\pi i\xi x}f(x)\,dx \;=\; \big(\chi_{[0,+\infty)}\cdot f\big)^\vee(\xi)$$

By Fourier inversion,

$$\mathcal{H}\widetilde{f} \;=\; \Big(-i\cdot\mathrm{sgn}\cdot(\widetilde{f})\widehat{\ }\Big)^\vee \;=\; (-i\cdot\mathrm{sgn}\cdot\chi_{[0,+\infty)}\cdot f)^\vee \;=\; -i\cdot(\chi_{[0,+\infty)}\cdot f)^\vee \;=\; -i\cdot\widetilde{f}$$

[7.6] Example: In the families of distributions $1/|x|^s$ and $\mathrm{sgn}(x)/|x|^s$, the *residue* of $1/|x|^s$ at $s=1$ is a determinable constant multiple of δ. Further, $\widehat{\delta}=1$, so for some determinable constant c,

$$\mathcal{H}\delta \;=\; \Big(-i\,\mathrm{sgn}\cdot 1\Big)^\vee \;=\; -i\cdot\mathrm{sgn}^\vee \;=\; c\cdot\eta \qquad\qquad \text{(with } \eta \text{ the principal-value integral of } 1/x)$$

from parity-preserving and homogeneity-transforming properties of Fourier transform. In the context of the traditional belief that δ acts as an identity in convolution, we anticipate that the constant is 1.

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8. *Appendix: uniqueness of odd/even homogeneous distributions*

After clarification of notions of parity and positive-homogeneity, we recall the iconic proof of uniqueness of distributions of given parity and positive-homogeneity degree.

[8.1] Claim: For a positive-homogeneous distribution u of degree s, $x\frac{d}{dx}u$ is again positive-homogeneous of degree s, and with the same parity.

Proof: For test function f,

$$((x\frac{d}{dx}u)\circ t)(f) \;=\; \frac{1}{t}(x\frac{d}{dx}u)(f\circ t^{-1}) \;=\; \frac{1}{t}(\frac{d}{dx}u)(x\cdot f\circ t^{-1}) \;=\; -\frac{1}{t}u(\frac{d}{dx}(x\cdot(f\circ t^{-1})))$$

$$=\; -\frac{1}{t}u(f\circ t^{-1}+x\cdot t^{-1}\cdot(f'\circ t^{-1})) \;=\; -\frac{1}{t}u(f\circ t^{-1}+(x\cdot f')\circ t^{-1})) \;=\; -\frac{1}{t}u((f+xf')\circ t^{-1})$$

$$=\; -(u\circ t)(f+xf') \;=\; -(u\circ t)(\frac{d}{dx}(xf)) \;=\; t^s\cdot\big(-u(\frac{d}{dx}(xf))\big) \;=\; t^s\cdot(x\frac{d}{dx}u)(f)$$

as asserted. Preservation of parity is similar: writing $f^-(x)=f(-x)$ for functions f, and $u^-(f)=u(f^-)$ for distributions u,

$$(x\frac{d}{dx}u)^-(f) \;=\; (x\frac{d}{dx}u)(f^-) \;=\; (\frac{d}{dx}u)(x\cdot f^-) \;=\; -u(\frac{d}{dx}(x\cdot f^-)) \;=\; u(\frac{d}{dx}((x\cdot f)^-))$$

$$=\; u(-(\frac{d}{dx}(x\cdot f))^-) \;=\; -u^-(\frac{d}{dx}(x\cdot f)) \;=\; ((x\frac{d}{dx})u^-)(f)$$

as claimed. ///

The *Euler operator* $x\frac{d}{dx}$ is the infinitesimal form of *dilation* $f(x)\to f(tx)$:

[8.2] Corollary: $x\frac{d}{dx}u=s\cdot u$ for positive-homogeneous u of degree s.

Proof: Differentiate the distribution-valued equality $u\circ t=t^s\cdot u$ with respect to t and set $t=1$: the right-hand side is $s\cdot u$. For the left-hand side, for test function f, since $t\to u\circ t$ is a smooth distribution-valued function of t,

$$(\frac{\partial}{\partial t}(u\circ t))(f) \;=\; \frac{\partial}{\partial t}((u\circ t)(f)) \;=\; \frac{\partial}{\partial t}(\frac{1}{t}u(f\circ\frac{1}{t})) \;=\; u(\frac{\partial}{\partial t}(\frac{1}{t}f\circ\frac{1}{t})) \;=\; u\Big(\frac{-1}{t^2}(f\circ\frac{1}{t})-\frac{x}{t^2}\cdot(f'\circ\frac{1}{t})\Big)$$

Evaluating at $t=1$ gives

$$(s\cdot u)(f) \;=\; u\Big(-f-x\cdot f'\Big) \;=\; -u(f)-(x\cdot u)(f') \;=\; -u(f)+\Big(\frac{d}{dx}(x\cdot u)\Big)(f)$$

$$= \; -u(f) + (1 \cdot u + x \cdot u')(f) \; = \; \Big(x\frac{d}{dx}u\Big)(f)$$

as claimed. ///

[8.3] Theorem: Up to constant multiples, the principal-value distribution η is the unique *odd* distribution of positive-homogeneity degree -1. More generally, except for $s = -n$ with $\varepsilon = (-1)^{n+1}$ with $n = 1, 2, \ldots$, there is a unique (up to constant multiples) distribution of parity ε and of positive-homogeneity degree s.

Proof: Fix parity $\varepsilon = \pm$, and let V be the set of test functions of parity ε vanishing to infinite order at 0. This is the closure, in the space $\mathcal{D}_\varepsilon = \mathcal{D}_\varepsilon(\mathbb{R})$ of test functions of parity ε, of the space of test functions on $\mathbb{R}$ with support not containing 0 and of parity ε. On $C_c^\infty(0, +\infty)$ there is a unique continuous functional on $C_c^\infty(0, +\infty)$ positive homogeneous of degree s (as distribution), given by integration against $|x|^s$. Given a choice of parity ε, there is a unique extension to a continuous linear functional v_s^ε on $\mathcal{D}_o^\varepsilon$, and of positive-homogeneity degree s. Dualize the short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{D}_\varepsilon \longrightarrow \mathcal{D}_\varepsilon/V \longrightarrow 0$$

to a short exact sequence (invoking Hahn-Banach for exactness at the right-most joint)

$$0 \longrightarrow (\mathcal{D}_\varepsilon/V)^* \longrightarrow \mathcal{D}_\varepsilon^* \longrightarrow V^* \longrightarrow 0$$

The quotient $\mathcal{D}_\varepsilon/V$ is identifiable with germs of smooth functions of parity ε supported at 0 modulo germs of smooth functions vanishing identically at 0 of parity ε. The dual is the collection of distributions supported at 0 of parity ε, which, by the theory of Taylor-Maclaurin series, consists of finite linear combinations of Dirac δ and its derivatives, or parity ε. Of course, δ is even, δ' is odd, and so on.

Let $T = x\frac{d}{dx} - s$, and consider the very small (vertical) complex of short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{D}_\varepsilon/V)^* & \longrightarrow & \mathcal{D}_\varepsilon^* & \longrightarrow & V^* & \longrightarrow & 0 \\
 & & \downarrow T & & \downarrow T & & \downarrow T & & \\
0 & \longrightarrow & (\mathcal{D}_\varepsilon/V)^* & \longrightarrow & \mathcal{D}_\varepsilon^* & \longrightarrow & V^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The corresponding long (co-) homology sequence is

$$0 \to \ker T\Big|_{(\mathcal{D}_\varepsilon/V)^*} \to \ker T\Big|_{\mathcal{D}_\varepsilon^*} \to \ker T\Big|_{V^*} \to \frac{(\mathcal{D}_\varepsilon/V)^*}{T((\mathcal{D}_\varepsilon/V)^*)} \to \frac{\mathcal{D}_\varepsilon^*}{T(\mathcal{D}_\varepsilon^*)} \to \frac{V^*}{T(V^*)} \to 0$$

The functional v_s^ε is in $\ker T|_{V^*}$, and we hope to extend it uniquely to a functional in $\ker T|_{\mathcal{D}_\varepsilon^*}$. It would suffice that $\ker T_{(\mathcal{D}_\varepsilon/V)^*} = 0$ and $T((\mathcal{D}_\varepsilon/V)^*) = (\mathcal{D}_\varepsilon/V)^*$, that is, T is an isomorphism on distributions supported on $\{0\}$ of parity ε and positive-homogeneity degree s. The only failures are $s = -n$ with $\varepsilon = (-1)^{n+1}$, because

$$\Big(x\frac{d}{dx} - (-n)\Big)\delta^{(n)} \; = \; 0 \qquad\qquad \text{(for } n = 1, 2, \ldots)$$

For the principal-value functional, although $s = -1$ does occur as a pole for parity $+1$, the parity is $\varepsilon = -1$, so there is no obstruction to the extension, and it is unique. ///

For $\mathrm{Re}(s) > -1$, let u_s^ε be the distributions given by integration against $|x|^s$ for $\varepsilon = +1$ and $\mathrm{sgn}\, x \cdot |x|^s$ for $\varepsilon = -1$. For $\mathrm{Re}(s) > -1$, these are locally integrable, and of respective parities $\varepsilon = \pm 1$ and of positive-homogeneity degree s.

[8.4] Corollary: The distributions u_s^ε have meromorphic continuations to tempered-distribution-valued functions of $s \in \mathbb{C}$, with only simple poles. For $\varepsilon = +1$, the poles are at $-n = -1, -3, -5, \ldots$, with residues constant multiples of $\delta^{(n)}$. For $\varepsilon = -1$, the poles are at $-n = -2, -4, -6, \ldots$, with residues constant multiples of $\delta^{(n)}$. The meromorphic continuations maintain the positive-homogeneity and parity. For $s \neq -1$, $u_s^\varepsilon = \frac{1}{s+1} \frac{d}{dx} u_{s+1}^{-\varepsilon}$. For $s = -1$,

$$\frac{\partial}{\partial x} u_0^{-1} \;=\; 2 \cdot \delta \qquad \text{and} \qquad \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial s} u_s^{+1}\Big|_{s=0}\Big) \;=\; u_{-1}^{-1} \;=\; \eta$$

Proof: The functionals v_s^ε on V are *entire*, since there is no convergence issue for test functions vanishing to infinite order at 0. Thus, at pole $-n$, $\lim_{s\to -n}(s+n)u_s^\varepsilon$ vanishes on test functions vanishing to infinite order at 0. Thus, the residue $\lim_{s\to -n}(s+n)u_s^\varepsilon$ is a distribution supported on $\{0\}$. It has the same parity ε, and is positive-homogeneous of degree $-n$, in the distributional sense. Thus, there is *no* pole unless $\varepsilon = (-1)^n$, for $n = 0, -1, -2, \ldots$.

For $\mathrm{Re}(s) \gg 1$, the integrate-against functionals u_s^ε given by $|x|^s$ for $\varepsilon = +1$ and $\mathrm{sgn} x \cdot |x|^s$ for $\varepsilon = -1$ are of (distributional) positive-homogeneity degree s, and parity ε. For each $\mathrm{Re}(s) \gg 1$, u_s^ε is the unique extensions of the distributions v_s^ε, up to constants possibly depending on s. By the vector-valued form of the identity principle from complex analysis, the same homogeneity and parity properties hold for any meromorphic continuation. From $x \frac{d}{dx} u_s^\varepsilon = s \cdot u_s^\varepsilon$,

$$\frac{d}{dx} u_s^{-\varepsilon} \;=\; \frac{1}{x} \cdot s \cdot u_s^{-\varepsilon} \;=\; s \cdot u_{s-1}^\varepsilon$$

Thus, $u_s^\varepsilon = \frac{1}{s+1} \frac{d}{dx} u_{s+1}^{-\varepsilon}$. Since $u_{s+1}^{-\varepsilon}$ is a holomorphic distribution-valued function in $\mathrm{Re}(s) > -2$, this extends u_s^ε to $\mathrm{Re}(s) > -2$, except for possible pole at $s = -1$. By induction, $u_s^\pm$ has a meromorphic continuation to all of $\mathbb{C}$ except for possible (simple) poles at negative integers. As above, the only possible residues are distributions supported at 0, which can only occur at non-positive even integers, or non-positive odd integers, depending on parity.

For $\varepsilon = +1$, the relation $\frac{d}{dx} u_s^{-\varepsilon} = s \cdot u_{s-1}^\varepsilon$ evaluated at $s = 0$ yields the *residue* of u_s^{+1} at -1, namely, a constant multiple of δ. For $\varepsilon = -1$, there is no pole, and evaluation of the relation at $s = 0$ just gives $\frac{d}{dx} 1 = 0$. Thus, we differentiate in s before evaluation: the relation $\frac{\partial}{\partial x} u_s^{-\varepsilon} = s \cdot u_{s-1}^\varepsilon$ gives

$$\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial s}\Big|_{s=0} u_s^{+1}\Big) \;=\; \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial x} u_s^{+1} \;=\; \frac{\partial}{\partial s}\Big|_{s=0}\Big(s \cdot u_{s-1}^{-1}\Big) \;=\; (u_{s-1}^{-1} + s \cdot \frac{\partial}{\partial s} u_s^{-1})\Big|_{s=0} \;=\; u_{-1}^{-1}$$

Of course,

$$\frac{\partial}{\partial s}\Big|_{s=0} u_s^{+1} \;=\; \frac{\partial}{\partial s}\Big|_{s=0} |x|^s \;=\; (|x|^s \cdot \log|x|)\Big|_{s=0} \;=\; \log|x|$$

This completes the discussion. ///

[8.5] Remark: In particular, the meromorphic continuations are *tempered* distributions. That is, all positive-homogeneous distributions are tempered. Also, away from 0, homogeneous distributions are given locally by smooth functions.

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9. *Appendix: $\mathcal{E}^*$ acting on $\mathcal{E}$*

This is another action of the space $\mathcal{E}^*$ of compactly-supported distributions on a space of nice functions, namely the smooth functions $\mathcal{E}$. This does not apply to the Hilbert transform, but is informative and relevant to other scenarios, for example, pseudo-differential and Fourier-integral operators.

With translation $T_x f(y) = f(x+y)$, a (traditional) natural action of $u \in \mathcal{E}^*$ on $f \in \mathcal{E}$ is

$$(u \cdot f)(x) \;=\; u(T_x f) \qquad\qquad (\text{for } x \in \mathbb{R}^n)$$

Certainly $x \to (u \cdot f)(x)$ is well-defined pointwise. The $\mathcal{E}$-valued function $x \to T_x f$ is a *smooth* $\mathcal{E}$-valued function (from the definitions), so $x \to u(T_x f)$ is a smooth $\mathbb{C}$-valued function, since u is linear and continuous. *Convolution* $u * v$ on $\mathcal{E}^*$ is characterized by *associativity*

$$(u * v)(f) \;=\; u \cdot (v \cdot f) \qquad\qquad (\text{for all } f \in \mathcal{E})$$

The associativity requirement yields an expression for convolution:

[9.1] Claim: $u * v \;=\; (u \otimes v) \circ a$, with[1] $(af)(y \times z) = f(y+z)$.

Proof: Compute directly:

$$(u \cdot (v \cdot f))(x) \;=\; u(T_x(v \cdot f)) \;=\; u(T_x(y \to v(T_y f))) \;=\; u(y \to v(T_{x+y} f))$$

$$= \; u\big(y \to v(z \to T_{x+y} f(z))\big) \;=\; (u \otimes v)\big(y \times z \to T_x f(y+z))\big) \;=\; (u \otimes v)\big(y \times z \to (s(T_x f))(y,z))\big)$$

$$= \; (u \otimes v)(a(T_x f)) \qquad\qquad (\text{with } (af)(y \times z) = f(y+z))$$

giving the desired expression for $u * v$. ///

[9.2] Claim: This convolution on $\mathcal{E}^*$ is associative.

Proof: Expanding the expression for convolution just derived,

$$(u * v) * w \;=\; ((u \otimes v) \circ a) * w \;=\; (((u \otimes v) \circ a) \otimes w) \circ a$$

Applied to $f \in \mathcal{E}(\mathbb{R}^n)$, not necessarily of the form $T_x f$, this is

$$\big((((u \otimes v) \circ a) \otimes w) \circ a\big)(f) \;=\; \big((((u \otimes v) \circ a) \otimes w) \circ a\big)(x \to f(x)) \;=\; \big((((u \otimes v) \circ a) \otimes w)(af)\big)(y \times z \to y+z)$$

For $F \in \mathcal{E}(\mathbb{R}^n \times \mathbb{R}^n)$, letting $(bF)(x \times y \times z) = F(x+y, z)$,

$$\big((((u \otimes v) \circ a) \otimes w) F\big)(y \times z \to y+z) \;=\; \big(((u \otimes v) \otimes w)(bF)\big)((x \times y) \times z \to (x+y)+z)$$

That is,

$$((u * v) * w)(f) \;=\; ((u \otimes v) \otimes w)(tf) \qquad\qquad (\text{three-fold (co-addition) } tF(x \times y \times z) = F(x+y+z))$$

The associativity of tensor products, and a similar computation for $u * (v * w)$, gives the associativity of convolution on $\mathcal{E}^*$. ///

[9.3] Remark: A similar argument applies to real Lie groups G in place of $\mathbb{R}^n$, with due attention to non-commutativity.

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