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Inducing cuspidal representations from compact opens

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We follow
• H. Jacquet, *Representation des groups lineaires p-adic*, Theory of Group Representations and Harmonic Analysis, CIME, II Ciclo, Mentcatini Terme 1970, 119-220, Edizioni Cremonese, Roma 1971

to see how to construct some supercuspidal representations of p-adic reductive groups from cuspidal (in a slightly different sense) representations of a maximal compact subgroup.

The further idea that *all* supercuspidal representations appear in this fashion occurred in
• R. Howe, *Tamely ramified supercuspidal representations of GL(n)*, Pac. J. Math. 73 (1977), 437-460.
• R. Howe, *Some qualitative results on the representation theory of GL(n) over a p-adic field*, Pac. J. Math. 73 (1977), 479-538.

This was brought to a certain fruition in
• C. Bushnell, P. Kutzko, *The Admissible Dual of GL(N) via Compact-Open Subgroups*, Ann. of Math. Studies no. 129, Princeton Univ. Press, 1993.

Let G be a p-adic reductive group, with special maximal compact K. For example, $G = GL(n, \mathbf{Q}_p)$ and $K = GL(n, \mathbf{Z}_p)$. Let σ be a an irreducible representation of K with the *cuspidal* property that, for every parabolic P of G and for N the unipotent radical of P

$$\int_{N\cap K} \sigma(n)\, dn = 0 \in \mathrm{End}_{\mathbf{C}}(\sigma)$$

For $G = GL(2, \mathbf{Q}_p)$ and $K = GL(2, \mathbf{Z}_p)$, for example, let H be the normal subgroup of K consisting of matrices congruent to 1 mod p, so we have $GL(2, \mathbf{Z}/p) \approx K/H$. Then finite group theory (explicit counting of conjugacy classes versus the irreducibles constructed via parabolic induction, etc.) shows that that $GL(2, \mathbf{Z}/p)$ has many cuspidal representations in this sense. That is, since there are $(q + 1)(q - 1)$ conjugacy classes there are $(q + 1)(q - 1)$ distinct irreducibles, of which $(q^2 - q)/2$ are cuspidal. [1]

Let Z be the center of G, and extend [2] σ to ZK. Let

$$\pi = \mathrm{Ind}_{ZK}^{G} \sigma$$

be the uniformly locally constant induction (the smooth dual to the compactly-supported induction). Fix a choice A of maximal split torus in a choice of minimal parabolic.

Proposition: For $f \in \pi$, for sufficiently small $\varepsilon > 0$, for $a \in A^-(\varepsilon)$ we have

$$f(a^{-1}) = 0$$

Thus, $a \to f(a^{-1})$ has compact support on A modulo the center Z.

Proof: Let Δ^+ be the positive roots on A corresponding to the choice of minimal parabolic. For $\varepsilon > 0$, let

$$A^-(\varepsilon) = \{a \in A : |\alpha(a)| < \varepsilon, \;\; \text{for all } \alpha \in \Delta^+\}$$

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[1] The count of cuspidal representations is not completely trivial, but does follow from counting the irreducible principal series, and the *special* representations and one-dimensional representations that occur in the reducible principal series. E.g., see http://www.math.umn.edu/˜garrett/m/v/toy_GL2.dvi

[2] Such an extension exists because $K \cap Z$ is open in Z, so $Z/Z \cap K$ is discrete.

Call another parabolic *standard* if it contains that fixed minimal one. Let $f \in \pi^{K'}$ for a compact open subgroup K'. For sufficiently small $\varepsilon > 0$, for all unipotent radicals N of standard parabolics,

$$a(N \cap K')a^{-1} \supset N \cap K, \quad \text{for all } a \in A^-(\varepsilon)$$

Since f is right K'-invariant it is certainly right $(K' \cap N)$-invariant. Then

$$f(a^{-1}) = \int_{N\cap K'} f(a^{-1}n)\, dn = \int_{N\cap K'} f(a^{-1}na \cdot a^{-1})\, dn = \int_{N\cap K'} \sigma(a^{-1}na)\, dn \cdot f(a^{-1})$$

Replacing n by ana^{-1}, up to a change-of-measure constant this is

$$\int_{a^{-1}(N\cap K')a} \sigma(n)\, dn \cdot f(a^{-1}) = \int_{a^{-1}(N\cap K')a/(N\cap K)} \left(\int_{K\cap N} \sigma(n'n)\, dn\right) dn' \cdot f(a^{-1})$$

The inner integral is 0, since σ is cuspidal on K. Thus,

$$f(t^{-1}) = 0$$

for $t \in A^-(\varepsilon)$ depending on K'. ///

Theorem: Every function $f \in \pi$ is compactly supported modulo the center Z of G.

Proof: Let $f \in \pi^{K'}$. For fixed compact open K', let X be a (finite) collection of representatives for K/K'. By the Cartan decomposition

$$G = KAK = \bigcup_x KAxK'$$

Let

$$A^+ = \{a \in A : |\alpha(a)| \ge 1, \quad \text{for all } \alpha \in \Delta^+\}$$

$$A^- = \{a \in A : |\alpha(a)| \le 1, \quad \text{for all } \alpha \in \Delta^+\}$$

Since K contains (representatives for) the Weyl group W of A, we have

$$A = \bigcup_{w\in W} wA^+w^{-1}$$

so in fact

$$G = KA^+K = \bigcup_x KA^+xK'$$

For $x \in X$, the function $f_x(g) = f(gx)$ is in $\pi^{xK'x^{-1}}$. Thus, for $x \in X$ there is ε_x such that $f(a^{-1}x = 0$ for $a \in A^-(\varepsilon_x)$. Let $\varepsilon > 0$ be the minimum of all the ε_x. Then for any $x \in X$, $k \in K$, $k' \in K'$, for all $a \in A^-(\varepsilon)$,

$$f(ka^{-1}xk') = 0$$

The set

$$C_{K'} = A^- - A^-(\varepsilon)$$

is compact modulo the center, and f is 0 off $KC_{K'}K$, which is compact modulo the center since K is compact. ///

Corollary: π is *admissible*: for a compact open subgroup K' of G,

$$\dim_{\mathbf{C}} \pi^{K'} < \infty$$

Proof: Again, any set

$$C_{K'} = A^- - A^-(\varepsilon)$$

is compact modulo Z, and f is zero off some set $KC_{K'}K$, and by the compactness mod Z

$$KC_{K'}K/ZK' = \text{ finite}$$

A function $f \in \pi^{K'}$ is well-defined on such a quotient, so lies in a finite-dimensional space. ///

Theorem: The induced representation π is *supercuspidal*.

Proof: We show that all Jacquet modules (co-isotypes of the trivial representation of N)

$$\pi_N = \pi/(\text{subspace generated by all } v - \pi(n)\cdot v,\ n \in N,\ v \in \pi)$$

are 0, for N the unipotent radical of a standard parabolic $P = MN$ (a Levi decomposition). Given $f \in \pi$, by the Iwasawa decomposition, there are compacta $C_M \subset M$ and $C_N \subset N$ such that

$$\mathrm{spt} f \subset KZC_MC_N$$

Given $m \in ZC_M$, take N' a compact open subgroup of N such that $N' \supset C_N$ and

$$N' \supset \bigcup_{m \in ZC_M} m^{-1}(N \cap K)m$$

This is possible since the latter union is compact, being a continuous image of the compact $C_M \times N$, noting that Z acts trivially by conjugation. For $g \in G$ let $g = kmn$ with $k \in K$, $m \in ZC_M$, and $n \in C_N$. Then

$$\int_{N'} f(gn')\,dn' = \int_{N'} f(kmnn')\,dn' = \sigma(k) \cdot \int_{N'} f(mnn')\,dn'$$

And since $N' \supset N_C$, we can replace n' by $n^{-1}n'$, and the integral becomes

$$\int_{N'} f(mn')\,dn' = \int_{N'} f(mn'm^{-1} \cdot m)\,dn'$$

Since $mN'm^{-1} \supset N \cap K$, this is

$$\int_{(N\cap K)\backslash N'} \left(\int_{N\cap K} f(n \cdot mn'm^{-1} \cdot m)\,dn \right) dn'$$

The inner integral is

$$\int_{N\cap K} \sigma(n)\,dn \cdot f(mn'm^{-1} \cdot m) = 0 \cdot f(mn'm^{-1} \cdot m)$$

by the cuspidality of σ. Thus, the whole is 0. ///
