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# *An iconic error ... computing Maaß-Selberg relations*

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Long ago, in 1980, I witnessed an argument about computation of inner products of truncated Eisenstein series. The claim made in 1980, which already has substance on the upper half-plane with $SL_2(\mathbb{Z})$ acting, is that *truncated Eisenstein series* $\wedge^T E_s$ *are essentially eigenfunctions for the Laplace-Beltrami operator* $\Delta$.

This is not true: truncated Eisenstein series are *not* eigenfunctions in any relevant sense.

Yet it *is* true that $\big(\Delta - s(s-1)\big) \wedge^T E_s$ vanishes away from the discontinuity of the truncated Eisenstein series.

The (false) assumption that truncated Eisenstein series are *essentially* eigenfunctions *does* permit a plausible-looking and traditional (!) derivation of the *correct* inner product of two truncated Eisenstein series... but, of course, also of any other assertion, true or false. This outcome can be and has been (erroneously) construed as validating the (incorrect) argument that truncated Eisenstein series are effectively eigenfunctions.

To my surprise, I witnessed a reiteration of this erroneous argument recently, in 2009. I conclude that the error and its supporting rationalizations have achieved iconic status.

To assert that $\wedge^T E_s$ is an eigenfunction *except for a set of measure* 0 is at least *misleading*, because differentiation of a discontinous function such as $\wedge^T E_s$ only has sense *distributionally*, and non-trivial distributions can easily have support on sets of measure 0.

About Schwartz' distributions: on many occasions, in response to a comment that a seemingly nonsensical computational outcome must be reconsidered in the context of *distributions*, one may hear "But I'm not *doing* distribution theory," as though the viewpoint were an *option*. My preferred and sincere response to the latter is: we are *always* doing distribution theory. The question is whether or not we *realize* this, and whether or not we do so *competently*.

In fact, truncated Eisenstein series are *not* $\Delta$-eigenfunctions in any useful sense: we show that this assumption would imply that truncated Eisenstein series are almost everywhere zero. Suppose that

$$(\Delta - s(s-1))\big(\wedge^T E_s\big) \;=\; (L^1_{\text{loc}} \text{ function vanishing a.e.}) \qquad\qquad \text{(Note: this is false!)}$$

Let $\alpha, \beta$ be complex numbers such that $\alpha(\alpha-1) \neq \beta(\beta-1)$. Then the obvious argument would prove orthogonality of the two truncated Eisenstein series:

$$\langle \wedge^T E_\alpha, \wedge^T E_\beta\rangle \;=\; \langle \frac{\Delta}{\alpha(\alpha-1)} \wedge^T E_\alpha, \wedge^T E_\beta\rangle \;=\; \frac{1}{\alpha(\alpha-1)}\langle \Delta \wedge^T E_\alpha, \wedge^T E_\beta\rangle \;=$$

$$\frac{1}{\alpha(\alpha-1)}\langle \wedge^T E_\alpha, \Delta \wedge^T E_\beta\rangle \;=\; \frac{1}{\alpha(\alpha-1)}\langle \wedge^T E_\alpha, \beta(\beta-1) \wedge^T E_\beta\rangle \;=\; \frac{\overline{\beta}(\overline{\beta}-1)}{\alpha(\alpha-1)}\langle \wedge^T E_\alpha, \wedge^T E_\beta\rangle$$

Thus, the inner product is 0. Being meromorphic in $\alpha$ and in $\overline{\beta}$, the inner product is *identically* 0. Thus, the (square-integrable) truncated Eisenstein series would be zero in $L^2(\Gamma\backslash\mathfrak{H})$. ///

Another hint of the impossibility of any useful eigenfunction property is that, if $\wedge^T E_s$ were essentially an eigenfunction, then the eigenvalue $s(s-1)$ would necessarily be *real*, which is certainly not so for typical $s \in \mathbb{C}$.

Further: purported eigenfunctions $\wedge^T E_s$ would provide a *continuum* of linearly independent eigenfunctions, impossible in the separable Hilbert space $L^2(\Gamma\backslash\mathfrak{H})$.

Perhaps more to the point, since the truncated Eisenstein series are in $L^2(\Gamma\backslash\mathfrak{H})$, they *do* have an $L^2$ spectral decomposition in terms of *genuine* eigenfunctions, which is obviously at odds with the notion that they themselves are eigenfunctions. Truncated Eisenstein series are orthogonal to cuspforms, and their $L^2$ spectral expansion is

$$\wedge^T E_\alpha \;=\; \frac{\langle \wedge^T E_\alpha, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \langle \wedge^T E_\alpha, E_s\rangle\, E_s\, ds$$

where a correct computation via Maaß-Selberg relations gives

$$\frac{\langle \wedge^T E_\alpha, 1\rangle \cdot 1}{\langle 1, 1\rangle} \;=\; \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathfrak{H})}\Big(\frac{T^{s-1}}{s-1} + c_\alpha \frac{T^{-s}}{-s}\Big)$$

and

$$\langle \wedge^T E_\alpha, E_s\rangle \;=\; \frac{T^{\alpha+\overline{s}-1}}{\alpha+\overline{s}-1} + c_\alpha \frac{T^{(1-\alpha)+\overline{s}-1}}{(1-\alpha)+\overline{s}-1} + c_{\overline{s}}\,\frac{T^{\alpha+(1-\overline{s})}}{\alpha+(1-\overline{s})-1} + c_\alpha\, c_{\overline{s}}\,\frac{T^{(1-\alpha)+(1-\overline{s})-1}}{(1-\alpha)+(1-\overline{s})-1}$$

and

$$c_\beta \;=\; \frac{\xi(2\beta-1)}{\xi(2\beta)}$$

Correct derivations of Maaß-Selberg relations are not difficult. For example, see

http://www.math.umn.edu/~garrett/m/v/maass_selberg_trivial.pdf

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A treatment of Maaß-Selberg relations emphasizing notions of generalized functions is in

[Casselman 1993] W. Casselman, *Extended automorphic forms on the upper half-plane*, Math. Ann. **296** (1993) no. 4, pp. 755-762.

The latter was partly motivated by

[Zagier 1982] D. Zagier, *The Rankin-Selberg method for automorphic functions which are not of rapid decay*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, pp. 415-437 (1982).

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